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## A min-max theorem for multiple integrals of the Calculus of Variations and applications

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**Analisi matematica.** — *A min-max theorem for multiple integrals of the Calculus of Variations and applications.* Nota di DAVID ARCOYA e LUCIO BOCCARDO, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — In this paper we deal with the existence of critical points for functionals defined on the Sobolev space $W_0^{1,2}(\Omega)$ by $J(v) = \int_\Omega \mathfrak{J}(x, v, Dv)\,dx$, $v \in W_0^{1,2}(\Omega)$, where $\Omega$ is a bounded, open subset of $\boldsymbol{R}^N$. Since the differentiability can fail even for very simple examples of functionals defined through multiple integrals of Calculus of Variations, we give a suitable version of the Ambrosetti-Rabinowitz Mountain Pass Theorem, which enables us to the study of critical points for functionals which are not differentiable in all directions. Then we present some applications of this theorem to the study of the existence and multiplicity of nonnegative critical points for multiple integrals of the Calculus of Variations.

KEY WORDS: Critical points; Multiple integrals of Calculus of Variations; Quasilinear equations.

RIASSUNTO. — *Un teorema di min-max per integrali multipli del Calcolo delle Variazioni.* In questa *Nota* si studia l'esistenza di punti critici per funzionali definiti nello spazio di Sobolev $W_0^{1,2}(\Omega)$ da $J(v) = \int_\Omega \mathfrak{J}(x, v, Dv)\,dx$, $v \in W_0^{1,2}(\Omega)$, dove $\Omega$ è un aperto limitato di $\boldsymbol{R}^N$. Esempi molto semplici (e ben noti) di funzionali mostrano che la differenziabilità può mancare anche se la funzione $\mathfrak{J}(x, v, Dv)$ è molto regolare. Questo motiva l'interesse a un teorema astratto di punto critico del tipo di quello di Ambrosetti-Rabinowitz («Mountain Pass Theorem» o «Teorema del cratere») per funzionali che non sono differenziabili in tutte le direzioni. Presentiamo alcune applicazioni di tale teorema allo studio dell'esistenza e della molteplicità di punti critici positivi di integrali multipli del Calcolo delle Variazioni.

## INTRODUCTION

We deal with the existence of critical points of functionals defined on Banach spaces. It is well known that existence of critical points and solvability of Euler-Lagrange equations are related, as well as that there is an extensive literature about the critical points which are minima, specially if the functionals are defined on the Sobolev space $W_0^{1,p}(\Omega)$ by

$$\int_\Omega \mathfrak{J}(x, v, Dv)\,dx\,, \qquad v \in W_0^{1,p}(\Omega)\,, \qquad p > 1\,,$$

where $\Omega$ is a bounded, open subset of $\boldsymbol{R}^N$ (see [9, 11]). Besides variational problems whose solutions correspond to minima, there is a broad variety of cases where one looks for critical points different from minima. Some methods have been developed to find critical points: one of the most important is the Ambrosetti-Rabinowitz minimax theorem (the so called Mountain Pass Theorem). Its statement (see [1, 2, 14]) involves, in addition to geometrical conditions, a technical assumption (the Palais-Smale condition) on the functional $J$ which often occurs in critical points theory: *any sequence* $(u_n)$

(*) Nella seduta del 16 giugno 1994.

*in the Banach space E for which $J(u_n)$ is bounded and $J'(u_n)$ converges to zero in the dual space $E'$, possesses a convergent subsequence*. In the previous statement, the differentiability on $E$ of the functional $J$ is needed. Unfortunately, the differentiability can fail even for very simple examples of functionals defined through multiple integrals of Calculus of Variations. For example, if one considers

$$I(v) = \int_\Omega A(x, v)|Dv|^2\, dx, \qquad v \in W_0^{1,2}(\Omega),$$

with $0 < \alpha_1 \leqslant A(x, z)$, $A_z'(x, z) \leqslant \alpha_2 < \infty$ and $N > 1$, then $I$ is not Gateaux-differentiable. It is only differentiable along directions of $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, even for smooth functions $A(x, z)$ (see [11, 13]). The first approach to the critical points of such kind of functional is given in [3]. For this reason, we prove a suitable version of the Ambrosetti-Rabinowitz Mountain Pass Theorem, which enables us to the study of critical points of functionals which are not differentiable in all directions. Then we apply this theorem to study existence and multiplicity of nonnegative critical points for some classes of functionals of the Calculus of Variations.

In this *Note*, we present, for a simple example, the results which we shall prove in [4]. Specifically, we consider the functional $J$ defined in $W_0^{1,2}(\Omega)$ by setting

$$J(v) = (1/2)\int_\Omega A(x, v)|Dv|^2\, dx - \int_\Omega F(x, v)\, dx, \qquad v \in W_0^{1,2}(\Omega), \tag{1}$$

where $A\colon \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is a Carathéodory function and $F(x, z) = \int_0^z f(x, t)\, dt$ is the primitive of a Carathéodory function $f$ on $\Omega \times \boldsymbol{R}$ satisfying:

$(A_1)$ There exist $\beta_1 > \alpha_1 > 0$ such that $\alpha_1 \leqslant A(x, z) \leqslant \beta_1$, for almost every $x \in \Omega$ and every $z \in \boldsymbol{R}$.

$(A_2)$ There exists the partial derivative $A_z'(x, z)$ of $A(x, z)$ which is also assumed to be a Carathéodory function such that $|A_z'(x, z)| \leqslant \beta_2$, for almost every $x \in \Omega$, $\forall z \in \boldsymbol{R}$, for some $\beta_2 > 0$.

$(A_3)$ $zA_z'(x, z) \geqslant 0$ for almost every $x \in \Omega$, for every $z \in \boldsymbol{R}$.

$(f_1)$ $f(x, 0) = 0$ $(x \in \Omega)$, and there exist $C_1$, $C_2 > 0$ such that $|f(x, z)| \leqslant C_1 |z|^\sigma + C_2$, for almost every $x \in \Omega$, $\forall z \in \boldsymbol{R}$, with $\sigma + 1 < 2^*$, $(2^* = 2N/(N-2)$, if $2 < N$, while $2^* = \infty$, if $N = 2)$.

Under these conditions, the functional $J$ is, in general, not Gateaux differentiable. However, it is well known (see [9]) that $J$ has a directional derivative $\langle J'(v), w\rangle$ at each $v \in W_0^{1,2}(\Omega)$ along any direction $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Indeed,

$$\langle J'(v), w\rangle = \int_\Omega A(x, v)Dw\, dx + \int_\Omega (1/2)A_z'(x, v)|Dv|^2 w\, dx - \int_\Omega f(x, v)w\, dx$$

for every $v \in W_0^{1,2}(\Omega)$ and $w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

DEFINITION 0.1. *A function $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is called a critical point of $J$ if $\langle J'(v), w\rangle = 0$, $\forall w \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.*

Therefore, the critical points $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of $J$ are solutions of the boundary value problem:

$$\text{(P)} \qquad \begin{cases} -\operatorname{div}(A(x,u)Du) + \dfrac{1}{2} A_z'(x,u)|Du|^2 = \dfrac{\partial F}{\partial z}(x,u) \equiv f(x,u), \\ u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \end{cases}$$

in the sense that

$$\int_\Omega A(x,u)\, Du\, Dv\, dx + (1/2)\int_\Omega A_z'(x,u)|Du|^2 v\, dx = \int_\Omega f(x,u) v\, dx,$$

for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Observe that $u = 0$ is a trivial solution. Depending on different assumptions on the growth of $F(x,z)$ (*e.g.*, superquadratic or subquadratic growth) with respect to $|z|$, we obtain different results of existence of nontrivial and nonnegative solutions. From the variational point of view, these distinct growths mean that the functional is either non-coercive or coercive.

## 1. A MOUNTAIN PASS THEOREM FOR SOME NON-DIFFERENTIABLE FUNCTIONALS

We devote this section to give a suitable version of the well-known Mountain Pass Theorem of A. Ambrosetti and P. H. Rabinowitz [2] which enables us to study the existence of critical points for functionals which are not differentiable in all directions. Specifically, and in order to obtain further results in the future, we prove a general min-max theorem from which it may be deduced the version of the Mountain Pass Theorem it is needed as well as new versions of others min-max theorems (see [10]). The theorem is the following:

THEOREM 1.1. *Let $(X, \|\cdot\|_X)$ be a Banach space and $Y \subset X$ a subspace which is a Banach space endowed with a norm $\|\cdot\|_Y$ such that $\|y\|_X \leq \|y\|_Y$, for every $y \in Y$. Assume that $J: X \to \mathbf{R}$ is a functional on X such that $J|_{(Y, \|\cdot\|_Y)}$ is lower semicontinuous* (l.s.c.) *and that it satisfies the following hypotheses:*

*a) $J$ has a directional derivative $\langle J'(u), v\rangle$ at each $u \in X$ through any direction $v \in Y$.*

*b) For fixed $u \in X$, the function $\langle J'(u), v\rangle$ is linear in $v \in Y$, and for fixed $v \in Y$, the function $\langle J'(u), v\rangle$ is continuous in $u \in X$.*

*Let also K be a compact metric space, $K_0 \subset K$ a closed subset and $\gamma_0 : K_0 \to (Y, \|\cdot\|_Y)$ a continuous map. Consider $\Gamma = \{\gamma : K \to (Y, \|\cdot\|_Y) / \gamma$ is a continuous and $\gamma|_{K_0} = \gamma_0\}$.*

*If*

$$(2) \qquad c = \inf_{\gamma \in \Gamma} \max_{t \in K} J(\gamma(t)) > c_1 = \max_{t \in K_0} J(\gamma_0(t)),$$

*then, for every* $\varepsilon > 0$ *and* $\gamma \in \Gamma$ *such that*

$$c \leq \max_{t \in K} J(\gamma(t)) \leq c + \varepsilon/2 \tag{3}$$

*there exist* $\bar{\gamma}_\varepsilon \in \Gamma$ *and* $u_\varepsilon \in \bar{\gamma}_\varepsilon(K) \subset Y$ *satisfying*

$$c \leq \max_{t \in K} J(\bar{\gamma}_\varepsilon(t)) \leq \max_{t \in K} J(\gamma(t)) \leq c + \varepsilon/2\,, \qquad \max_{t \in K} \|\bar{\gamma}_\varepsilon(t) - \gamma(t)\|_Y \leq \sqrt{\varepsilon}\,,$$

$$c - \varepsilon \leq J(u_\varepsilon) \leq c + \varepsilon/2\,, \qquad |\langle J'(u_\varepsilon), v\rangle| \leq \sqrt{\varepsilon}\,\|v\|_Y\,, \qquad \forall v \in Y\,.$$

REMARK 1.2. In the literature, there are different proofs of the classical Mountain Pass Theorem (see [2, 5, 7, 10, 12]). The original one [2] is based on a suitable deformation lemma. H. Brézis gave an alternative proof using Ekeland variational principle [8] (see [5, 10]). Our proof of this theorem follows closely the ideas of [12] which also applies this principle.

## 2. APPLICATIONS

In this section, we present the main results of existence of nontrivial and nonnegative solutions of (P). In the proofs, we shall use the abstract Theorem 1.1, with $Y = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, combined with some techniques of [6] which are useful to prove our modified Palais-Smale condition. We start with the case $f(x, z)$ superlinear at $+\infty$, *i.e.*, $f(x,z)$ satisfies

($f_2$) $\lim_{z \to +\infty} z^{-1} f(x,z) = +\infty$, uniformly in $x \in \Omega$.

In addition to ($f_2$), we assume ($f_1$) and the following condition on $f$:

($f_3$) There exist $m > 2$, $\alpha_2 > 0$ and $R_1 > 0$ such that $mF(x,z) \leq zf(x,z) + \alpha_2 \lambda_1 |z|^2$, for almost every $x \in \Omega$ and $|z| \geq R_1$; where $\lambda_1 > 0$ denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions on $\Omega$.

Also, let us suppose the following hypothesis on $A$:

($A_4$) There exists $\alpha_3 > \alpha_2$ such that $(m/2 - 1)A(x,z) - (1/2)zA_z'(x,z) \geq \alpha_3$, for almost every $x \in \Omega$, $z \in \mathbf{R}$ ($m$ and $\alpha_2$ are given by ($f_3$)).

Before to continue, some remarks about ($f_3$) and ($A_3$) are needed:

REMARKS 2.1. *i*) Hypothesis ($f_3$) is slightly more general than the one introduced by A. Ambrosetti and P. H. Rabinowitz [2] in the semilinear case, *i.e.* $A(x,z) = 1$. In fact, these authors assume the existence of $m > 2$ and $R_1 > 0$ such that

$$mF(x,z) \leq zf(x,z)\,, \tag{4}$$

for almost every $x \in \Omega$ and $|z| \geq R_1$.

*ii*) Observe also that if we assume $(A_1)$ then a sufficient condition for $(A_4)$ to hold is the following:

(5) $$zA_z'(x,z) \leqslant \delta A(x,z), \qquad \forall x \in \Omega, \qquad \forall z \in \mathbf{R},$$

for some $\delta \in (0, m-2)$.

*iii*) Very simple examples show the meaning of hypothesis $(A_4)$ and point out its relationship with the Ambrosetti-Rabinowitz condition (4) for semilinear (superlinear) equations. In fact, consider the example:

$$J(u) = (1/2)\int_\Omega b(u)^2 |Du|^2\,dx - \int_\Omega |u|^\mu\,dx, \qquad u \in W_0^{1,2}(\Omega),$$

where $\mu > 2$ and $b(z)$ is a smooth function such that $0 < \alpha_1 \leqslant b(z) \leqslant \beta_1$, $|b'(z)| \leqslant \beta_3$, $\forall z \in \mathbf{R}$.

Define

$$B(z) = \int_0^z b(t)\,dt.$$

For every $u \in W_0^{1,2}(\Omega)$, we put $\hat{u} = B(u)$. Since the function $B$ is increasing and lipschitz continuous, the map $u \to \hat{u}$ is a (bicontinuous) bijection on $W_0^{1,2}(\Omega)$. If we study the modified functional

$$\hat{J}(\hat{u}) = (1/2)\int_\Omega |D\hat{u}|^2\,dx - \int_\Omega |B^{-1}(\hat{u})|^\mu\,dx, \qquad \hat{u} \in W_0^{1,2}(\Omega),$$

we see that, in this case, condition (4) becomes the following: *there exists* $\overline{m} > 2$ *and* $R_1 > 0$ *such that*

$$0 < \overline{m}(B^{-1}(z))^\mu \leqslant \mu z \frac{d}{dz}(B^{-1}(z))^\mu, \qquad \forall z \geqslant R_1,$$

that is,

(6) $$0 < \overline{m} z b(z) \leqslant \mu B(z), \qquad \forall z \geqslant R_1.$$

Conversely, if we write the Euler-Lagrange equation for $J$, our assumptions $(f_3)$ and (5) become

$$m = \mu > 2, \qquad 2zb(z)b'(z) \leqslant \delta(b(z))^2, \qquad \forall z \in \mathbf{R},$$

for some $\delta \in (0, m-2)$. Integrating, we get

(7) $$zb(z) \leqslant ((\delta/2)+1)B(z), \qquad \forall z \in \mathbf{R}.$$

We can deduce (6) from (7) if there exists $\overline{m} > 2$ such that $(\delta/2)+1 = \mu/\overline{m}$.

But, taking into account that $\mu/((\delta/2)+1) > m/((m/2)-1+1) = 2$ it is clear that this choice of $\overline{m}$ is possible. So, condition (5) allows to study the functional $J$ in the framework of [2].

THEOREM 2.2. *Assume* $(A_{1\text{-}4})$, $(f_{1\text{-}3})$ *and*

$(f_4)$ $f(x, |z|) = o(|z|)$ *at* $z = 0$ *uniformly in* $x \in \Omega$.

*Then problem* (P) *has, at least, one nontrivial and nonnegative solution.*

Now, we shall present some results concerning the case of a *sublinear* function $f$.

THEOREM 2.3. *Assume* $(A_{1\text{-}3})$. *Let* $f(x,z)$ *be such that* $f(x,z) = \lambda g(x,z)$ *with* $\lambda > 0$ *and* $g: \Omega \times \mathbf{R} \to \mathbf{R}$ *a Carathéodory function satisfying* $(f_4)$ *and* $g(x,0) = 0$, $g(x,z) > 0$ *for almost every* $x \in \Omega$, *and every* $z > 0$. *Assume also* $\lim_{z \to +\infty} z^{-1} g(x,z) = 0$, *uniformly in* $x \in \Omega$. *Then* (P) *has, at least, two nontrivial and nonnegative solutions.*

In following result, we consider the existence of solution for the problem:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\operatorname{div}(A(x,u)Du) + (1/2)A_z'(x,u)|Du|^2 = \lambda u^r - u^s, \\ u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \end{cases}$$

with $1 < r < s < 2^* - 1$.

THEOREM 2.4. *Let* $(A_{1\text{-}2})$ *hold. There exists* $\bar{\lambda} > 0$ *such that for every* $\lambda > \bar{\lambda}$ *problem* $(\mathrm{P}_\lambda)$ *has, at least, two nonzero and nonnegative solutions* $u_\lambda$, $\bar{u}_\lambda \in W_0^{1,2}(\Omega) \cap$ $\cap L^\infty(\Omega)$.

REMARK 2.5. We remark explicitely that, in contrast with the previous existence results, condition $(A_3)$ is not needed here.

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