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On maximal subgroups of minimax groups

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Teoria dei gruppi. — *On maximal subgroups of minimax groups.* Nota di SILVANA FRANCIOSI e FRANCESCO DE GIOVANNI, presentata (*) dal Socio G. Zappa.

ABSTRACT. — It is proved that a soluble residually finite minimax group is finite-by-nilpotent if and only if it has only finitely many maximal subgroups which are not normal.

KEY WORDS: Minimax group; Frattini subgroup; Finite-by-nilpotent group.

RIASSUNTO. — *Sui sottogruppi massimali dei gruppi minimax.* Si dimostra che un gruppo risolubile minimax residualmente finito è finito-per-nilpotente se e soltanto se contiene solo un numero finito di sottogruppi massimali che non sono normali.

1. INTRODUCTION

It has been proved by Robinson [4] that a finitely generated soluble group is nilpotent if and only if all its finite homomorphic images are nilpotent. It follows in particular that nilpotency is a Frattini property of the class of finitely generated soluble groups, *i.e.* if G is a finitely generated soluble group such that the Frattini factor group $G/\Phi(G)$ is nilpotent then also G is nilpotent. This result was extended by Lennox [2], who proved that a finitely generated soluble group G is finite-by-nilpotent if and only if $G/\Phi(G)$ has the same property. On the other hand, it is easily seen that $G/\Phi(G)$ is finite-by-nilpotent if and only if G contains only finitely many maximal subgroups which are not normal. Thus Lennox's result shows that finite-by-nilpotent groups can be recognized in the class of finitely generated soluble groups by the behaviour of their maximal subgroups. The aim of this short article is to obtain a similar statement for soluble residually finite minimax groups. Recall that a group is called *minimax* if it has a series of finite length whose factors either satisfy the minimal or the maximal condition on subgroups.

THEOREM. *Let G be a soluble residually finite minimax group. Then G is finite-by-nilpotent if and only if it has finitely many maximal subgroups which are not normal.*

Since the finite residual of a soluble minimax group is periodic, it follows from the above theorem that, if G is a soluble minimax group with finitely many non-normal maximal subgroups, then G is Černikov-by-nilpotent. Another obvious consequence is that the property of being finite-by-nilpotent is a Frattini property of the class of all soluble residually finite minimax groups.

(*) Nella seduta del 3 novembre 1994.

COROLLARY. *Let G be a soluble residually finite minimax group such that the factor group $G/\Phi(G)$ is finite-by-nilpotent. Then G is finite-by-nilpotent.*

Our result cannot be extended to the wider class of residually finite $\mathfrak{S}_1$-groups (here a soluble group is said to be an $\mathfrak{S}_1$-*group* if it has finite rank and contains elements of only finitely many distinct prime orders). In fact, Robinson [3] constructed a metabelian residually finite $\mathfrak{S}_1$-group G such that all maximal subgroups of G are normal, but G is not finite-by-nilpotent.

Most of our notation is standard and can be found in [5].

2. Proof of the Theorem

A well-known result of P. Hall states that a group G is finite-by-nilpotent if and only if the factor group $G/Z_n(G)$ is finite for some non-negative integer n (see [5, Part 1, Theorem 4.25]). In this case, it is clear that every maximal subgroup of G which is not normal contains $Z_n(G)$, so that G has only finitely many non-normal maximal subgroups. Thus the necessity of the condition in our theorem is proved.

Let G be a group. We shall denote by $L(G)$ the intersection of all maximal subgroups of G which are not normal. Clearly $L(G)$ is a characteristic subgroup of G containing the Frattini subgroup $\Phi(G)$, and it is easy to show that $L(G)/\Phi(G)$ is the centre of the factor group $G/\Phi(G)$ (see for instance [8, p. 64]). It follows in particular that, if G is any group, for the Frattini factor group $G/\Phi(G)$ the properties of being finite-by-nilpotent and central-by-finite are equivalent.

The structure of soluble residually finite minimax groups has been investigated in [3]. Our first lemma is an easy application of results from that article.

LEMMA 1. *Let G be a soluble residually finite minimax group, and let F be the Fitting subgroup of G. Then $F/\Phi(G)$ is the Fitting subgroup of $G/\Phi(G)$.*

PROOF. The Frattini subgroup $\Phi(G)$ of G is nilpotent (see [3, Theorem 5.12]), and so is contained in F. Let $K/\Phi(G)$ be the Fitting subgroup of $G/\Phi(G)$, and let N be a G-invariant subgroup of K such that K/N is finite. Put $\overline{G} = G/N$. Then $\overline{K}$ is a finite normal subgroup of $\overline{G}$ and $\overline{K}/(\overline{K} \cap \Phi(\overline{G}))$ is nilpotent, so that $\overline{K}$ is nilpotent by the Frattini argument. Since G is a soluble minimax group, it follows that every finite homomorphic image of K is nilpotent. Therefore K is nilpotent (see [3, Theorem 5.11]), and hence $F = K$. ■

In [1] Lennox has proved that, if G is a finitely generated soluble group such that $G/\Phi(G)$ is finite, then also G is finite. The following lemma gives a corresponding result for soluble minimax groups.

LEMMA 2. *Let G be a soluble minimax group such that $G/\Phi(G)$ is periodic. Then G is a Černikov group. In particular, if G is residually finite, then it is finite.*

Proof. It is well-known that the finite residual of G is a Černikov group (see [5, Part 2, Theorem 10.33]), so that without loss of generality it can be assumed that G is residually finite. Then the Frattini subgroup $\Phi(G)$ is nilpotent (see [3, Theorem 5.12]). Since every maximal subgroup of G has finite index, the factor group $G/\Phi(G)$ is residually finite, and hence even finite. Let p be a prime such that $\Phi(G)^p \neq \Phi(G)$. As p divides the order of the finite group $G/\Phi(G)^p$, it is well-known that p divides also the order of $G/\Phi(G)$. Thus $\Phi(G)^q = \Phi(G)$ for all but finitely many primes q, and hence $\Phi(G)$ is periodic (see [5, Part 2, Theorem 10.34]). Therefore G is periodic, and hence even finite. ∎

It is well-known that, if A is a torsion-free abelian group and Γ is a group of automorphisms of A such that $A/C_A(\Gamma)$ is periodic, then $\Gamma = 1$. In the proof of the Theorem we will also need the following slight extension of this property.

Lemma 3. *Let A be a torsion-free abelian group, and let Γ be a group of automorphisms of A. If A contains a Γ-invariant subgroup B such that A/B is periodic and* $[B,\underbrace{\Gamma,\dots,\Gamma}_{n}] = 1$ *for some positive integer n, then* $[A,\underbrace{\Gamma,\dots,\Gamma}_{n}] = 1$.

Proof. The result is clear if $n = 1$. Suppose that $n > 1$, and put $C = C_A(\Gamma)$. Let T/C be the subgroup of A/C consisting of all elements of finite order. Since $[C,\Gamma] = 1$, we have also that $[T,\Gamma] = 1$, so that $T = C$, and $\bar{A} = A/C$ is torsion-free. Moreover,

$$[\bar{B}, \underbrace{\Gamma, \dots, \Gamma}_{n-1}] = 1$$

and by induction on n it follows that also

$$[\bar{A}, \underbrace{\Gamma, \dots, \Gamma}_{n-1}] = 1 .$$

Thus $[A, \underbrace{\Gamma, \dots, \Gamma}_{n-1}]$ is contained in C, and hence $[A,\underbrace{\Gamma,\dots,\Gamma}_{n}] = 1$. ∎

Let G be a group. A G-module A is said to be *polytrivial* if there exists in A a finite series of G-submodules $0 = A_0 \leqslant A_1 \leqslant \dots \leqslant A_t = A$ such that each element of G acts like the identity automorphism on A_{i+1}/A_i for every $i \leqslant t-1$. If A is a G-module, the G-submodule consisting of all elements of finite order of A will be denoted by $\mathrm{tor}(A)$.

Lemma 4. *Let G be a group, and let A and B be G-modules such that* $A/\mathrm{tor}(A)$ *and* $B/\mathrm{tor}(B)$ *are polytrivial G-modules. Then also the G-module* $(A \otimes B)/\mathrm{tor}(A \otimes B)$ *is polytrivial.*

Proof. Put $A_0 = \mathrm{tor}(A)$ and $B_0 = \mathrm{tor}(B)$. It is well-known that the tensor product $(A/A_0) \otimes (B/B_0)$ is a polytrivial G-module (see [7, 5.2.11]). Moreover, there is an exact sequence

$$(A_0 \otimes B) \oplus (A \otimes B_0) \xrightarrow{\mu} A \otimes B \to (A/A_0) \otimes (B/B_0) \to 0 .$$

Since $\operatorname{Im}\mu$ is obviously contained in $\operatorname{tor}(A \otimes B)$, it follows that also $(A \otimes B)/\operatorname{tor}(A \otimes B)$ is a polytrivial G-module. ■

Proof of the Theorem. Assume that the result is false, and consider a counterexample G with minimal torsion-free rank r. Since every periodic subgroup of G is finite, it can be assumed without loss of generality that G has no non-trivial periodic normal subgroups. In particular, the Fitting subgroup F of G is a torsion-free nilpotent group, and $F/\Phi(G)$ is the Fitting subgroup of $G/\Phi(G)$ by Lemma 1. Clearly every maximal subgroup of G has finite index, so that $G/L(G)$ is finite, and $G/\Phi(G)$ is central-by-finite. It follows that F has finite index in G, and so G is nilpotent-by-finite. Assume that F has nilpotency class $c > 1$, and put $K = \gamma_c(F)$. If J/K is the finite residual of G/K, the minimax group G/J is residually finite and has torsion-free rank less than r, so that G/J is finite-by-nilpotent, and hence the factor group G/K is Černikov-by-nilpotent. In particular, the group G/F' is Černikov-by-nilpotent, so that by Lemma 4 the tensor product $X = (F/F') \otimes (\gamma_{c-1}(F)/K)$ is a G-module such that $X/\operatorname{tor}(X)$ is polytrivial. On the other hand, the torsion-free group K is a G-homomorphic image of X, so that K is a polytrivial G-module, and there exists a positive integer n such that $[K, \underbrace{G, \ldots, G}_{n}] = 1$. Since F has finite index in G, the subgroup J is contained in F, so that K lies in $Z(J)$ and $J/Z(J)$ is periodic, as the finite residual of a soluble minimax group is periodic. Thus also J' is periodic, and hence $J' = 1$ and J is torsion-free abelian. It follows from Lemma 3 that $[J, \underbrace{G, \ldots, G}_{n}] = 1$, so that $J \leqslant Z_n(G)$ and G is finite-by-nilpotent. This contradiction shows that $c = 1$, so that F is abelian. Let V/F be the Fitting subgroup of the finite soluble group G/F, and let $R/Z(V)$ be the finite residual of $G/Z(V)$. Clearly $Z(V)$ is contained in F, so that also R lies in F. Since F is torsion-free we obtain by Lemma 3 that $[R, V] = 1$, so that $R = Z(V)$ and $G/Z(V)$ is residually finite. Assume that $Z(V) \neq 1$. Then $G/Z(V)$ has torsion free rank less than r, and hence is finite-by-nilpotent. It follows that V is finite-by-nilpotent, and so even nilpotent, as G has no non-trivial periodic normal subgroups. Thus $V = F$, a contradiction. Therefore $Z(V) = 1$, so that in particular $H^0(V/F, F) = 0$, and the homology group $H_0(V/F, F)$ is finite (see [6, Lemma 5.12]). This means that $F/[F, V]$ is finite, so that also the factor group $G/[F, V]$ is finite. In particular G/G' is finite, so that also $G/\Phi(G)$ is finite, since $L(G) \cap G' \leqslant \Phi(G)$. Thus G is finite by Lemma 2, and this contradiction completes the proof. ■

References

[1] J. C. Lennox, *Finite Frattini factors in finitely generated soluble groups.* Proc. Amer. Math. Soc., 41, 1973, 356-360.

[2] J. C. Lennox, *Finite by nilpotent Frattini factors in finitely generated hyper(abelian by finite) groups.* Arch. Math. (Basel), 25, 1974, 463-465.

[3] D. J. S. Robinson, *Residual properties of some classes of infinite soluble groups.* Proc. London Math. Soc., (3), 18, 1968, 495-520.
[4] D. J. S. Robinson, *A theorem on finitely generated hyperabelian groups.* Invent. Math., 10, 1970, 38-43.
[5] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups.* Springer, Berlin 1972.
[6] D. J. S. Robinson, *The vanishing of certain homology and cohomology groups.* J. Pure Appl. Algebra, 7, 1976, 145-167.
[7] D. J. S. Robinson, *A Course in the Theory of Groups.* Springer, Berlin 1982.
[8] B. A. F. Wehrfritz, *Infinite Linear Groups.* Springer, Berlin 1973.

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