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Solvable finite groups with a particular configuration of Fitting sets

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Teoria dei gruppi. — *Solvable finite groups with a particular configuration of Fitting sets.* Nota di DANIELA BUBBOLONI, presentata (*) dal Socio G. ZAPPA.

ABSTRACT. — A Fitting set is called elementary if it consists of the subnormal subgroups of the conjugates of a given subgroup. In this paper we analyse the structure of the finite solvable groups in which every Fitting set is the insiemistic union of elementary Fitting sets whose intersection is the subgroup 1.

KEY WORDS: Solvable finite groups; Fitting sets; Nilpotent groups.

RIASSUNTO. — *Gruppi risolubili finiti con una particolare configurazione degli insiemi di Fitting.* Un insieme di Fitting si dice elementare se è costituito dai sottogruppi subnormali dei coniugati di un dato sottogruppo. In questo lavoro si analizza la struttura dei gruppi finiti risolubili in cui ogni insieme di Fitting è unione insiemistica di insiemi di Fitting elementari la cui intersezione si riduce al sottogruppo unità.

INTRODUCTION

By group we shall always mean finite group and we shall use throughout the notations of [2]; in particular if G is a group and $T \leqslant G$ we set $sT = \{S \leqslant G: S \leqslant T\}$, $sT^G = \{S \leqslant G: S \leqslant T^g$ for some $g \in G\}$, $snT = \{S \leqslant G: S \operatorname{sn} T\}$, $snT^G = \{S \leqslant G: S \operatorname{sn} T^g$ for some $g \in G\}$ where $S \operatorname{sn} T$ means that S is a subnormal subgroup of T.

A *Fitting set* of a group G is a collection $\mathcal{F}$ of subgroups of G such that: *i*) if $T \trianglelefteq S \in \mathcal{F}$, then $T \in \mathcal{F}$; *ii*) if $T, S \in \mathcal{F}$ and $S, T \trianglelefteq ST$, then $ST \in \mathcal{F}$; *iii*) if $S \in \mathcal{F}$ and $g \in G$, then $S^g \in \mathcal{F}$. This definition was introduced and developed by Anderson in [1]. The most familiar example of Fitting set of a group G is given by the set of the p-subgroups of G; more generally, given a Fitting class $\mathfrak{F}$, the so called *trace* of $\mathfrak{F}$ in G $\mathrm{tr}_{\mathfrak{F}}(G) = \{H \leqslant G: H \in \mathfrak{F}\}$ is a Fitting set of G. We shall focus on the case $\mathfrak{F} = \mathfrak{N}^k$ with $k \in \mathbf{N}$, where $\mathfrak{N}^1 = \mathfrak{N}$ is the class of nilpotent groups and $\mathfrak{N}^k$ is defined inductively by $\mathfrak{N}^k = (G: G/F(G) \in \mathfrak{N}^{k-1})$. Let G be a group and $\mathcal{F}$ a Fitting set of G: $V \leqslant G$ is called $\mathcal{F}$-maximal if $V \in \mathcal{F}$ and from $V \leqslant U \leqslant G$ with $U \in \mathcal{F}$, it follows $U = V$; $V \leqslant G$ is called an $\mathcal{F}$-injector if for every $K \operatorname{sn} G$, $V \cap K$ is $\mathcal{F}$-maximal in K; $V \leqslant G$ is called an injector if V is a $\mathcal{F}$-injector for some Fitting set of G.

A fundamental result in the theory of Fitting sets guarantees that if G is a solvable group and $\mathcal{F}$ a Fitting set of G, then $\mathcal{F}$-injectors exist and constitute a conjugacy class [2, VIII, 2.9]. This means that the theory of Fitting sets is, in particular, a generalization of the classical theory of Sylow and Hall subgroups.

There is a very strong link between Fitting sets and injectors: namely if G is a solvable group and $H \leqslant G$, H is an injector of G if and only if snH^G is a Fitting set of G [2, VIII, 3.3].

(*) Nella seduta del 3 novembre 1994.

We shall call *elementary* a Fitting set $\mathcal{F}$ of a group G if there exists $H \leqslant G$ such that $\mathcal{F} = snH^G$. By the result quoted above, we can deduce that every Fitting set of a solvable group contains an elementary Fitting set; moreover most of the well-known Fitting sets are elementary. These two facts have been the initial motivation for our research. To be more precise let us introduce the following definition: if G is a group and $\mathcal{F}_i$ for $i = 1, \ldots, n$ are Fitting sets of G such that $\mathcal{F}_i \cap \mathcal{F}_j = \{1\}$ for $i \neq j$, then the set $\mathcal{F} = \bigcup_{i=1}^{n} \mathcal{F}_i$ of subgroups of G is called the *disjoint union* of the $\mathcal{F}_i$ and it is denoted by $\dot{\bigcup}_{i=1}^{n} \mathcal{F}_i$. Then the problem is the following: how many Fitting sets can we construct via the disjoint union of elementary Fitting sets or, from another point of view, can we classify those solvable groups for which every Fitting set is given by the disjoint union of elementary Fitting sets? A first useful observation is that a solvable group is nilpotent if and only if every Fitting set is an elementary one. The next step is to investigate the structure of solvable groups in which the Fitting set of the nilpotent subgroups is a disjoint union of elementary Fitting sets. This will be described in section 1. In the next section 2 we shall treat the analogous problem for the Fitting set $\mathrm{tr}_{\mathfrak{N}^2}(G)$ and this will shortly lead to the solution of our general problem. These topics and others related to them also constitute a section of my PhD thesis on Fitting sets [5].

1. Solvable groups in which the trace of $\mathfrak{N}$ is the disjoint union of elementary Fitting sets

In what follows if π_i for $i = 1, \ldots, n$ are sets of primes with $\pi_i \cap \pi_j = \emptyset$ for $i \neq j$, then we shall write $\dot{\bigcup}_{i=1}^{n} \pi_i$ instead of $\bigcup_{i=1}^{n} \pi_i$.

Lemma 1.1. Let G be a solvable group with $\mathrm{tr}_{\mathfrak{N}}(G) = \dot{\bigcup}_{i=1}^{t} sM_i^G$, where sM_i^G are elementary Fitting sets of G and $\pi_i = \pi(|M_i|)$. Then the M_i are nilpotent Hall subgroups of G and $\pi(|G|) = \dot{\bigcup}_{i=1}^{t} \pi_i$.

Proof. First of all we observe that if P is a p-subgroup of G, then $P \leqslant M_i^g$ for some $i = 1, \ldots, n$ and $g \in G$. Therefore $\pi(|G|) = \bigcup_{i=1}^{t} \pi_i$. We show now that the M_i are Hall subgroups of G. If $p \mid |M_i|$, then there exists a p-group $P_i \neq 1$ with $P_i \leqslant M_i$. Let $P \in \mathrm{Syl}_p(G)$ with $P \geqslant P_i$; then $P \leqslant M_k^g$ for some $k = 1, \ldots, n$ and $g \in G$. If $k \neq i$, then $P_i \in sM_i^G \cap sM_k^g = 1$ contrary to the assumption; it follows that $k = i$ and $|P| = |G|_p = |M_i|$. To show that $\pi_i \cap \pi_j = \emptyset$ for $i \neq j$, let p be a prime such that $p \mid |M_i|, |M_j|$. Then, since M_i, M_j are Hall subgroups of G, there exist $1 \neq P \in \mathrm{Syl}_p(G)$ and $x \in G$ such that $P \leqslant M_i$ and $P^x \leqslant M_j$. It follows $P \leqslant M_i \cap M_j^{x^{-1}}$ and consequently $P \in sM_i^G \cap sM_j^g = 1$, a contradiction. ■

LEMMA 1.2. Let G be a solvable group and let N_i for $i = 1, \dots, t$ be nilpotent Hall subgroups of G such that $\pi(|G|) = \dot{\bigcup}_{i=1}^{t} \pi_i$, with $\pi_i = \pi(|N_i|)$. Then the following statements are equivalent:

i) $\mathrm{tr}_{\mathfrak{N}}(G) = \dot{\bigcup}_{i=1}^{t} sN_i^G$, with sN_i^G elementary Fitting sets of G;

ii) $C_G(x)$ is a π_i-group for every $i = 1, \dots, t$ and $1 \neq x \in G$ π_i-element;

iii) every element in G is a π_i-element for some $i = 1, \dots, t$.

PROOF. *i*) $\Rightarrow$ *ii*) Let $1 \neq x \in G$ be a π_i-element and $r \mid |C_G(x)|$ a prime; then there exists $y \in C_G(x)$ with $o(y) = r$ and $\langle x, y\rangle = \langle x\rangle \times \langle y\rangle \in \mathfrak{N}$. It follows that $\langle x, y\rangle \leq N_j^g$ for some $j = 1, \dots, t$ and $g \in G$: we cannot have $j \neq i$ because x would be a π_j-element contradicting $\pi_i \cap \pi_j = \emptyset$ and $x \neq 1$. Hence $i = j$ and $r \mid |N_i|$, that is $r \in \pi_i$.

ii) $\Rightarrow$ *iii*) Let $1 \neq x \in G$, with $o(x) = m_i m$ where $1 \neq m_i$ is π_i-number and $1 \neq m$ is a π_i'-number. Then there exist $y \neq 1$ and $z \neq 1$ in G such that $yz = zy$ with y a π_i-element and z a π_i'-element, contrary to *ii*).

iii) $\Rightarrow$ *i*) By assumption N_i is a nilpotent π_i-Hall subgroup of G. It follows that sN_i^G is the Fitting set of the π_i-subgroups of G and $\mathrm{tr}_{\mathfrak{N}}(G) \supseteq \bigcup_{i=1}^{t} sN_i^G$. Let $M \in \mathfrak{N}$ with $M \leq G$; then we have $M = M_1 \times \dots \times M_t$, with M_i a π_i-Hall subgroup of M making use of $\dot{\bigcup}_{i=1}^{t} \pi_i = \pi(|G|)$. Assume $M_i \neq 1 \neq M_j$ for $i \neq j$; then there exist $x_i \in M_i$, $x_j \in M_j$ with $o(x_i) = p_i \in \pi_i$, $o(x_j) = p_j \in \pi_j$ and $x_i x_j = x_j x_i$. Therefore $o(x_i x_j) = p_i p_j$ contrary to *iii*). Hence $M = M_i$ for some $i = 1, \dots, t$ and then $M \leq N_i^g$ for some $g \in G$. This means $\mathrm{tr}_{\mathfrak{N}}(G) \subseteq \bigcup_{i=1}^{t} sN_i^G$. Finally the fact that $sN_i^G \cap sN_j^G = 1$ for $i \neq j$ is a trivial consequence of $\pi_i \cap \pi_j = \emptyset$. ■

LEMMA 1.3. Let G be a solvable group such that $\mathrm{tr}_{\mathfrak{N}}(G) = \dot{\bigcup}_{i=1}^{t} sN_i^G$ with $N_i \neq 1$. Then $t \leq 2$.

PROOF. Let $\pi_i = \pi(|N_i|)$ for $i = 1, \dots, t$. We have $\pi_i \neq \emptyset$ and, by Lemma 1.1, it follows that $\pi_i \cap \pi_j = \emptyset$ for $i \neq j$. Assume $t \geq 3$; let $p_i \in \pi_i$ for $i = 1, 2, 3$ and let H be a $\{p_1, p_2, p_3\}$-Hall subgroup of G. By 1.2 every element in G is a π_i-element for a suitable $i \in \{1, \dots, t\}$, hence every element in H is a p_i-element for a suitable $i \in \{1, 2, 3\}$. Then using the theorem on page 172 in [4], we deduce that the order of H is divisible at most by two primes, a contradiction. ■

This lemma allows us to consider only those solvable groups for which the trace of $\mathfrak{N}$ is a disjoint union of two elementary Fitting sets. It also leads to a very useful result about the behaviour of quotients of this type of group.

Corollary 1.4. If G is a solvable group with $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \,\dot{\cup}\, sN_2^G$ and $N \trianglelefteq G$, then $\mathrm{tr}_{\mathfrak{N}}(\overline{G}) = s\overline{N}_1^{\overline{G}} \,\dot{\cup}\, s\overline{N}_2^{\overline{G}}$, where $\overline{H}$ stands for HN/N for every $H \leqslant G$.

Proof. Let G be a solvable group with $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \,\dot{\cup}\, sN_2^G$. By Lemma 1.1 N_1, N_2 are nilpotent Hall subgroups of G such that $\pi_1 \,\dot{\cup}\, \pi_2 = \pi(|G|)$, where $\pi_i = \pi(|N_i|)$, and therefore $G = N_1 N_2$. Moreover, by Lemma 1.2, every element in G is a π_1-element or a π_2-element. We set $\overline{H} = HN/N$ for each $H \leqslant G$, $\overline{\pi}_i = \pi(|\overline{N}_i|)$ for $i = 1, 2$ and observe that $\overline{N}_1$ and $\overline{N}_2$ are nilpotent Hall subgroups in $\overline{G}$. Then, by $\overline{\pi}_i \subseteq \pi_i$ and $\overline{G} = \overline{N}_1\overline{N}_2$, it follows that $\overline{\pi}_1 \,\dot{\cup}\, \overline{\pi}_2 = \pi(|\overline{G}|)$. Now choose $\overline{x} = xN \neq 1$ in $\overline{G}$; then we have $o(\overline{x}) \,|\, o(x)$ and $\overline{x}$ is a $\overline{\pi}_1$-element or a $\overline{\pi}_2$-element. Therefore, by Lemma 1.2, we obtain that $\mathrm{tr}_{\mathfrak{N}}(\overline{G}) = s\overline{N}_1^{\overline{G}} \,\dot{\cup}\, s\overline{N}_2^{\overline{G}}$. ∎

In order to prove our main result, that is Theorem 1.7, we need the following two lemmas. We shall omit the proof of the second which may be obtained by induction on the order of the group.

Lemma 1.5. Let G be a solvable group such that $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \,\dot{\cup}\, sN_2^G$ and let $\pi_i = \pi(|N_i|)$ for $i = 1, 2$. If $1 \neq L \leqslant G$ is a π_i-group and $1 \neq M \leqslant N_G(L)$ is a π_2-group, then LM is a Frobenius group with Frobenius complement M.

Proof. Obviously $LM \leqslant G$ and $1 < L \trianglelefteq LM$. On the other hand if $1 \neq x \in L$, then we have $C_M(x) = M \cap C_G(x) = 1$ because, by assumption, M is a π_2-group, while by Lemma 1.2 $C_G(x)$ is a π_i-group and, by 1.1, $\pi_1 \cap \pi_2 = \emptyset$. ∎

Lemma 1.6. Let G be a solvable group such that $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \,\dot{\cup}\, sN_2^G$ and let $\pi_i = \pi(|N_i|)$ for $i = 1, 2$. If $O_{\pi_1}(G) > 1$, then the ascending nilpotent series coincides with the $\pi_1'\pi_1$-series: in particular $F(G) = O_{\pi_1}(G)$, $F_2(G) = O_{\pi_1\pi_1'}(G)$, $F_3(G) = O_{\pi_1\pi_1'\pi_1}(G)$.

Theorem 1.7. Let $G \neq 1$ be a solvable group. Then the following statements are equivalent:

i) there exist $1 < N_1$, $N_2 < G$ such that $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \,\dot{\cup}\, sN_2^G$;

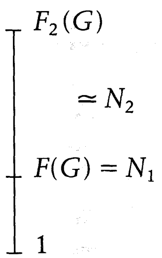
ii) N_1, N_2 are nilpotent Hall-subgroups of G, $G = N_1N_2$ and $\pi_1 \,\dot{\cup}\, \pi_2 = \pi(|G|)$, where $\pi_i = \pi(|N_i|)$ for $i = 1, 2$.

If $O_{\pi_1}(G) > 1$, then there are two possibilities for the structure of G:

a) $G = F_2(G)$ is a Frobenius group with Frobenius complement N_2;

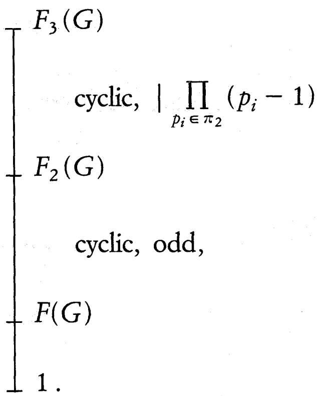
b) $G = F_3(G)$, $F_2(G)$ is a Frobenius group whose complement, N_2, is cyclic of odd order and $G/F(G)$ is a Frobenius group whose complement $N_1/F(G)$ is cyclic of order dividing $\prod_{p_i \in \pi_2} (p_i - 1)$.

a)



;

b)



Proof. *i*) $\Rightarrow$ *ii*) Let $G \neq 1$ solvable with $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \dot{\cup} sN_2^G$ and $1 < N_i < G$. Let $\pi_i = \pi(|N_i|)$ for $i = 1, 2$. By Lemma 1.1, the N_i are nilpotent Hall subgroups of G and $\pi_1 \dot{\cup} \pi_2 = \pi(|G|)$, hence $G = N_1 N_2$ and $\pi_2 = \pi_1'$. The solvability of G implies $O_{\pi_1}(G) > 1$ or $O_{\pi_2}(G) > 1$ and, reordering the π_i, we can assume $O_{\pi_1}(G) > 1$. Therefore by Lemma 1.6, $F(G) = O_{\pi_1}(G)$, $F_2(G) = O_{\pi_1 \pi_1'}(G)$, $F_3(G) = O_{\pi_1 \pi_1' \pi_1}(G)$. Now N_1 is a π_1-Hall subgroup of G: this implies $N_1 \geq F(G) = F$ and, by Lemma 1.5, N_2 is a Frobenius complement in the Frobenius group $N_2 F$. Therefore the Sylow subgroups of N_2 are cyclic or generalized quaternion [3, V. 8.7]; but $N_2 \in \mathfrak{N}$ and therefore we can have either N_2 cyclic or $N_2 \simeq C \times Q$ with C cyclic, Q generalized quaternion and $(|C|, 2) = 1$.

Now set $\overline{H} = HF/F$ for each $H \leq G$ and $\overline{\pi}_i = \pi(|\overline{N}_i|)$ for $i = 1, 2$. Obviously $\overline{\pi}_1 \subseteq \pi_1$ and $\overline{\pi}_2 = \pi_2$. Because $N_i \neq 1$, G is not a nilpotent group and therefore $1 \neq \overline{G} = \overline{N}_1 \overline{N}_2$; moreover, by 1.4, $\mathrm{tr}_{\mathfrak{N}}(\overline{G}) = s\overline{N}_1^{\overline{G}} \dot{\cup} s\overline{N}_2^{\overline{G}}$.

Let us show that $\overline{N}_2 \trianglelefteq \overline{G}$. Let $\overline{L} \leq \overline{G}$ minimal normal: then $\overline{L}$ is nilpotent and this implies $\overline{L} \leq F(\overline{G}) = O_{\pi_2}(\overline{G}) < \overline{N}_2$, since $\overline{N}_2$ is a π_2-Hall subgroup of $\overline{G}$. In particular the elementary abelian p-group $\overline{L}$ is contained in a p-Sylow $\overline{P}$ of $\overline{N}_2 \simeq N_2$; but $\overline{P}$ is cyclic or generalized quaternion. Consequently it contains only one subgroup of order p which moreover is inside the centre of $\overline{P}$. This gives $\overline{L}$ cyclic and $\overline{L} \leq Z(\overline{P})$; then by nilpotency of $\overline{N}_2$, we have $\overline{L} \leq Z(\overline{N}_2)$. Assume $\overline{N}_2 \ntrianglelefteq \overline{G}$. Then $g \in \overline{G}$ exists so that $\overline{N}_2^g \neq \overline{N}_2$ and from $\overline{L} = \overline{L}^g \subseteq Z(\overline{N}_2)^g = Z(\overline{N}_2^g)$, it follows that $C_{\overline{G}}(\overline{L}) \geq \langle \overline{N}_2, \overline{N}_2^g \rangle > \overline{N}_2$. Now $\overline{G} = \overline{N}_1 \overline{N}_2$, hence there exists $1 \neq h \in C_{\overline{G}}(\overline{L}) \cap \overline{N}_1$. Taking $1 \neq l \in \overline{L}$, we obtain a $\overline{\pi}_2$-element whose centralizer in $\overline{G}$ is not a $\overline{\pi}_2$-group, contrary to Lemma 1.2.

Thus we have $\overline{N}_2 \trianglelefteq \overline{G}$ and $\overline{N}_2 \in \mathfrak{N}$, hence $F(\overline{G}) \geq \overline{N}_2$; on the other hand we have already observed that $F = O_{\pi_1}(G)$ and $F_2(G) = O_{\pi_1 \pi_1'}(G)$, therefore $F(\overline{G}) = O_{\pi_2}(\overline{G}) \leq \overline{N}_2$. It follows $F(\overline{G}) = \overline{N}_2$, namely $F_2(G) = N_2 F$ and $F_2(G)$ is a Frobenius group with Frobenius complement N_2.

If $\overline{N}_1 = 1$, we obtain $\overline{G} = \overline{N}_2$ and therefore $G = F_2(G)$ has the structure a).

If $\overline{N}_1 \neq 1$, we obtain $\overline{G} = F(\overline{G})\overline{N}_1$ and, applying Lemma 1.5 to $\overline{G}$, we get that $\overline{G}$ is a Frobenius group with nilpotent Frobenius complement $\overline{N}_1$; hence $G = F_3(G)$.

We observe that $2 \nmid |N_2|$: otherwise $\overline{N}_2 \simeq N_2$ would contain only one subgroup $\langle i \rangle$ of order 2, characteristic in $\overline{N}_2 \trianglelefteq \overline{G}$ and therefore normal in $\overline{G}$. It follows that $\langle i \rangle \leqslant Z(\overline{G})$; hence $2 \mid |C_{\overline{G}}(x)|$ for every $x \in \overline{G}$. Then, by Lemma 1.2, $\overline{G}$ does not contain non-trivial $\overline{\pi}_1$-elements and, in particular $\overline{N}_1 = 1$, a contradiction.

It follows then that N_2 is cyclic of odd order.

Finally we consider $\overline{N}_1$: this is a nilpotent Frobenius complement in $\overline{G}$, hence $\overline{N}_1$ is cyclic or $\overline{N}_1 \simeq C \times Q$ with C cyclic, Q generalized quaternion and $(|C|, 2) = 1$. We observe now that $\overline{N}_1$ is embedded in the automorphism group of the Frobenius kernel $\overline{F_2(G)} \simeq N_2$ of $\overline{G}$; but $\overline{N}_2$ cyclic implies $\mathrm{Aut}(N_2)$ abelian and therefore $\overline{N}_1$ is cyclic.

Moreover if $|N_2| = \prod_{p_i \in \pi_2} p_i^{\alpha_i}$, we obtain $|N_1| \mid |\mathrm{Aut}(N_2)| = \varphi(|N_2|) = \prod_{p_i \in \pi_2} p_i^{\alpha_i - 1}(p_i - 1)$, and from this, considering that $\overline{\pi}_1 \subseteq \pi_1$ and $\pi_1 \cap \pi_2 = \emptyset$, we have $|N_1| \mid \prod_{p_i \in \pi_2} (p_i - 1)$. This means that if $\overline{N}_1 \neq 1$, then G has the structure described in b).

ii) $\Rightarrow$ *i*) We start with G a solvable group and N_1, N_2 two nilpotent Hall subgroups of G such that $G = N_1 N_2$ with $\pi_1 \dot{\cup} \pi_2 = \pi(|G|)$, where $\pi_i = \pi(|N_i|)$. Taking G of type a) or b) we can assume, without loss of generality, that $O_{\pi_1}(G) > 1$. By Lemma 1.2, proving *ii*) is equivalent showing that each element in G is a π_1-element or a π_2-element. We analyse separately the cases a) and b).

If G is of type a), then $G = F_2(G)$ is a Frobenius group with complement N_2 and kernel $F(G)$; hence we have $|F(G)| = |G|/|N_2| = |N_1|$. This means that $F(G)$ is the only π_1-Hall subgroup of G and therefore $F(G) = N_1$. By the Frobenius partition, each element in G is in $F(G)$ or in a Frobenius complement. Then it is either a π_1-element or a π_2-element.

If G is of type b), then the Frobenius complement N_2 of the Frobenius group $F_2(G)$ is a π_2-Hall of G. Therefore the Frobenius kernel $F(F_2(G)) = F(G)$ is a π_1-Hall of $F_2(G)$ while $F_2(G)/F(G) \simeq N_2$, Frobenius kernel of $G/F(G)$, is a π_2-group. Furthermore each non-trivial π_2-element in G belongs to $F_2(G) \setminus F(G)$: namely the elements in $F(G)$ are π_1-elements and $F_2(G) \trianglelefteq G$ contains N_2 which is a π_2-Hall of G and therefore $F_2(G)$ contains each π_2-element of G.

Assume now that there exist $1 \neq x \in G, p \in \pi_1$ and $q \in \pi_2$ such that $p, q \mid o(x)$. Then there exist also a π_1-element $y \neq 1$ and a π_2-element $z \in F_2(G) \setminus F(G)$ such that $yz = zy = x$. Notice that $F_2(G)$ is a Frobenius group with Frobenius kernel $F(G)$, hence the only element of $F(G)$ centralized by $z \in F_2(G) \setminus F(G)$ is 1; then $y \notin F(G)$, that is $1 \neq \overline{y} = yF(G) \in \overline{G} = G/F(G)$. By hypothesis, $\overline{G}$ is a Frobenius group whose kernel is, as already observed, a π_2-group; therefore the π_1-element $\overline{y}$ must lie in a Frobenius complement and then it does not centralize $\overline{z} \in F_2(G)/F(G) - \{1\}$ contrary to the fact that y centralizes z. ■

2. Solvable groups in which every Fitting set is a disjoint union of elementary Fitting sets

If G is a group and $\mathcal{F}$ is a Fitting set of G, we denote by $G_{\mathcal{F}}$ the $\mathcal{F}$-radical of G, that is the union of all the normal subgroups of G belonging to $\mathcal{F}$. We begin this section with two easy but useful remarks.

Remark 2.1. Let $\mathcal{F}$ be a Fitting set of the group G and suppose that snM_i^G are elementary Fitting sets of G for $i = 1, \dots, k$ such that $\mathcal{F} \subseteq \dot{\bigcup}_{i=1}^{k} snM_i^G$. Then $\mathcal{F} = \dot{\bigcup}_{i=1}^{k} sn[(M_i)_{\mathcal{F}}]^G$, with $sn[(M_i)_{\mathcal{F}}]^G$ elementary Fitting sets of G.

Remark 2.2. Let G be a solvable group.

a) If $\mathcal{F} \supseteq \mathrm{tr}_{\mathfrak{N}}(G)$ is a Fitting set of G such that $\mathcal{F} = \dot{\bigcup}_{i=1}^{k} snM_i^G$, with $snM_i^G \neq \mathrm{I}$ elementary Fitting sets, then $k \leq 2$ and the $N_i = F(M_i)$ are such that $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \dot{\cup} sN_2^G$.

b) If there exists an elementary Fitting set of G containing $\mathrm{tr}_{\mathfrak{N}}(G)$, then G is nilpotent.

Theorem 2.3. Let G be a solvable, but not nilpotent group. Then the following two statements are equivalent:

i) $\mathrm{tr}_{\mathfrak{N}^2}(G)$ is a disjoint union of elementary Fitting sets;

ii) $G \simeq \left\{ \begin{pmatrix} x & \\ \alpha x + \beta \end{pmatrix} : \alpha \in H \leq GF(p^n)^{\times}, \beta \in GF(p^n) \right\}$ where $n = \mathrm{ord}\ p(q)$, for each $q \mid |H|$.

Proof. *i*) $\Rightarrow$ *ii*) Let G be a non-nilpotent solvable group with $\mathrm{tr}_{\mathfrak{N}^2}(G)$ the disjoint union of elementary Fitting sets. $\mathrm{Tr}_{\mathfrak{N}^2}(G) \supseteq \mathrm{tr}_{\mathfrak{N}}(G)$ and Remark 2.2 imply that $\mathrm{tr}_{\mathfrak{N}^2}(G) = snM_1^G \dot{\cup} snM_2^G$ with $M_i \neq 1$. Moreover, setting $N_i = F(M_i)$, we have $\mathrm{tr}_{\mathfrak{N}}(G) = sN_1^G \dot{\cup} sN_2^G$. By Theorem 1.7, if we put $\pi_i = \pi(|N_i|)$ for $i = 1, 2$, then we obtain $G = N_1 N_2$ with $1 \neq N_i \neq G$ nilpotent π_i-Hall subgroups of G and $\pi_1 \dot{\cup} \pi_2 = \pi(|G|)$. Furthermore G is of type *a*) or of type *b*) as in Theorem 1.7 *ii*). We consider separately the two types explaining the corresponding structure of G in the case *a*) and the impossibility of case *b*).

G of type a). In this case G is a Frobenius group with kernel $N_1 = F(G)$ and complement N_2. Then $N_2 = F(M_2) \leq M_2$; it cannot be that $N_2 < M_2$ otherwise we would have $M_2 \cap N_1 \trianglelefteq M_2$ with $M_2 \cap N_1 \in \mathfrak{N}$ which gives $N_2 = F(M_2) \geq M_2 \cap N_1 \neq 1$ contrary to $\pi_1 \cap \pi_2 = \emptyset$; therefore $M_2 = N_2$ is a Frobenius complement in G. We observe that $G \in \mathfrak{N}^2$, that is $G \in \mathrm{tr}_{\mathfrak{N}^2}(G) = snM_1^G \dot{\cup} snM_2^G$; on the other hand $G \notin snM_2^G$, otherwise $G = N_2$, contrary to $N_1 \neq 1$. Hence $G = M_1$ and, using the fact that $\mathfrak{N}^2$ is closed with respect to subgroups, we obtain $sG = snG \dot{\cup} sH^G$ with H a Frobenius complement of G. But $snG = \{T \leqslant F(G)\} \cup \{T \leqslant G: T > F(G)\}$ [3, V, 8.16] hence if $T \leqslant G$, setting $F = F(G)$, the alternatives are: $T \leqslant F$, $T > F$, $T \leqslant H^g$ for some $g \in G$. We show that this implies F minimal normal in K for each $K \leqslant G$ with $K > F$. Assume that there exists $K \leqslant G$ with $K > F > \overline{F}$ and $1 \neq \overline{F} \lhd K$. Then $K \cap H = \overline{H} \neq 1$, $\overline{F}\overline{H} \leqslant G$ and also $\overline{F}\overline{H} \nleqslant F$ as well as $\overline{F}\overline{H} \ngeqslant F$, because $\overline{F}\overline{H} \cap F = \overline{F}(F \cap K \cap H) = \overline{F} \neq F$. Finally $\overline{F}\overline{H} \nleqslant H^g$ since $1 \neq \overline{F} \nleqslant H^g$ for every $g \in G$. It follows that $\overline{F}\overline{H} \notin snG \dot{\cup} sH^G$, a contradiction. In particular F is elementary abelian, say $|F| = p^n$ with $\{p\} = \pi_1$, $n \geqslant 1$ and if we consider H imbedded in $\mathrm{Aut}(F) \simeq GL(n,p)$, then F is an irreducible H-module.

If $2 \nmid |H|$ then H is a nilpotent complement in the Frobenius group G, hence H is cyclic. Then by II, 3.10 in [3], there exists a monomorphism $a: H \to GF(p^n)^\times$ such that $x^h = a(h)x$ for each $x \in F$, where we identify F with $GF(p^n)$ and the operation on the right side is the product in the field. This gives

$$G \simeq \left\{ \begin{pmatrix} x \\ \alpha x + \beta \end{pmatrix} : \beta \in GF(p^n),\ \alpha \in a(H) \leqslant GF(p^n)^\times \right\}.$$

Now, due to the fact that F is minimal normal in K for each $K \leqslant G$ with $K > F$, the same argument applies to every subgroup in H and enables us to deduce that n is the order of p modulo q, for each $q \mid |H|$.

If $2 \mid |H|$, then applying the same argument again to $C \leqslant H$ with $|C| = 2$, we obtain $n = \mathrm{ord}p(2)$; but $(|H|, |F|) = 1$ implies p odd and therefore $n = 1$, thus $F \simeq C_p$ and H is embedded in $\mathrm{Aut}(C_p) \simeq C_{p-1}$. In particular H is cyclic and $|H| \mid p-1$; then the argument applies to H itself and this leads to

$$G \simeq \left\{ \begin{pmatrix} x \\ \alpha x + \beta \end{pmatrix} : \beta \in GF(p),\ \alpha \in H \leqslant GF(p)^\times \right\},$$

with $q \mid p-1$ for each $q \mid |H|$.

G of type b). In this case it is $G = F_3(G)$, $F_2(G)$ is a Frobenius group with complement N_2 and $G/F(G)$ is a Frobenius group with complement $N_1/F(G)$. We observe, first of all, that $N_1 \in \mathfrak{N}$ and therefore $N_1 \in \mathfrak{N}^2$. On the other hand $N_{G/F(G)}(N_1/F(G)) = N_1/F(G)$ and, due to the fact that a Frobenius complement is selfnormalizing, it follows that $N_G(N_1) = N_1$. Hence $N_1 = F(M_1)$ is not subnormal in any subgroup of G in which it is properly included and therefore $N_1 = M_1$. Now from $F_2(G) \in \mathfrak{N}^2$ and $F_2(G) \nleqslant N_1$, we obtain $F_2(G) \in snM_2^G$, hence $snF_2(G) \subseteq snM_2^G$: in particular we have $1 \neq F(G) \in snM_2^G \cap snM_1^g$, a contradiction.

ii) $\Rightarrow$ *i*) Let G be a group as in *ii*), that is, up to isomorphism, $G = GF(p^n) \rtimes H$ with $H \leqslant GF(p^n)^\times$, $x^h = xh$ for each $x \in GF(p^n)$ and $h \in H$, where n is the order of p modulo q for each q such that $q \mid |H|$. Obviously G is a Frobenius group with kernel $F = GF(p^n)$ and complement H. We show that F is minimal normal in K for each $K \leqslant G$ with $K > F$. Assume that there exists $K = FL$ with $1 \neq L \leqslant H$ and $N < F$ minimal normal in FL; without loss of generality we can assume $|L| = q$ with q a prime. Then N is an irreducible L-module. Moreover, if $|N| = p^k$, again by II, 3.10 in [3], we have $k = \mathrm{ord}\,p(q)$; but $q \mid |H|$, therefore $\mathrm{ord}\,p(q) = n$ and $N = F$, a contradiction.

Now let $T \leqslant G$ with $T \nleqslant F$, $T \ngtr F$ and consider $T \cap F = \overline{T} \trianglelefteq T$. We have $\overline{T} < F$ and $N_G(\overline{T}) \geqslant T, F$ since F is abelian; hence $N_G(\overline{T}) \geqslant TF$ and thus $\overline{T} \lhd K = TF$ with $K > F$. Using the fact that F is minimal normal in K, it follows that $\overline{T} = 1$. Then $p \nmid |T|$: otherwise, considering $P \in \mathrm{Syl}_p(T)$ we would have $1 \neq P \leqslant T \cap F$ since F is the only p-Sylow in G, contrary to $T \cap F = 1$. Thus T is included in a p'-Hall subgroup of G, namely in H^g for a suitable $g \in G$. Considering that $snG = \{T \leqslant F\} \cup \{T \leqslant G\colon T > F\}$ [3, V, 8.16], this shows that $sG = snG \,\dot{\cup}\, sH^G$. But snG is a Fitting set of G and H nilpotent Hall subgroup of G implies that $sH^G = snH^G$ is a Fitting set of G. Since $G \in \mathfrak{N}^2$, this means that $\mathrm{tr}_{\mathfrak{N}^2}(G)$ is a disjoint union of elementary Fitting sets of G. ■

Corollary 2.4. Let G be a non-nilpotent solvable group. Then the two following statements are equivalent:

i) $\mathrm{tr}_{\mathfrak{N}^k}(G)$ is a disjoint union of elementary Fitting sets for some $k \in \mathbf{N}$, $k \geqslant 2$;

ii) $G \simeq \left\{ \begin{pmatrix} x & \\ \alpha x + \beta & \end{pmatrix} \colon \beta \in GF(p^n),\ \alpha \in H \leqslant GF(p^n)^\times \right\}$, where $n = \mathrm{ord}\,p(q)$, for each $q \mid |H|$.

Proof. *i*) $\Rightarrow$ *ii*) Since $\mathrm{tr}_{\mathfrak{N}^k}(G) \supseteq \mathrm{tr}_{\mathfrak{N}^2}(G)$ for each $k \geqslant 2$ we obtain, by Remark 2.1, that $\mathrm{tr}_{\mathfrak{N}^2}(G)$ is a disjoint union of elementary Fitting sets and then, by Theorem 2.3, G has the structure described in *ii*).

ii) $\Rightarrow$ *i*) If the structure of G is as in *ii*), then $G \in \mathfrak{N}^2$ and so $\mathrm{tr}_{\mathfrak{N}^k}(G) = \mathrm{tr}_{\mathfrak{N}^2}(G)$ for every $k \geqslant 2$. Thus, by Theorem 2.3, $\mathrm{tr}_{\mathfrak{N}^k}(G)$ is disjoint union of elementary Fitting sets, for each $k \geqslant 2$. ■

Theorem 2.5. Let G be a non-nilpotent solvable group. Then the following statements are equivalent:

i) every Fitting set of G is disjoint union of elementary Fitting sets of G;

ii) $G \simeq \left\{ \begin{pmatrix} x & \\ \alpha x + \beta & \end{pmatrix} \colon \beta \in GF(p^n),\ \alpha \in H \leqslant GF(p^n)^\times \right\}$, where $n = \mathrm{ord}\,p(q)$ for each $q \mid |H|$;

iii) every Fitting set of G is a disjoint union of at most two elementary Fitting sets.

Proof. *i*) $\Rightarrow$ *ii*) It is a trivial consequence of Theorem 2.3.

ii) $\Rightarrow$ *iii*) Let $G \simeq \left\{ \begin{pmatrix} x & \\ \alpha x + \beta & \end{pmatrix} \colon \beta \in GF(p^n),\ \alpha \in H \leqslant GF(p^n)^\times \right\}$ with $n = \mathrm{ord}\,p(q)$ for each $q \mid |H|$ and $\mathfrak{F}$ a Fitting set of G. Due to the facts established in the proof of

Theorem 2.3 *ii*) $\Rightarrow$ *i*) we have $sG = snG \dot{\cup} snH^G$, with snG, snH^G Fitting sets of G. But $\mathcal{F} \subseteq sG$ and then, by Remark 2.1, $\mathcal{F} = snG_{\mathcal{F}} \dot{\cup} sn[H_{\mathcal{F}}]^G$ is the disjoint union of at most two elementary Fitting sets.

iii) $\Rightarrow$ *i*) Straightforward. ■

REMARK 2.6. In the groups as in Theorem 2.5 *ii*), the condition $n = \mathrm{ord}\, p(q)$ for each $q \mid |H|$, does not force $|H|$ to be prime. For example consider $p = 29$, $n = 2$. Then $|GF(p^n)^\times| = 2^3 \cdot 3 \cdot 5 \cdot 7$ and there exists $H \leqslant GF(p^n)^\times$ with $|H| = 15$. The group $G = GF(29^2) \rtimes H$, where the action is given by the product in the field, is of type *ii*), since $2 = \mathrm{ord}\, 29(5) = \mathrm{ord}\, 29(3)$.

REFERENCES

[1] W. ANDERSON, *Fitting sets in finite soluble groups.* Ph. D. thesis, Michigan State University, 1973.

[2] K. DOERK - T. HAWKES, *Finite Soluble Groups.* Berlin - New York 1992.

[3] B. HUPPERT, *Endliche Gruppen I.* Berlin - Heidelberg - New York 1967.

[4] G. ZACHER, *Sull'ordine di un gruppo finito risolubile somma dei suoi sottogruppi di Sylow.* Atti Acc. Lincei Rend. fis., s. 8, vol. 20, 1956, 171-174.

[5] D. BUBBOLONI, *Contributi alla teoria degli insiemi di Fitting.* Tesi di dottorato, Università degli Studi di Firenze, 1992.

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