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The existence of angular derivatives of holomorphic maps of Siegel domains in a generalization of C^*-algebras

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Geometria. — *The existence of angular derivatives of holomorphic maps of Siegel domains in a generalization of C^*-algebras.* Nota di Kazimierz Włodarczyk, presentata (*) dal Socio E. Vesentini.

Abstract. — The aim of this paper is to start a systematic investigation of the existence of angular limits and angular derivatives of holomorphic maps of infinite dimensional Siegel domains in J^*-algebras. Since J^*-algebras are natural generalizations of C^*-algebras, B^*-algebras, JC^*-algebras, ternary algebras and complex Hilbert spaces, various significant results follow. Examples are given.

Key words: Holomorphic maps; Angular limits; Angular derivatives; Infinite dimensional Siegel domains; Generalizations of C^*-algebras.

Riassunto. — *L'esistenza di derivate angolari di mappe olomorfe di domini di Siegel in una generalizzazione di algebre C^*.* Questo articolo ha lo scopo di avviare uno studio sistematico dell'esistenza di limiti e derivate angolari di mappe olomorfe di domini di Siegel di dimensione infinita in algebre J^*. Poiché le algebre J^* sono generalizzazioni naturali di algebre C^*, algebre B^*, algebre JC^*, algebre ternarie e spazi di Hilbert complessi, ne seguono diversi risultati significativi. Vengono esaminati alcuni esempi.

1. Introduction

The basic fact of classical complex analysis, of importance to the theory of automorphic functions and hyperbolic geometry, is the biholomorphic equivalence between the open unit disk $\Delta = \{x \in \boldsymbol{C}: |x| < 1\}$ and the right half-plane $\Pi = \{x \in \boldsymbol{C}: \mathrm{Re}\, x > 0\}$ via the Cayley transformation $x \to (1+x)(1-x)^{-1}$ (often the upper half-plane is used instead). Recently, Pjatetskij-Shapiro has used multivariable «half-planes» in $\boldsymbol{C}^n$, the so-called tube domains and Siegel domains, in the theory of automorphic maps in several variables [33]. The associated Cayley transformations can be described either Lie theoretically [25] or, somewhat more directly, by using Jordan algebras and triple systems [29].

For $k > 0$, let $\Sigma_k = \{x \in \boldsymbol{C}: |\mathrm{Im}\, x| < k\, \mathrm{Re}\, x\}$.

The following theorem is well known in complex analysis:

Theorem 1.1. *Let f be a map holomorphic on Π, such that $f(\Pi) \subset \Pi$. If $a = \inf\{[\mathrm{Re} f(x)]/[\mathrm{Re}\, x]: x \in \Pi\}$ then, for any $k > 0$, we have*

$$\lim [f(x)]/x = \lim [\mathrm{Re} f(x)]/[\mathrm{Re}\, x] = \lim Df(x) = a$$

as $x \to \infty$, $x \in \Sigma_k$.

This result and its version when $f: \Delta \to \Delta$, concerning the existence of angular limits and angular derivatives of holomorphic maps, discovered by Carathéodory and developed particularly by Julia, Landau, Nevanlinna, Valiron, Warschawski, Wolff, Eke, Goldberg, Kin, Sarason, Pommerenke and Cowen, are important tools in the study of the boundary behaviour of holomorphic maps in $\boldsymbol{C}$ (in particular, they are a nice way to de-

(*) Nella seduta del 16 giugno 1994.

scribe approximative Denjoy-Wolff boundary fixed points [2, 7, 8, 10, 41, 46]) and have drawn interest for a long time. A survey appears in [2-5, 9, 13, 23, 27, 32, 34, 36, 37, 40, 42]. From a number of theoretical points of view, it is desirable to possess analogues of Carathéodory results in higher dimensions. The case of holomorphic maps of the Euclidean unit balls into themselves in C^n was studied by Rudin [35] and MacCluer and Shapiro [30]. Carathéodory's work was extended by Ky Fan [11] who proved the following elegant result:

THEOREM 1.2. *Let H denote a complex Hilbert space and let f be an operator-valued holomorphic map on the open half-plane Π, such that, for each $x \in \Pi$, $f(x)$ is an operator on H with $\mathrm{Re} f(x) > 0$. Suppose there is a Hermitian operator A on H satisfying $[\mathrm{Re} f(x)]/[\mathrm{Re}\, x] > A$ for all $x \in \Pi$ and, for any $\varepsilon > 0$, there is $z \in \Pi$ such that $\|[\mathrm{Re} f(z)]/[\mathrm{Re}\, z] - A\| < \varepsilon$. Then, for any $k > 0$, we have*

$$\lim \|[f(x)]/x - A\| = \lim \|[\mathrm{Re} f(x)]/[\mathrm{Re}\, x] - A\| = \lim \|Df(x) - A\| = 0$$

as $x \to \infty$, $x \in \Sigma_k$.

However, generally, the above-mentioned settings of investigations, concerning the existence of angular limits and derivatives of holomorphic maps of Siegel domains, exclude infinite dimensional situations. One of our goals here is to use the ideas of functional analysis, operator theory and infinite dimensional holomorphy in order to initiate those omitting the settings of investigations. For information about infinite dimensional holomorphy, the reader is referred to [8, 12, 31].

The history of the classifying of homogeneous domains begins with E. Cartan's famous paper [6] in C^n, $n > 1$, and continues with numerous contributions culminating in the works by L. Harris [14-18]. He introduced J^*-algebras and discovered a setting in which a large number of bounded and unbounded convex homogeneous domains in various finite and infinite dimensional complex Banach spaces can be studied simultaneously.

For complex Hilbert spaces H and K, let $\mathcal{L}(H, K)$ be the Banach space of all bounded linear operators from H to K with the operator norm. A closed complex linear subspace $\mathfrak{B}$ of $\mathcal{L}(H, K)$ is a J^*-algebra if $XX^*X \in \mathfrak{B}$ whenever $X \in \mathfrak{B}$.

Let $\mathfrak{B} \subset \mathcal{L}(H, K)$ be a J^*-algebra. For a partial isometry $V \in \mathfrak{B}$, let

$$\mathfrak{M}_V = \{X \in \mathfrak{B}\colon 2\,\mathrm{Re}\, V^*X - X^*(I_K - VV^*)X + I_H - V^*V > 0\}.$$

If $V \neq 0$, the unbounded convex domains $\mathfrak{M}_V$ are identical, by a simple rotation $X \to \to iX$, with operator Siegel domains [14-18]. If $V = 0$, this set reduces to the open unit ball $\mathfrak{B}_0 = \{X \in \mathfrak{B}\colon \|X\| < 1\}$. As is known, the open unit balls $\mathfrak{B}_0$ are bounded symmetric homogeneous domains [14-18]. For an isometry $V \in \mathfrak{B}$, let $\mathfrak{N}_V = \{X \in \mathfrak{B}\colon \mathrm{Re}\, V^*X - X^*(I_K - VV^*)X > 0\}$.

Finite and infinite dimensional Siegel-Harris domains $\mathfrak{M}_V$ and $\mathfrak{N}_V$ and Cartan-Harris bounded symmetric homogeneous domains $\mathfrak{B}_0$ were treated of by several authors (see *e.g.* [1, 6, 14-22, 24-26, 28, 29, 38, 39]).

The main objective of this paper is to define and characterize in $\mathfrak{M}_V$ (V-a partial isometry) and in $\mathfrak{N}_V$ (V-an isometry) infinite dimensional angular sets and solve the general problems concerning the existence of angular limits and angular derivatives for holomorphic maps $F: \mathfrak{M}_V \to \mathfrak{M}_V$ and $F: \mathfrak{N}_V \to \mathfrak{N}_V$ in such angular sets, respectively. Since J^*-algebras are natural generalizations of C^*-algebras, B^*-algebras, JC^*-algebras, ternary algebras, complex Hilbert spaces and others, therefore, in particular, various important results follow from this fact. Examples are given. The principal tool we use are general results of the Pick-Julia type for Siegel domains in J^*-algebras. This paper is a continuation of the studies in [44, 47, 49, 50].

2. Main results for $F: \mathfrak{M}_V \to \mathfrak{M}_V$

Before formulating our main results in $\mathfrak{M}_V$ in detail, we briefly review some background material.

For a partial isometry $V \in \mathfrak{B}$, let

(2.1) $$\mathfrak{M}_V = \{X \in \mathfrak{B}: \mathrm{RE}\, V^* X > 0\}$$

where

(2.2) $$\mathrm{RE}\, V^* X = 2\, \mathrm{Re}\, V^* X - X^* (I_K - VV^*) X + I_H - V^* V.$$

For X, $Z \in \mathfrak{M}_V$, V – *a* non-zero partial isometry in $\mathfrak{B}$, let

(2.3) $$P_{V,X,Z} = V^* X + Z^* V - Z^* (I_H - VV^*) X + I_H - V^* V.$$

It is evident that $P_{V,X,Z} = P^*_{V,Z,X}$ and $P_{V,X,X} = \mathrm{RE}\, V^* X$.

Further, set

(2.4) $$\mathfrak{p} = \{X \in \mathfrak{L}(H, H): \mathrm{Re} X > 0\}.$$

Our first result of Pick-Julia type is

Theorem 2.1. *Let $\mathfrak{B} \subset \mathfrak{L}(H, K)$ be a J^*-algebra containing a non-zero partial isometry V. If $F: \mathfrak{M}_V \to \mathfrak{p}$ is a holomorphic map such that $F(Z) = I_H$ for some $Z \in \mathfrak{M}_V$, then*

(2.5) $$\|F(X)\| \leq 4 \cdot \|(\mathrm{RE}\, V^* Z)^{-1/2} P_{V,X,Z} (\mathrm{RE}\, V^* X)^{-1/2}\|^2$$

for all $X \in \mathfrak{M}_V$. Here $\mathfrak{M}_V$, $\mathrm{RE}\, V^* X$, $\mathfrak{p}$ and $P_{V,X,Z}$ are defined by (2.1), (2.2), (2.4) *and* (2.3), *respectively.*

If a J^*-algebra $\mathfrak{B} \subset \mathfrak{L}(H, K)$ contains an isometry U and a non-zero partial isometry V, let, for $X \in \mathfrak{M}_V$,

(2.6) $$A_{U,V}(X) = (I_H + U^* V) A_V^{-1/2} + (V^* - U^*) B_V^{-1/2} X$$

and

(2.7) $$B_{U,V}(X) = (I_H - U^* V) A_V^{-1/2} + (V^* + U^*) B_V^{-1/2} X,$$

where

(2.8) $$A_V = I_H + V^* V \quad \text{and} \quad B_V = I_K + VV^*.$$

Moreover, for $\alpha > 1$, let

(2.9) $$D_\alpha(U, V) = \{X \in \mathfrak{B}: C(U, V; X) < (\alpha/2) \cdot C(V; X)\}$$

where

$$C(U, V; X) = \|[A_{U,V}(X)](I_H + V^* X)^{-1} A_V^{1/2}\| \tag{2.10}$$

and

$$C(V; X) = \|A_V^{-1/2}(I_H + V^* X)(\mathrm{RE}\, V^* X)^{-1}(I_H + X^* V) A_V^{-1/2}\|^{-1}. \tag{2.11}$$

We require an easy fact.

Proposition 2.1. *If* $\alpha > 1$, *then* $D_\alpha(U, V) \subset \mathfrak{M}_V$. *Moreover, if* $X \in D_\alpha(U, V)$, *then*

$$\|B_V^{-1/2}(X - V)(I_H + V^* X)^{-1} A_V^{1/2}\| \to 1 \quad \textit{or, equivalently}, \quad C(V; X) \to 0 \tag{2.12}$$

if and only if

$$(X - V)(I_H + V^* X)^{-1} \to B_V^{1/2} U A_V^{-1/2}. \tag{2.13}$$

If $\alpha \leqslant 1$, *then* $D_\alpha(U, V) = \emptyset$.

For $\alpha > 1$, we call the sets $D_\alpha(U, V) \subset \mathfrak{M}_V$ angular sets determined by U.

For $X \in \mathfrak{M}_V$, let

$$\mathcal{M}_{U,V}(X) = [B_{U,V}(X)][A_{U,V}(X)]^{-1}. \tag{2.14}$$

The next proposition will be most useful.

Proposition 2.2. *For all* $X \in \mathfrak{M}_V$, *the operator* $\mathcal{M}_{U,V}(X)$ *is invertible,* i.e. $[\mathcal{M}_{U,V}(X)]^{-1}$ *exists,* $\mathcal{M}_{U,V}(X) \in \mathfrak{p}$ *for all* $X \in \mathfrak{M}_V$ *and*

$$\mathrm{Re}\, \mathcal{M}_{U,V}(X) = [A_{U,V}(X)^*]^{-1}\{X^* B_V^{-1/2}(VV^* - UU^*) B_V^{-1/2} X + \tag{2.15}$$

$$+ 2\,\mathrm{Re}\, X^* B_V^{-1/2}(V + UU^* V) A_V^{-1/2} + A_V^{-1/2}(I_H - V^* UU^* V) A_V^{-1/2}\}[A_{U,V}(X)]^{-1}.$$

Moreover, for $X \in \mathfrak{M}_V$ *and* $P \in \mathfrak{B}$,

$$D([\mathcal{M}_{U,V}(X)]^{-1})(P) = -2A_V^{-1/2}(I_H + V^* X)[B_{U,V}(X)]^{-1} \cdot \tag{2.16}$$

$$\cdot U^* B_V^{1/2}[I_K - X(I_H + V^* X)^{-1} V^*] P [B_{U,V}(X)]^{-1}$$

and

$$D(\mathcal{M}_{U,V}(X))(P) = 2A_V^{-1/2}(I_H + V^* X)[A_{U,V}(X)]^{-1} \cdot \tag{2.17}$$

$$\cdot U^* B_V^{1/2}[I_K - X(I_H + V^* X)^{-1} V^*] P [A_{U,V}(X)]^{-1}.$$

Let us observe that, in particular, from (2.6)-(2.11) and (2.14)-(2.17) we get

$$D_\alpha(U, U) = \{X \in \mathfrak{B}: \|(I_H + U^* X)^{-1}\| < \tag{2.18}$$

$$< (\alpha/2)\,\|(I_H + U^* X)(\mathrm{RE}\, U^* X)^{-1}(I_H + X^* U)\|^{-1}\}$$

for $\alpha > 1$, and that, for $X \in \mathfrak{M}_U$ and $P \in \mathfrak{B}$,

$$\mathcal{M}_{U,U}(X) = U^* X, \qquad \mathrm{Re}\, \mathcal{M}_{U,U} = \mathrm{Re}\, U^* X, \tag{2.19}$$

$$D([\mathcal{M}_{U,U}(X)]^{-1})(P) = \tag{2.20}$$

$$= -(I_H + U^* X)(U^* X)^{-1}(I_H + U^* X)^{-1} U^* P (U^* X)^{-1} = -(U^* X)^{-1} U^* P (U^* X)^{-1}$$

and

$$D(\mathcal{M}_{U,U}(X))(P) = U^*P \tag{2.21}$$

since $U^* B_U^{1/2} = A_U^{1/2} U^*$ and $U^*(I_K + XU^*)^{-1} = (I_H + U^*X)^{-1} U^*$. Moreover, (2.13) may be replaced by $(X - U)(I_H + U^*X)^{-1} \to U$.

We are now able to formulate our main result.

THEOREM 2.2. *Let $\mathfrak{B} \subset \mathcal{L}(H, K)$ be a J^*-algebra containing an isometry U and a non-zero partial isometry V, and let $F: \mathfrak{M}_V \to \mathfrak{M}_V$ be a map holomorphic in $\mathfrak{M}_V$.*

(*a*) *Suppose there is a Hermitian operator $A \in \mathcal{L}(H, H)$ satisfying*

$$\operatorname{Re}(\mathcal{M}_{U,V} \circ F)(X) > A^{1/2}[\operatorname{Re} \mathcal{M}_{U,V}(X)] A^{1/2} \tag{2.22}$$

for all $X \in \mathfrak{M}_V$. If $D_\beta(U, V)$, $\beta > 1$, stands for an angular set such that, for any $\varepsilon > 0$, there exists a point $Z \in D_\beta(U, V)$ for which the inequality

$$\|[\operatorname{Re} \mathcal{M}_{U,V}(Z)]^{-1/2} \cdot A^{-1/2}[\operatorname{Re}(\mathcal{M}_{U,V} \circ F)(Z)] A^{-1/2} [\operatorname{Re} \mathcal{M}_{U,V}(Z)]^{-1/2} - I_H\| < \varepsilon \tag{2.23}$$

holds, then, for any $\alpha > 1$, we have

$$\lim \|[\mathcal{M}_{U,V}(X)]^{-1/2} \cdot A^{-1/2}[(\mathcal{M}_{U,V} \circ F)(X)] A^{-1/2} [\mathcal{M}_{U,V}(X)]^{-1/2} - I_H\| = 0, \tag{2.24}$$

$$\lim \|[\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1/2} \cdot A^{-1/2}[\operatorname{Re}(\mathcal{M}_{U,V} \circ F)(X)] A^{-1/2} [\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1/2} - I_H\| = 0, \tag{2.25}$$

$$\lim \|D\{(\mathcal{M}_{U,V} \circ F)(X) - A^{1/2}[\mathcal{M}_{U,V}(X)] A^{1/2}\}(U)\| = 0 \tag{2.26}$$

and

$$\lim \|D\{A^{1/2}[(\mathcal{M}_{U,V} \circ F)(X)]^{-1} A^{1/2} - [\mathcal{M}_{U,V}(X)]^{-1}\}(U)\| = 0 \tag{2.27}$$

as $C(V; X) \to 0$, $X \in D_\alpha(U, V)$.

(*b*) *Suppose there is a Hermitian operator $A \in \mathcal{L}(H, H)$ satisfying*

$$\operatorname{Re}(\mathcal{M}_{U,V} \circ F)(X) = A^{1/2}[\operatorname{Re} \mathcal{M}_{U,V}(X)] A^{1/2} \tag{2.28}$$

for all $X \in \mathfrak{M}_V$. Then, for any $\alpha > 1$, assertion (2.24)-(2.27) *holds as $C(V; X) \to 0$, $X \in D_\alpha(U, V)$. Here $\mathfrak{M}_V$, $\mathcal{M}_{U,V}$, $D_\alpha(U, V)$ and $C(V; X)$ are defined by* (2.1), (2.14), (2.9) *and* (2.11), *respectively.*

3. MAIN RESULTS FOR $F: \mathfrak{N}_V \to \mathfrak{N}_V$

The statement of the results for $\mathfrak{N}_V$ requires some definitions.

For an isometry $V \in \mathfrak{B}$, let

$$\mathfrak{N}_V = \{X \in \mathfrak{B}: \operatorname{Re} V^*X - X^*(I_K - VV^*)X > 0\}. \tag{3.1}$$

For $X,\ Z \in \mathfrak{N}_V$, let

$$P_{V,X,Z} = 2V^*X + 2Z^*V - Z^*(I_K - VV^*)X. \tag{3.2}$$

It is evident that $P_{V,X,Z} = P^*_{V,Z,X}$.

We can show

THEOREM 3.1. *Let $\mathfrak{B} \subset \mathfrak{L}(H,K)$ be a J^*-algebra containing an isometry V. If $F\colon \mathfrak{N}_V \to \mathfrak{p}$ is a holomorphic map such that $F(Z) = I_H$ for some $Z \in \mathfrak{N}_V$, then*

$$\|F(X)\| \leq 4\cdot\|(P_{V,Z,Z})^{-1/2}P_{V,X,Z}(P_{V,X,X})^{-1/2}\|^2 \tag{3.3}$$

for all $X \in \mathfrak{N}_V$. Here $\mathfrak{N}_V$, $\mathfrak{p}$ and $P_{V,X,Z}$ are defined by (3.1), (2.4) *and* (3.2), *respectively.*

If a J^*-algebra $\mathfrak{B} \subset \mathfrak{L}(H,K)$ contains isometries U and V, let, for $X \in \mathfrak{N}_V$,

$$A_{U,V}(X) = I_H + U^*V + (V^* - U^*)X \tag{3.4}$$

and

$$B_{U,V}(X) = I_H - U^*V + (V^* + U^*)X. \tag{3.5}$$

Moreover, for $\alpha > 1$, let

$$D_\alpha(U,V) = \{X \in \mathfrak{B}\colon C(U,V;X) < (\alpha/2)\cdot C(V;X)\} \tag{3.6}$$

where

$$C(U,V;X) = \|[A_{U,V}(X)](I_H + V^*X)^{-1}\| \tag{3.7}$$

and

$$C(V;X) = \|(I_H + V^*X)(P_{V,X,X})^{-1}(I_H + X^*V)\|^{-1}. \tag{3.8}$$

We shall need the following

PROPOSITION 3.1. *If $\alpha > 1$, then $D_\alpha(U,V) \subset \mathfrak{N}_V$. Moreover, if $X \in D_\alpha(U,V)$, then*

$$\|(X - V)(I_H + V^*X)^{-1}\| \to 1 \quad \text{or, equivalently,} \quad C(V;X) \to 0 \tag{3.9}$$

if and only if

$$(X - V)(I_H + V^*X)^{-1} \to U. \tag{3.10}$$

If $\alpha \leq 1$, then $D_\alpha(U,V) = \emptyset$.

For $\alpha > 1$, we call the sets $D_\alpha(U,V) \subset \mathfrak{N}_V$ angular sets determined by U.

For $X \in \mathfrak{N}_V$, let

$$\mathcal{N}_{U,V}(X) = [B_{U,V}(X)][A_{U,V}(X)]^{-1}. \tag{3.11}$$

We then have

PROPOSITION 3.2. *For all $X \in \mathfrak{N}_V$, the operator $\mathcal{N}_{U,V}(X)$ is invertible,* i.e. $[\mathcal{N}_{U,V}(X)]^{-1}$ *exists, $\mathcal{N}_{U,V}(X) \in \mathfrak{p}$ for all $X \in \mathfrak{N}_V$ and*

$$\operatorname{Re}\mathcal{N}_{U,V}(X) = [A_{U,V}(X)^*]^{-1}\{X^*(VV^* - UU^*)X + {}+ 2\operatorname{Re}X^*(V + UU^*V) + I_H - V^*UU^*V)\}[A_{U,V}(X)]^{-1}. \tag{3.12}$$

Moreover, for $X \in \mathfrak{R}_V$ *and* $P \in \mathfrak{B}$,

$$(3.13)\quad D([\mathcal{N}_{U,V}(X)]^{-1})(P) = = -2(I_H + V^*X)[B_{U,V}(X)]^{-1}U^*B_V[I_K - X(I_H + V^*X)^{-1}V^*]P[B_{U,V}(X)]^{-1}$$

and

$$(3.14)\quad D(\mathcal{N}_{U,V}(X))(P) = = 2(I_H + V^*X)[A_{U,V}(X)]^{-1}U^*B_V[I_K - X(I_H + V^*X)^{-1}V^*]P[A_{U,V}(X)]^{-1}.$$

Let us observe that, in particular, from (3.4)-(3.8) and (3.11)-(3.14) we get

$$(3.15)\quad D_\alpha(U, U) = \{X \in \mathfrak{B}\colon 2\|(I_H + U^*X)^{-1}\| < < (\alpha/2)\|(I_H + U^*X)(P_{U,X,X})^{-1}(I_H + X^*U)\|^{-1}\}$$

for $\alpha > 1$, and that, for $X \in \mathfrak{R}_U$ and $P \in \mathfrak{B}$,

$$(3.16)\quad \mathcal{N}_{U,U}(X) = U^*X, \qquad \operatorname{Re}\mathcal{N}_{U,U} = \operatorname{Re}U^*X,$$

$$(3.17)\quad D([\mathcal{N}_{U,U}(X)]^{-1})(P) = -(U^*X)^{-1}U^*P(U^*X)^{-1}$$

and

$$(3.18)\quad D(\mathcal{N}_{U,U}(X))(P) = U^*P.$$

The basic result of this section is

THEOREM 3.2. *Let* $\mathfrak{B} \subset \mathcal{L}(H, K)$ *be a* J^**-algebra containing isometries* U *and* V, *and let* $F\colon \mathfrak{R}_V \to \mathfrak{R}_V$ *be a map holomorphic in* $\mathfrak{R}_V$.

(*a*) *Suppose there is a Hermitian operator* $\boldsymbol{A} \in \mathcal{L}(H, H)$ *satisfying*

$$(3.19)\quad \operatorname{Re}(\mathcal{N}_{U,V}\circ F)(X) > \boldsymbol{A}^{1/2}[\operatorname{Re}\mathcal{N}_{U,V}(X)]\boldsymbol{A}^{1/2}$$

for all $X \in \mathfrak{R}_V$. *If* $D_\beta(U, V)$, $\beta > 1$, *stands for an angular set such that, for any* $\varepsilon > 0$, *there exists a point* $Z \in D_\beta(U, V)$ *for which the inequality*

$$(3.20)\quad \|[\operatorname{Re}\mathcal{N}_{U,V}(Z)]^{-1/2}\cdot \cdot\boldsymbol{A}^{-1/2}[\operatorname{Re}(\mathcal{N}_{U,V}\circ F)(Z)]\boldsymbol{A}^{-1/2}[\operatorname{Re}\mathcal{N}_{U,V}(Z)]^{-1/2} - I_H\| < \varepsilon$$

holds, then, for any $\alpha > 1$, *we have*

$$(3.21)\quad \lim\|[\mathcal{N}_{U,V}(X)]^{-1/2}\boldsymbol{A}^{-1/2}[(\mathcal{N}_{U,V}\circ F)(X)]\boldsymbol{A}^{-1/2}[\mathcal{N}_{U,V}(X)]^{-1/2} - I_H\| = 0,$$

$$(3.22)\quad \lim\|[\operatorname{Re}\mathcal{N}_{U,V}(X)]^{-1/2}\cdot \cdot\boldsymbol{A}^{-1/2}[\operatorname{Re}(\mathcal{N}_{U,V}\circ F)(X)]\boldsymbol{A}^{-1/2}[\operatorname{Re}\mathcal{N}_{U,V}(X)]^{-1/2} - I_H\| = 0,$$

$$(3.23)\quad \lim\|D\{(\mathcal{N}_{U,V}\circ F)(X) - \boldsymbol{A}^{1/2}[\mathcal{N}_{U,V}(X)]\boldsymbol{A}^{1/2}\}(U)\| = 0$$

and

$$(3.24)\quad \lim\|D\{\boldsymbol{A}^{1/2}[(\mathcal{N}_{U,V}\circ F)(X)]^{-1}\boldsymbol{A}^{1/2} - [\mathcal{N}_{U,V}(X)]^{-1}\}(U)\| = 0$$

as $C(V; X) \to 0$, $X \in D_\alpha(U, V)$.

(*b*) *Suppose there is a Hermitian operator* $A \in \mathcal{L}(H, H)$ *satisfying*

$$\mathrm{Re}\,(\mathcal{N}_{U,V} \circ F)(X) = A^{1/2}[\mathrm{Re}\,\mathcal{N}_{U,V}(X)]A^{1/2} \tag{3.25}$$

for all $X \in \mathfrak{N}_V$. *Then, for any* $\alpha > 1$, *assertion* (3.21)-(3.24) *holds as* $C(V; X) \to 0$, $X \in D_\alpha(U, V)$. *Here* $\mathfrak{N}_V$, $\mathcal{N}_{U,V}(X)$, $D_\alpha(U, V)$ *and* $C(V; X)$ *are defined by* (3.1), (3.11), (3.6) *and* (3.8), *respectively*.

4. Proof of Theorem 2.1

Let $T_Y: \mathfrak{B}_0 \to \mathfrak{B}_0$, $Y \in \mathfrak{B}_0$, denote the Möbius biholomorphic map of the form (see [15, Theorem 2, p. 20])

$$T_Y(X) = B_Y^{-1/2}(X - Y)(I_H - Y^* X)^{-1} A_Y^{1/2}, \qquad X \in \mathfrak{B}_0, \tag{4.1}$$

where

$$A_Y = I_H - Y^* Y \quad \text{and} \quad B_Y = I_K - YY^* \quad \text{for } Y \in \mathfrak{B}_0. \tag{4.2}$$

The biholomorphic map f_V of $\mathfrak{B}_0$ onto $\mathfrak{M}_V$ is defined by the formula (see [43, Theorem 5, p. 499])

$$f_V(X) = B_V^{-1/2}(X + V)(I_H - V^* X)^{-1} A_V^{1/2}, \qquad X \in \mathfrak{B}_0, \tag{4.3}$$

and, moreover,

$$f_V^{-1}(Y) = B_V^{-1/2}(Y - V)(I_H + V^* Y)^{-1} A_V^{1/2}, \qquad Y \in \mathfrak{M}_V, \tag{4.4}$$

where $A_V = I_H + V^* V$ and $B_V = I_K + VV^*$.

Let $\mathfrak{D}_0 = \{X \in \mathcal{L}(H, H): \|X\| < 1\}$ and let $f: \mathfrak{D}_0 \to \mathfrak{p}$ be a Cayley biholomorphic map of the form $f(X) = (I_H + X)(I_H - X)^{-1}$, $X \in \mathfrak{D}_0$.

Let $R = f_V^{-1}(Z)$. Using the Schwarz lemma to the holomorphic map $g: \mathfrak{B}_0 \to \mathfrak{D}_0$, $g(0) = 0$, defined by the formula $g(Y) = (f^{-1} \circ F \circ f_V \circ T_R^{-1})(Y)$, $Y \in \mathfrak{B}_0$, we obtain $\|g(Y)\| \leqslant \|Y\|$, $Y \in \mathfrak{B}_0$. In particular, for $Y = (T_R \circ f_V^{-1})(X)$, $X \in \mathfrak{M}_V$, we get $\|(f^{-1} \circ F)(X)\| \leqslant \|(T_R \circ f_V^{-1})(X)\|$, $X \in \mathfrak{M}_V$. Thus

$$\{(f^{-1} \circ F)(X)\}^* \{(f^{-1} \circ F)(X)\} \leqslant \eta^2 I_H \quad \text{where} \quad \eta = \|(T_R \circ f_V^{-1})(X)\|,$$

and, in particular,

$$F(X)^* F(X) - (1 + \eta)(1 - \eta)^{-1} F(X)^* - (1 + \eta)(1 - \eta)^{-1} F(X) + I_H \leqslant 0$$

or, equivalently,

$$\|F(X) - (1 + \eta)(1 - \eta)^{-1} I_H\| \leqslant 2\eta^{1/2}(1 - \eta)^{-1}$$

because

$$(f^{-1} \circ F)(X) = [F(X) - I_H][I_H + F(X)]^{-1}.$$

In consequence, we have $\|F(X)\| \leqslant (1 + \eta)(1 - \eta)^{-1}$. Hence it follows that $\|F(X)\| \leqslant$ $\leqslant 4(1 - \|T_R(S)\|^2)^{-1}$, $X \in \mathfrak{M}_V$, $S = f_V^{-1}(X)$. This, together with the identity [45, formula (18), p. 247] $(1 - \|T_R(S)\|^2)^{-1} = T(R, S)$ where

$$T(R, S) = \|A_R^{-1/2}(I_H - R^* S) A_S^{-1}(I_H - S^* R) A_R^{-1/2}\|, \tag{4.5}$$

is equivalent to

$$(4.6)\qquad \|F(X)\| \leq 4\cdot T(R,S), \quad S = f_V^{-1}(X), \quad R = f_V^{-1}(Z), \quad X \in \mathfrak{M}_V.$$

Moreover, by (4.2) and (4.4),

$$A_R = A_V^{1/2}(I_H + Z^*V)^{-1}(\mathrm{RE}\,V^*Z)(I_H + V^*Z)^{-1}A_V^{1/2},$$

$$A_S = A_V^{1/2}(I_H + X^*V)^{-1}(\mathrm{RE}\,V^*X)(I_H + V^*X)^{-1}A_V^{1/2},$$

$$I_H - R^*S = A_V^{1/2}(I_H + Z^*V)^{-1}P_{V,X,Z}(I_H + V^*X)^{-1}A_V^{1/2}.$$

Consequently,

$$(4.7)\qquad T(R,S) =$$

$$= \|W_Z^{-1}(\mathrm{RE}\,V^*Z)^{-1/2}P_{V,X,Z}(\mathrm{RE}\,V^*X)^{-1}P_{V,Z,X}(\mathrm{RE}\,V^*Z)^{-1/2}(W_Z^{-1})^*\|$$

where W_Z is a unitary operator of the form

$$W_Z = (\mathrm{RE}\,V^*Z)^{-1/2}(I_H + Z^*V)A_V^{-1/2}\cdot$$

$$\cdot\{A_V^{1/2}(I_H + Z^*V)^{-1}(\mathrm{RE}\,V^*Z)(I_H + V^*Z)^{-1}A_V^{1/2}\}^{1/2},$$

which, by (4.6) and (4.7), yields (2.5).

5. Proof of Proposition 2.1

For $\alpha > 1$, we let

$$(5.1)\quad D_\alpha(U,V) = \{X \in \mathfrak{B}: \|I_H - U^*f_V^{-1}(X)\| < (\alpha/2)(1 - \|f_V^{-1}(X)\|^2)\}.$$

Of course, $D_\alpha(U,V) \subset \mathfrak{M}_V$ for all $\alpha > 1$. When $\alpha \leq 1$, this set is empty.

Now, let us observe that

$$(5.2)\qquad I_H - U^*f_V^{-1}(X) =$$

$$= [(I_H + U^*V)A_V^{-1/2} + (V^* - U^*)B_V^{-1/2}X](I_H + V^*X)^{-1}A_V^{1/2};$$

using the spectrum σ, we show that

$$(5.3)\qquad (1 - \|f_V^{-1}(X)\|^2)^{-1} = \sup\sigma\{[I_H - f_V^{-1}(X)^*f_V^{-1}(X)]^{-1}\} =$$

$$= \|A_V^{-1/2}(I_H + V^*X)(\mathrm{RE}\,V^*X)^{-1}(I_H + X^*V)A_V^{-1/2}\|.$$

Consequently, (5.1) and (2.9) are identical.

Now, note that if $X \in D_\alpha(U,V)$ and $\|f_V^{-1}(X)\| \to 1$ or, by (5.3), equivalently $C(V;X) \to 0$, then, by (5.1), $C(U,V;X) \to 0$ or, equivalently, $\|I_H - U^*f_V^{-1}(X)\| \to 0$, which implies that $S = f_V^{-1}(X) \to U$ since U is an isometry. Consequently, (2.12) implies (2.13).

6. Proof of Proposition 2.2

We define a holomorphic map $\mathcal{M}_{U,V}: \mathfrak{M}_V \to \mathcal{L}(H,H)$ by the formula

(6.1) $\mathcal{M}_{U,V}(X) = [I_H + U^* f_V^{-1}(X)][I_H - U^* f_V^{-1}(X)]^{-1}, \quad X \in \mathfrak{M}_V.$

Let us observe that

(6.2) $\operatorname{Re}\mathcal{M}_{U,V}(X) =$

$$= [I_H - f_V^{-1}(X)^* U]^{-1}[I_H - f_V^{-1}(X)^* UU^* f_V^{-1}(X)][I_H - U^* f_V^{-1}(X)]^{-1},$$

$X \in \mathfrak{M}_V$. Obviously, the operator $\mathcal{M}_{U,V}(X)$ is invertible, *i.e.* $[\mathcal{M}_{U,V}(X)]^{-1}$ exists and $\mathcal{M}_{U,V}(X) \in \mathfrak{p}$ for all $X \in \mathfrak{M}_V$.

Using (4.4) and (5.2), we get that (6.1) and (6.2) are identical with (2.14) and (2.15), respectively.

Finally, let us notice that

(6.3) $D([\mathcal{M}_{U,V}(X)]^{-1})(P) =$

$$= -2[I_H + U^* f_V^{-1}(X)]^{-1} U^* Df_V^{-1}(X)(P)[I_H + U^* f_V^{-1}(X)]^{-1},$$

(6.4) $D(\mathcal{M}_{U,V}(X))(P) =$

$$= 2[I_H - U^* f_V^{-1}(X)]^{-1} U^* Df_V^{-1}(X)(P)[I_H - U^* f_V^{-1}(X)]^{-1}$$

and, by [43, p. 510], we get

(6.5) $Df_V^{-1}(X)(P) = B_V^{1/2}(I_K + XV^*)^{-1} P (I_H + V^* X)^{-1} A_V^{1/2}, \quad X \in \mathfrak{M}_V, \quad P \in \mathfrak{B}.$

Consequently, using (4.4) and the fact that $(I_K + XV^*)^{-1} = I_K - X(I_H + V^* X)^{-1} V^*$, we obtain (2.16) and (2.17).

7. Proof of Theorem 2.2

(*a*) Let $\varepsilon > 0$ be arbitrary and fixed. By (2.23), there exists $Z \in D_\beta(U, V)$ such that

$$\|[\operatorname{Re}\mathcal{M}_{U,V}(Z)]^{-1/2} \boldsymbol{A}^{-1/2}[\operatorname{Re}(\mathcal{M}_{U,V} \circ F)(Z)]\boldsymbol{A}^{-1/2}[\operatorname{Re}\mathcal{M}_{U,V}(Z)]^{-1/2} - I_H\| < \varepsilon.$$

We define maps E and G, holomorphic in $\mathfrak{M}_V$, by the formulae

$$E(X) = \boldsymbol{A}^{-1/2}[(\mathcal{M}_{U,V} \circ F)(X)]\boldsymbol{A}^{-1/2} - \mathcal{M}_{U,V}(X)$$

and

(7.1) $G(X) = [\operatorname{Re}E(Z)]^{-1/2}[E(X) - i \cdot \operatorname{Im}E(Z)][\operatorname{Re}E(Z)]^{-1/2},$

respectively. Let us observe that, by (2.22), $\operatorname{Re}E(X) > 0$ and $\operatorname{Re}G(X) > 0$ for all $X \in \mathfrak{M}_V$, and $G(Z) = I_H$. Applying Theorem 2.1 to the map G, we get, by (4.5) and (4.6),

(7.2) $\|G(X)\| \leqslant 4 \cdot T(R, S) = 4\|A_R^{-1/2}(I_H - R^* S) A_S^{-1}(I_H - S^* R) A_R^{-1/2}\|, \ X \in \mathfrak{M}_V,$

where $R = f_V^{-1}(Z)$, $S = f_V^{-1}(X)$. Now, from (7.1) we obtain

$\|[\mathcal{M}_{U,V}(X)]^{-1/2}[E(X)][\mathcal{M}_{U,V}(X)]^{-1/2}\| =$

$$= \|[\mathcal{M}_{U,V}(X)]^{-1/2}\boldsymbol{A}^{-1/2}[(\mathcal{M}_{U,V} \circ F)(X)]\boldsymbol{A}^{-1/2}[\mathcal{M}_{U,V}(X)]^{-1/2} - I_H\| \leqslant$$

$$\leqslant \|[\mathcal{M}_{U,V}(X)]^{-1}\| \|[\operatorname{Re} E(Z)]^{1/2} G(X) [\operatorname{Re} E(Z)]^{1/2} + i \cdot \operatorname{Im} E(Z)\| \leqslant$$
$$\leqslant \|[\mathcal{M}_{U,V}(X)]^{-1}\| \{\|\operatorname{Re} E(Z)\| \cdot \|G(X)\| + \|\operatorname{Im} E(Z)\|\} \leqslant$$
$$\leqslant \|[\mathcal{M}_{U,V}(X)]^{-1}\| \|[\operatorname{Re} \mathcal{M}_{U,V}(Z)]^{-1/2} [\operatorname{Re} E(Z)] [\operatorname{Re} \mathcal{M}_{U,V}(Z)]^{-1/2}\| \cdot$$
$$\cdot \|\operatorname{Re} \mathcal{M}_{U,V}(Z)\| \|G(X)\| + \|[\mathcal{M}_{U,V}(X)]^{-1}\| \|\operatorname{Im} E(Z)\| .$$

Let $\alpha > 1$ be arbitrary and fixed. Since

$$\|[\mathcal{M}_{U,V}(X)]^{-1}\| \leqslant \|I_H - U^* S\| \|(I_H + U^* S)^{-1}\| ,$$

by (6.2),

$$\|\operatorname{Re} \mathcal{M}_{U,V}(X)\| \leqslant \|(I_H - R^* U)^{-1} (I_H - R^* U U^* R)(I_H - U^* R)^{-1}\| \leqslant$$
$$\leqslant \|(I_H - R^* U)^{-1}\|^2 \|I_H - R^* U + R^* U (I_H - U^* R)\| \leqslant$$
$$\leqslant (1 - \|R\|)^{-2} \|I_H - R^* U\| (1 + \|R^* U\|) < 2(1 - \|R\|)^{-2} \|I_H - R^* U\| ,$$
$$\|I_H - U^* S\| (1 - \|S\|^2)^{-1} < \alpha/2 , \quad \|I_H - U^* R\| (1 - \|R\|^2)^{-1} < \beta/2$$

and, by (4.5) and (7.2),

$$\|G(X)\| \leqslant 4 \cdot T(R, S) \leqslant 4 \|I_H - S^* R\|^2 [(1 - \|S\|^2)(1 - \|R\|^2)]^{-1} ,$$

it follows that

$$\text{(7.3)} \quad \|[\mathcal{M}_{U,V}(X)]^{-1/2} [E(X)] [\mathcal{M}_{U,V}(X)]^{-1/2}\| \leqslant 2\varepsilon\alpha\beta \|(I_H + U^* S)^{-1}\| \cdot$$
$$\cdot \|I_H - S^* R\|^2 (1 - \|R\|^2)^{-1} + \|I_H - U^* S\| \|(I_H + U^* S)^{-1}\| \|\operatorname{Im} E(Z)\| .$$

Consequently, since the right-hand side of inequality (7.3), by (2.12) and (2.13) (or see section 5), tends to

$$\varepsilon\alpha\beta \|I_H - U^* R\|^2 (1 - \|R\|^2)^{-1} <$$
$$< \varepsilon\alpha\beta \|I_H - U^* R\|^2 (1 - \|R\|^2)^{-2} (1 - \|R\|)^2 < (1/2)\, \varepsilon\alpha\beta^3$$

and $\varepsilon > 0$ can be arbitrarily small, therefore

$$\lim \|[\mathcal{M}_{U,V}(X)]^{-1/2} \boldsymbol{A}^{-1/2} [(\mathcal{M}_{U,V} \circ F)(X)] \boldsymbol{A}^{-1/2} [\mathcal{M}_{U,V}(X)]^{-1/2} - I_H\| = 0$$

as $C(V; X) \to 0$, $X \in D_\alpha(U, V)$, *i.e.* (2.24) holds.

Now, let us observe that

$$\|[\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1/2} [\operatorname{Re} E(X)] [\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1/2}\| =$$
$$= \|[\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1/2} \boldsymbol{A}^{-1/2} [\operatorname{Re} (\mathcal{M}_{U,V} \circ F)(X)] \boldsymbol{A}^{-1/2} [\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1/2} - I_H\| \leqslant$$
$$\leqslant \|E(X)\| \|[\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1}\| \leqslant$$
$$\leqslant \|[\mathcal{M}_{U,V}(X)]^{-1/2} [E(X)] [\mathcal{M}_{U,V}(X)]^{-1/2}\| \|\mathcal{M}_{U,V}(X)\| \|[\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1}\| .$$

But

$$\|[\mathcal{M}_{U,V}(X)]\| \leqslant \|I_H + U^* S\| (1 - \|S\|)^{-1} < 2(1 - \|S\|)^{-1}$$

and

$$\|[\operatorname{Re} \mathcal{M}_{U,V}(X)]^{-1}\| \leqslant \|I_H - U^* S\|^2 (1 - \|S\|^2)^{-1} < (\alpha^2/4)(1 - \|S\|^2) .$$

Thus

$$\|[\operatorname{Re}\mathcal{M}_{U,V}(X)]^{-1/2}[\operatorname{Re}E(X)][\operatorname{Re}\mathcal{M}_{U,V}(X)]^{-1/2}\| <$$
$$< \alpha^2\|[\mathcal{M}_{U,V}(X)]^{-1/2}[E(X)][\mathcal{M}_{U,V}(X)]^{-1/2}\|.$$

Since, by (2.24), the right-hand side of the above inequality tends to zero, we have

$$\lim \|[\operatorname{Re}\mathcal{M}_{U,V}(X)]^{-1/2}A^{-1/2}[\operatorname{Re}(\mathcal{M}_{U,V}\circ F)(X)]A^{-1/2}[\operatorname{Re}\mathcal{M}_{U,V}(X)]^{-1/2} - I_H\| = 0$$

as $C(V;X)\to 0$, $X \in D_\alpha(U,V)$, *i.e.* (2.25) holds.

Let $1<\gamma<\alpha$. We shall need the following relation between $D_\alpha(U,V)$ and $D_\gamma(U,V)$. Assume that

$$(7.4)\qquad 1<\gamma<\alpha \quad \text{and} \quad \delta = (1/3)(1/\gamma - 1/\alpha)$$

and

$$(7.5)\qquad X \in D_\gamma(U,V), \quad i.e.\ \|I_H - U^* f_V^{-1}(X)\| < (\gamma/2)(1 - \|f_V^{-1}(X)\|^2).$$

If

$$(7.6)\qquad |\lambda| \leq \delta\|I_H - U^* f_V^{-1}(X)\|,$$

then

$$(7.7)\qquad f_V[f_V^{-1}(X) + \lambda U] \in D_\alpha(U,V),$$
$$i.e.\ \|I_H - U^*[f_V^{-1}(X) + \lambda U]\| < (\alpha/2)(1 - \|f_V^{-1}(X) + \lambda U\|^2).$$

Indeed, from (7.4) we have

$$(7.8)\qquad |\lambda|^2 < |\lambda|, \quad 2/\alpha < 2, \quad (5\delta + 2/\alpha) < 2/\gamma$$

whenever $|\lambda|$ is sufficiently small. From (7.5) we get

$$(7.9)\qquad \|f_V^{-1}(X)\|^2 + (2/\gamma)\|I_H - U^* f_V^{-1}(X)\| < 1.$$

Thus, using (7.8), (7.6) and (7.9), we obtain

$$\|f_V^{-1}(X) + \lambda U\|^2 + (2/\alpha)\|I_H - U^*[f_V^{-1}(X) + \lambda U]\| \leq$$
$$\leq \|f_V^{-1}(X)\|^2 + 3|\lambda| + (2/\alpha)\|I_H - U^* f_V^{-1}(X)\| + (2/\alpha)|\lambda| \leq$$
$$\leq \|f_V^{-1}(X)\|^2 + 5|\lambda| + (2/\alpha)\|I_H - U^* f_V^{-1}(X)\| \leq$$
$$\leq \|f_V^{-1}(X)\|^2 + (2/\gamma)\|I_H - U^* f_V^{-1}(X)\| < 1.$$

This immediately yields (7.7).

Now, we prove (2.26) and (2.27). By Proposition 2.2 and the Cauchy integral formula [31, Proposition 2, p. 21],

$$(7.10)\qquad D\{A^{1/2}[(\mathcal{M}_{U,V}\circ F)(X)]^{-1}[A^{1/2}\mathcal{M}_{U,V}(X) -$$
$$-(\mathcal{M}_{U,V}\circ F)(X)A^{-1/2}][\mathcal{M}_{U,V}(X)]^{-1}\}(U) =$$
$$= D\{A^{1/2}[(\mathcal{M}_{U,V}\circ F)(X)]^{-1}A^{1/2} - [\mathcal{M}_{U,V}(X)]^{-1}\}(U) =$$
$$= \frac{1}{2\pi i}\int_{|\lambda|=r}[(\mathcal{M}_{U,V}\circ f_V)(S+\lambda U)]^{-1/2}\{[(\mathcal{M}_{U,V}\circ f_V)(S+\lambda U)]^{1/2}A^{1/2}\cdot$$

$$\cdot[(\mathcal{M}_{U,V}\circ F\circ f_V)(S+\lambda U)]^{-1}A^{1/2}[(\mathcal{M}_{U,V}\circ f_V)(S+\lambda U)]^{1/2}-I_H\}\cdot$$
$$\cdot[(\mathcal{M}_{U,V}\circ f_V)(S+\lambda U)]^{-1/2}\lambda^{-2}\,d\lambda$$

where $S = f_V^{-1}(X)$, $\lambda = r\cdot e^{it}$, $r = r(X) = \delta\|I_H - U^*S\|$, $t\in[0;2\pi]$ and f_V is defined by (4.3). But

$$\|[(\mathcal{M}_{U,V}\circ f_V)(S+\lambda U)]^{-1}\||\lambda|^{-1}\leq[\|I_H-U^*S\|+|\lambda|]\cdot$$
$$\cdot\|[I_H+U^*(S+\lambda U)]^{-1}\||\lambda|^{-1}=(\delta^{-1}+1)\|[I_H+U^*(S+\lambda U)]^{-1}\|.$$

Since the right-hand side of the above inequality tends to $2^{-1}(\delta^{-1}+1)$, from (7.10) we get (2.26) and (2.27) by using (2.24), (7.4)-(7.7), (6.3)-(6.5), (2.16) and (2.17).

(*b*) If (2.28) holds for all $X\in\mathfrak{M}_V$, let $\varepsilon>0$ be arbitrary and fixed and let η be such that $0<\eta<\varepsilon$. Then

$$A^{-1/2}[\operatorname{Re}(\mathcal{M}_{U,V}\circ F)(X)]A^{-1/2}+\eta[\operatorname{Re}\mathcal{M}_{U,V}(X)]>\operatorname{Re}\mathcal{M}_{U,V}(X)$$

for all $X\in\mathfrak{M}_V$. Moreover, obviously, then there exists some $Z\in D_\beta(U,V)$ for which the inequality

$$\|[\operatorname{Re}\mathcal{M}_{U,V}(Z)]^{-1/2}A^{-1/2}[\operatorname{Re}(\mathcal{M}_{U,V}\circ F)(Z)]A^{-1/2}[\operatorname{Re}\mathcal{M}_{U,V}(Z)]^{-1/2}-I_H+\eta I_H\|=$$
$$=\eta<\varepsilon$$

holds. Now, we define maps E_η and G_η, holomorphic in $\mathfrak{M}_V$, by the formulae

$$E_\eta(X)=E(X)+\eta[\mathcal{M}_{U,V}(X)],\quad E(X)=A^{-1/2}[(\mathcal{M}_{U,V}\circ F)(X)]A^{-1/2}-\mathcal{M}_{U,V}(X)$$

and

$$G_\eta(X)=[\operatorname{Re}E_\eta(Z)]^{-1/2}[E_\eta(X)-i\cdot\operatorname{Im}E_\eta(Z)][\operatorname{Re}E_\eta(Z)]^{-1/2},$$

respectively. Let us note that $\operatorname{Re}E_\eta(X)>0$ and $\operatorname{Re}G_\eta(X)>0$ for all $X\in\mathfrak{M}_V$, and that $G_\eta(Z)=I_H$. Using analogous considerations as in part (*a*), we have, respectively, for $R=f_V^{-1}(Z)$ and $S=f_V^{-1}(X)$,

$$\|[\mathcal{M}_{U,V}(X)]^{-1/2}E(X)[\mathcal{M}_{U,V}(X)]^{-1/2}\|=$$
$$=\|[\mathcal{M}_{U,V}(X)]^{-1/2}A^{-1/2}[(\mathcal{M}_{U,V}\circ F)(X)]A^{-1/2}[\mathcal{M}_{U,V}(X)]^{-1/2}-I_H\|\leq$$
$$\leq\|[\mathcal{M}_{U,V}(X)]^{-1}\|\{\|\operatorname{Re}E_\eta(Z)\|\|G_\eta(X)\|+\|\operatorname{Im}E_\eta(Z)\|\}+\eta.$$

Thus, for any $\alpha>1$, using analogous arguments as in part (*a*), we obtain

$$\lim\|[\mathcal{M}_{U,V}(X)]^{-1/2}[E(X)][\mathcal{M}_{U,V}(X)]^{-1/2}\|\leq(1/2)\,\varepsilon\alpha\beta^3+\eta$$

as $C(V;X)\to 0$, $X\in D_\alpha(U,V)$. This implies (2.24). Using arguments similar to those given in part (*a*) we prove that also (2.25), (2.26) and (2.27) hold as $C(V;X)\to 0$ in all angular sets $D_\alpha(U,V)$, $\alpha>1$.

8. Proof of Theorem 3.1

The biholomorphic map f_V of $\mathfrak{B}_0$ onto $\mathfrak{R}_V$ is defined by the formula

$$f_V(X)=(X+V)(I_H-V^*X)^{-1},\qquad X\in\mathfrak{B}_0, \tag{8.1}$$

and, moreover,

(8.2) $$f_V^{-1}(Y) = (Y - V)(I_H + V^* Y)^{-1}, \qquad Y \in \mathfrak{R}_V.$$

Thus, for $R = f_V^{-1}(Z)$ and $S = f_V^{-1}(X)$, Z, $X \in \mathfrak{R}_V$, we obtain

$$A_R = I_H - R^* R = (I_H + Z^* V)^{-1} P_{V,Z,Z} (I_H + V^* Z)^{-1},$$

$$A_S = I_H - S^* S = (I_H + X^* V)^{-1} P_{V,X,X} (I_H + V^* X)^{-1},$$

$$I_H - R^* S = (I_H + Z^* V)^{-1} P_{V,X,Z} (I_H + V^* X)^{-1}$$

where $P_{V,X,Z}$ is defined by (3.2) and, using analogous considerations as in section 4 where the maps f_V and f_V^{-1} are defined by (8.1) and (8.2), respectively, we get

$$T(R, S) = \|A_R^{-1/2}(I_H - R^* S) A_S^{-1}(I_H - S^* R) A_R^{-1/2}\| =$$

$$= \|W_Z^{-1}(P_{V,Z,Z})^{-1/2} P_{V,X,Z}(P_{V,X,X})^{-1} P_{V,Z,X}(P_{V,Z,Z})^{-1/2}(W_Z^{-1})^*\|$$

where W_Z is a unitary operator of the form

$$W_Z = (P_{V,Z,Z})^{-1/2}(I_H + Z^* V)\{(I_H + Z^* V)^{-1} P_{V,Z,Z}(I_H + V^* Z)^{-1}\}^{1/2}.$$

Since $(W_Z^{-1})^* W_Z^{-1} = I_H$ we obtain $T(R, S) = \|(P_{V,Z,Z})^{-1/2} P_{V,X,Z}(P_{V,X,X})^{-1/2}\|^2$. This yields the desired requirement (3.3).

9. Proofs of Propositions 3.1 and 3.2

For f_V^{-1} defined by (8.2) and for $\alpha > 1$, let

(9.1) $$D_\alpha(U, V) = \{X \in \mathfrak{B}: \|I_H - U^* f_V^{-1}(X)\| < (\alpha/2)(1 - \|f_V^{-1}(X)\|^2)\}.$$

Then

$$I_H - U^* f_V^{-1}(X) = [I_H + U^* V + (V^* - U^*)X](I_H + V^* X)^{-1}$$

and, using the spectrum σ, we show that

$$(1 - \|f_V^{-1}(X)\|^2)^{-1} = \|(I_H + V^* X)(P_{V,X,X})^{-1}(I_H + X^* V)\|.$$

Thus, (9.1) and (3.6) are identical.

Now, if $X \in D_\alpha(U, V)$ and $\|f_V^{-1}(X)\| \to 1$, then, by (9.1), $\|I_H - U^* f_V^{-1}(X)\| \to 0$, which implies that $S = f_V^{-1}(X) \to U$ since U is an isometry. Thus (3.9) implies (3.10). The converse is obvious.

Moreover, if we define a holomorphic map $\mathcal{H}_{U,V}: \mathfrak{R}_V \to \mathcal{L}(H, H)$ by the formula

(9.2) $$\mathcal{H}_{U,V}(X) = [I_H + U^* f_V^{-1}(X)][I_H - U^* f_V^{-1}(X)]^{-1}, \qquad X \in \mathfrak{R}_V,$$

where f_V^{-1} is defined by (8.2), then

(9.3) $$\operatorname{Re} \mathcal{H}_{U,V}(X) =$$

$$= [I_H - f_V^{-1}(X)^* U]^{-1}[I_H - f_V^{-1}(X)^* U U^* f_V^{-1}(X)][I_H - U^* f_V^{-1}(X)]^{-1},$$

$X \in \mathfrak{N}_V$, the operator $\mathcal{N}_{U,V}(X)$ is invertible, *i.e.* $[\mathcal{N}_{U,V}(X)]^{-1}$ exists, $\mathcal{N}_{U,V}(X) \in \mathfrak{p}$ for all $X \in \mathfrak{N}_V$,

(9.4) $$D([\mathcal{N}_{U,V}(X)]^{-1})(P) = \\ = -2[I_H + U^* f_V^{-1}(X)]^{-1} U^* Df_V^{-1}(X)(P)[I_H + U^* f_V^{-1}(X)]^{-1},$$

(9.5) $$D(\mathcal{N}_{U,V}(X))(P) = 2[I_H - U^* f_V^{-1}(X)]^{-1} U^* Df_V^{-1}(X)(P)[I_H - U^* f_V^{-1}(X)]^{-1}$$

and

(9.6) $$Df_V^{-1}(X)(P) = B_V(I_K + XV^*)^{-1}P(I_H + V^*X)^{-1} \text{ for } X \in \mathfrak{N}_V \text{ and } P \in \mathfrak{B}.$$

Formulae (9.2)-(9.6) imply (3.11)-(3.14) and, in particular, (3.15)-(3.18).

10. Proof of Theorem 3.2

Applying (9.2)-(9.4), (8.1), (8.2), the notations $R = f_V^{-1}(Z)$ and $S = f_V^{-1}(X)$, $Z, X \in \mathfrak{N}_V$, conditions (3.19), (3.20) and (3.25) and using analogous argumentation as in section 7, we prove (3.21)-(3.24).

11. Examples

1. Let $\mathfrak{B} \subset \mathcal{L}(H, K)$ be a J^*-algebra containing an isometry U; let

$$\mathfrak{M}_U = \{X \in \mathfrak{B}: 2\,\mathrm{Re}\, U^*X - X^*(I_K - UU^*)X > 0\}$$

and let $F = f_U \circ f \circ f_U^{-1}: \mathfrak{M}_U \to \mathfrak{M}_U$ be a map holomorphic in $\mathfrak{M}_U$, where $f(X) = (X + U)/2$, $X \in \mathfrak{B}_0$, and f_U and f_U^{-1} are defined by (4.3) and (4.4), respectively. Then, for $A = 2I_H$, we have (when $X \in \mathfrak{M}_U$)

$$\mathrm{Re}\,\mathcal{N}_{U,U}(X) = [I_H - f_U^{-1}(X)^* U]^{-1}[I_H - f_U^{-1}(X)^* UU^* f_U^{-1}(X)][I_H - U^* f_U^{-1}(X)]^{-1},$$

$$A^{-1/2}[\mathrm{Re}\,(\mathcal{N}_{U,U} \circ F)(X)]A^{-1/2} = [I_H - f_U^{-1}(X)^* U]^{-1}\{I_H - f_U^{-1}(X)^* UU^* f_U^{-1}(X) + \\ + (1/2)[I_H - f_U^{-1}(X)^* U][I_H - U^* f_U^{-1}(X)]\}[I_H - U^* f_U^{-1}(X)]^{-1}$$

and, consequently,

$$[\mathrm{Re}\,\mathcal{N}_{U,U}(X)]^{-1/2}A^{-1/2}[\mathrm{Re}\,(\mathcal{N}_{U,U} \circ F)(X)]A^{-1/2}[\mathrm{Re}\,\mathcal{N}_{U,U}(X)]^{-1/2} - I_H = \\ = (1/2)[\mathrm{Re}\,\mathcal{N}_{U,U}(X)]^{-1} = \\ = -(1/2)\{I_H - [I_H - U^* f_U^{-1}(X)]^{-1} - [I_H - f_U^{-1}(X)^* U]^{-1}\}^{-1}.$$

Thus (2.24)-(2.27) holds in all angular sets $D_\alpha(U, U)$, $\alpha > 1$, defined by (2.18), for $A = 2I_H$ and when $f_U^{-1}(X) \to U$, $X \in D_\alpha(U, U)$.

2. Let $\mathfrak{B} \subset \mathcal{L}(H, K)$ be a J^*-algebra containing an isometry U; let $a \in \Delta \setminus \{0\}$ be arbitrary and fixed, let $U_1 = a|a|^{-1}U$ and $U_2 = -a|a|^{-1}U$ and let $F_i = f_{U_i} \circ T_{aU} \circ f_{U_i}^{-1}$ be biholomorphic maps of $\mathfrak{M}_{U_i}$ onto $\mathfrak{M}_{U_i}$ where (see (4.1), (4.2) and [48, Theo-

rem 2.1(c), p. 203])

$$\mathfrak{M}_{U_i} = \{X \in \mathfrak{B}: 2 \operatorname{Re} U_i^* X - X^* (I_K - U_i U_i^*) X > 0\},$$

$$T_{aU}(X) = (I_K - |a|^2 UU^*)^{-1/2}(X - aU)(I_H - \bar{a}U^* X)^{-1}(1 - |a|^2)^{1/2}, \quad X \in \mathfrak{B}_0,$$

and f_{U_i} and $f_{U_i}^{-1}$ are defined by (4.3) and (4.4) for $i = 1, 2$, respectively.

Let

$$\boldsymbol{A}_1 = (1 - |a|)(1 + |a|)^{-1} I_H \text{ and } \boldsymbol{A}_2 = (1 + |a|)(1 - |a|)^{-1} I_H.$$

Since, for $X \in \mathfrak{M}_{U_i}$,

$\operatorname{Re} \mathcal{M}_{U_i, U_i}(X) =$

$$= [I_H - f_{U_i}^{-1}(X)^* U_i]^{-1} [I_H - f_{U_i}^{-1}(X)^* U_i U_i^* f_{U_i}^{-1}(X)][I_H - U_i^* f_{U_i}^{-1}(X)]^{-1},$$

therefore

$$\boldsymbol{A}^{-1/2}[\operatorname{Re}(\mathcal{M}_{U_i, U_i} \circ F_i)(X)]\boldsymbol{A}^{-1/2} = \operatorname{Re} \mathcal{M}_{U_i, U_i}(X).$$

Moreover, for $X \in \mathfrak{B}_0$,

$U_i^* DT_{aU}(X) U_i =$

$$= U_i^*(I_K - |a|^2 UU^*)^{1/2}(I_K - \bar{a}XU^*)^{-1} U_i (I_H - \bar{a}U^* X)^{-1}(1 - |a|^2)^{1/2}.$$

Consequently, F_i satisfies all the assumptions and assertions of Theorem 2.2(b) for $\boldsymbol{A}_i$ in all angular sets $D_\alpha(U_i, U_i)$, $i = 1, 2$, respectively.

3. Let $\mathfrak{B} \subset \mathcal{L}(H, K)$ be a J^*-algebra containing a unitary operator U; let

$$\mathfrak{N}_U = \{X \in \mathfrak{B}: \operatorname{Re} U^* X > 0\}$$

and let $F = f_U \circ f \circ f_U^{-1}$ be a biholomorphic map of $\mathfrak{N}_U$ into $\mathfrak{N}_U$ where f_U and f_U^{-1} are defined by formulae (8.1) and (8.2), respectively, and (see [48, p. 206])

$$f(X) = [aU + (2 - a)X][(2 + a)I_H - aU^* X]^{-1}, \quad X \in \mathfrak{B}_0,$$

$a \in \Pi$ is arbitrary and fixed. Then, by (9.3), we have

$$\operatorname{Re} \mathcal{N}_{U,U}(X) = [I_H - f_U^{-1}(X)^* U]^{-1} [I_H - f_U^{-1}(X)^* f_U^{-1}(X)][I_H - U^* f_U^{-1}(X)]^{-1}$$

and

$$\operatorname{Re}(\mathcal{N}_{U,U} \circ F)(X) = [I_H - f_U^{-1}(X)^* U]^{-1} \{I_H - f_U^{-1}(X)^* UU^* f_U^{-1}(X) +$$
$$+ (\operatorname{Re} a)[f_U^{-1}(X)^* - U^*][f_U^{-1}(X) - U]\} [I_H - U^* f_U^{-1}(X)]^{-1}.$$

Consequently, (3.19)-(3.24) holds in all angular sets $D_\alpha(U, U)$, $\alpha > 1$, defined by (3.15), for $\boldsymbol{A} = I_H$ when $f_U^{-1}(X) \to U$, $X \in D_\alpha(U, U)$.

12. Remarks

1. Let $K = \boldsymbol{C}$ and $v = 1$, *i.e.* $\mathfrak{M}_1 = \Pi$.

(a) Then, for $u = e^{i\mu} \neq 1$, $\mu \in \boldsymbol{R}$, by (2.9)-(2.11), we obtain

$$D_\alpha(u, 1) = \{x \in \boldsymbol{C}: |1 + \bar{u} + (1 + \bar{u})x| < 2\alpha(\operatorname{Re} x)|1 + x|^{-1}\}.$$

Thus the condition

$$x \in D_\alpha(u, 1) \quad \text{and} \quad (x-1)(x+1)^{-1} \to u$$

implies

$$x \in D_\alpha(u, 1) \quad \text{and} \quad x \to (1+u)(1-u)^{-1} \in (\partial \Pi) \cap (\partial D_\alpha(u, 1)).$$

(*b*) If $u = 1$, then, by (2.18),

$$D_\alpha(1, 1) = \{x \in C\colon |1 - (x-1)(x+1)^{-1}| < (\alpha/2)[1 - |(x-1)(x+1)^{-1}|^2]\}.$$

Thus

$$x \in D_\alpha(1, 1) \text{ and } (x-1)(x+1)^{-1} \to 1 \quad \text{implies } x \in D_\alpha(1, 1) \text{ and } |x| \to \infty .$$

2. The following relations between $D_\alpha(1, 1)$ and Σ_k hold:

(*a*) If $x \in D_\alpha(1, 1)$, then $x \in \Sigma_k$ for $k = (\alpha^2 - 1)^{1/2}$. Indeed, then $|1 + x| < \alpha(\mathrm{Re}\, x)$ and, consequently,

$$(1 + \mathrm{Re}\, x)^2 + (\mathrm{Im}\, x)^2 < (k^2 + 1)(\mathrm{Re}\, x)^2 < [k^2 + (1 + \mathrm{Re}\, x)^2 (\mathrm{Re}\, x)^{-2}](\mathrm{Re}\, x)^2,$$

which implies that $x \in \Sigma_k$.

(*b*) If $x \in \Sigma_k$ and $\mathrm{Re}\, x > 1$, then $x \in D_\alpha(1, 1)$ for $\alpha = (k^2 + 4)^{1/2}$. Indeed, then $(\mathrm{Im}\, x)^2 < k^2(\mathrm{Re}\, x)^2$ and, consequently,

$$(1 + \mathrm{Re}\, x)^2 + (\mathrm{Im}\, x)^2 < (k^2 + 4)(\mathrm{Re}\, x)^2.$$

3. If $K = C = \mathcal{L}(C, C)$, then sets (2.18) and (3.15) are identical and inequalities (2.5) and (3.3) are identical with the original Pick-Julia inequalities (see *e.g.* [1]).

13. Special case $F\colon \mathfrak{B}_0 \to \mathfrak{B}_0$

We conclude this section with some other consequences of the arguments in sections 4-7 when $V = 0$.

Let $\mathfrak{B} \subset \mathcal{L}(H, K)$ be a J^*-algebra containing an isometry U.

For $\alpha > 1$, let

$$D_\alpha(U) = \{X \in \mathfrak{B}\colon \|I_H - U^* X\| < (\alpha/2)(1 - \|X\|^2)\}. \tag{13.1}$$

Of course, $D_\alpha(U) \subset \mathfrak{B}_0$ for all $\alpha > 1$. When $\alpha \leq 1$, this set is empty. We call $D_\alpha(U)$, $\alpha > 1$, angular sets.

For $Y \in \partial \mathfrak{B}_0$, we define a holomorphic map $\mathcal{M}_Y\colon \mathfrak{B}_0 \to \mathcal{L}(H, H)$ by the formula

$$\mathcal{M}_Y(X) = (I_H + Y^* X)(I_H - Y^* X)^{-1}, \qquad X \in \mathfrak{B}_0. \tag{13.2}$$

Let us observe that

$$\mathrm{Re}\, \mathcal{M}_Y(X) = (I_H - X^* Y)^{-1}(I_H - X^* Y Y^* X)(I_H - Y^* X)^{-1}, \quad X \in \mathfrak{B}_0.$$

Obviously, the operator $\mathcal{M}_Y(X)$ is invertible, *i.e.* $[\mathcal{M}_Y(X)]^{-1}$ exists and $\mathcal{M}_Y(X) \in \mathfrak{p}$ for all $X \in \mathfrak{B}_0$ and $Y \in \partial \mathfrak{B}_0$. Moreover, for $X \in \mathfrak{B}_0$, $P \in \mathfrak{B}$ and $Y \in \partial \mathfrak{B}_0$,

$$D([\mathcal{M}_Y(X)]^{-1})(P) = -2(I_H + Y^* X)^{-1} Y^* P (I_H + Y^* X)^{-1}$$

and

$$D(\mathcal{M}_Y(X))(P) = 2(I_H - Y^*X)^{-1}Y^*P(I_H - Y^*X)^{-1}.$$

Using arguments similar to those given in sections 4-7 and in [50], one obtains the following result.

THEOREM 13.1. *Let $\mathfrak{B} \subset \mathcal{L}(H, K)$ be a J^*-algebra containing an isometry U, let $F: \mathfrak{B}_0 \to \mathfrak{B}_0$ be a map holomorphic in $\mathfrak{B}_0$ and let $W \in \partial\mathfrak{B}_0$.*

(*a*) *Suppose there is a Hermitian operator $A \in \mathcal{L}(H, H)$ satisfying*

$$A^{1/2}[\operatorname{Re}(\mathcal{M}_W \circ F)(X)]A^{1/2} > \operatorname{Re}\mathcal{M}_U(X)$$

for all $X \in \mathfrak{B}_0$. If $D_\beta(U)$, $\beta > 1$, stands for an angular set such that, for any $\varepsilon > 0$, there exists a point $Z \in D_\beta(U)$ for which the inequality

$$\|[\operatorname{Re}\mathcal{M}_U(Z)]^{-1/2}A^{1/2}[\operatorname{Re}(\mathcal{M}_W \circ F)(Z)]A^{1/2}[\operatorname{Re}\mathcal{M}_U(Z)]^{-1/2} - I_H\| < \varepsilon$$

holds, then, for any $\alpha > 1$, we have

$$\lim \|[\mathcal{M}_U(X)]^{1/2}A^{-1/2}[(\mathcal{M}_W \circ F)(X)]^{-1}A^{-1/2}[\mathcal{M}_U(X)]^{1/2} - I_H\| = 0, \tag{13.3}$$

$$\lim \|[\operatorname{Re}\mathcal{M}_U(X)]^{1/2}\cdot A^{-1/2}[\operatorname{Re}(\mathcal{M}_W \circ F)(X)]^{-1}A^{-1/2}[\operatorname{Re}\mathcal{M}_U(X)]^{-1/2} - I_H\| = 0, \tag{13.4}$$

$$\lim \|D\{A^{-1/2}[(\mathcal{M}_W \circ F)(X)]^{-1}A^{-1/2} - [\mathcal{M}_U(X)]^{-1}\}(U)\| = 0 \tag{13.5}$$

and

$$\lim \|D\{(\mathcal{M}_W \circ F)(X) - A^{-1/2}[\mathcal{M}_U(X)]A^{-1/2}\}(U)\| = 0 \tag{13.6}$$

as $X \to U$, $X \in D_\alpha(U)$.

(*b*) *Suppose there is a Hermitian operator $A \in \mathcal{L}(H, H)$ satisfying*

$$A^{1/2}[\operatorname{Re}(\mathcal{M}_W \circ F)(X)]A^{1/2} = \operatorname{Re}\mathcal{M}_U(X)$$

for all $X \in \mathfrak{B}_0$. Then, for any $\alpha > 1$, assertion (13.3)-(13.6) *holds as $X \to U$, $X \in D_\alpha(U)$. Here $\mathcal{M}_Y$ and $D_\alpha(U)$ are defined by* (13.2) *and* (13.1), *respectively.*

REMARK 13.1. Examples which satisfy the assumptions of Theorem 13.1 are given in [50, section 3].

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