The existence of angular derivatives of holomorphic maps of Siegel domains in a generalization of $C^*$-algebras


Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1994_9_5_4_309_0>

**Abstract.** — The aim of this paper is to start a systematic investigation of the existence of angular limits and angular derivatives of holomorphic maps of infinite dimensional Siegel domains in $J^*$-algebras. Since $J^*$-algebras are natural generalizations of $C^*$-algebras, $B^*$-algebras, $JC^*$-algebras, ternary algebras and complex Hilbert spaces, various significant results follow. Examples are given.

**Key words:** Holomorphic maps; Angular limits; Angular derivatives; Infinite dimensional Siegel domains; Generalizations of $C^*$-algebras.


1. **Introduction**

The basic fact of classical complex analysis, of importance to the theory of automorphic functions and hyperbolic geometry, is the biholomorphic equivalence between the open unit disk $\Delta = \{x \in \mathbb{C}: |x| < 1\}$ and the right half-plane $\Pi = \{x \in \mathbb{C}: \text{Re} x > 0\}$ via the Cayley transformation $x \mapsto (1 + x)(1 - x)^{-1}$ (often the upper half-plane is used instead). Recently, Piatetskij-Shapiro has used multivariable «half-planes» in $\mathbb{C}^n$, the so-called tube domains and Siegel domains, in the theory of automorphic maps in several variables [33]. The associated Cayley transformations can be described either Lie theoretically [25] or, somewhat more directly, by using Jordan algebras and triple systems [29].

For $k > 0$, let $\Sigma_k = \{x \in \mathbb{C}: |\text{Im} x| < k \text{Re} x\}$.

The following theorem is well known in complex analysis:

**Theorem 1.1.** Let $f$ be a map holomorphic on $\Pi$, such that $f(\Pi) \subset \Pi$. If $a = \inf \{[\text{Re} f(x)]/[\text{Re} x]: x \in \Pi\}$ then, for any $k > 0$, we have

$$\lim_{x \to \infty} [f(x)]/[x] = \lim \frac{\text{Re} f(x)}{\text{Re} x} = \lim Df(x) = a$$

as $x \to \infty$, $x \in \Sigma_k$.

This result and its version when $f: \Delta \to \Delta$, concerning the existence of angular limits and angular derivatives of holomorphic maps, discovered by Carathéodory and developed particularly by Julia, Landau, Nevanlinna, Valiron, Warschawski, Wolff, Eke, Goldberg, Kin, Sarason, Pommerenke and Cowen, are important tools in the study of the boundary behaviour of holomorphic maps in $\mathbb{C}$ (in particular, they are a nice way to de-

scribe approximative Denjoy-Wolff boundary fixed points [2, 7, 8, 10, 41, 46]) and have
drawn interest for a long time. A survey appears in [2-5, 9, 13, 23, 27, 32, 34, 36, 37, 40,
42]. From a number of theoretical points of view, it is desirable to possess analogues of
Carathéodory results in higher dimensions. The case of holomorphic maps of the Eucli-
dean unit balls into themselves in $\mathbb{C}^n$ was studied by Rudin [35] and MacCluer and Sha-
piro [30]. Carathéodory's work was extended by Ky Fan [11] who proved the following
elegant result:

**Theorem 1.2.** Let $H$ denote a complex Hilbert space and let $f$ be an operator-valued ho-
lorphic map on the open half-plane $\Pi$, such that, for each $x \in \Pi$, $f(x)$ is an operator on $H
with \Re f(x) > 0$. Suppose there is a Hermitian operator $A$ on $H$ satisfying
$\frac{\Re f(x)}{\Re x} > A$ for all $x \in \Pi$ and, for any $\varepsilon > 0$, there is $z \in \Pi$ such that
$\|\frac{\Re f(z)}{\Re z} - A\| < \varepsilon$. Then, for any $k > 0$, we have

$$\lim \| f(x) - A \| = \lim \| \frac{\Re f(x)}{\Re x} - A \| = \lim \| Df(x) - A \| = 0$$
as $x \to \infty$, $x \in \Sigma_k$.

However, generally, the above-mentioned settings of investigations, concerning the
existence of angular limits and derivatives of holomorphic maps of Siegel domains,
exclude infinite dimensional situations. One of our goals here is to use the ideas of func-
tional analysis, operator theory and infinite dimensional holomorphy in order to initia-
te those omitting the settings of investigations. For information about infinite dimen-
sional holomorphy, the reader is referred to [8, 12, 31].

The history of the classifying of homogeneous domains begins with E. Cartan's fa-
mous paper [6] in $\mathbb{C}^n$, $n > 1$, and continues with numerous contributions culminating in
the works by L. Harris [14-18]. He introduced $J^*$-algebras and discovered a setting in
which a large number of bounded and unbounded convex homogeneous domains in
various finite and infinite dimensional complex Banach spaces can be studied
simultaneously.

For complex Hilbert spaces $H$ and $K$, let $\mathcal{L}(H, K)$ be the Banach space of all boun-
ded linear operators from $H$ to $K$ with the operator norm. A closed complex linear sub-
space $\mathcal{B}$ of $\mathcal{L}(H, K)$ is a $J^*$-algebra if $XX^* X \in \mathcal{B}$ whenever $X \in \mathcal{B}$.

Let $\mathcal{B} \subset \mathcal{L}(H, K)$ be a $J^*$-algebra. For a partial isometry $V \in \mathcal{B}$, let

$$\mathcal{M}_V = \{ X \in \mathcal{B} : 2 \Re V^* X - X^* (I_K - VV^*) X + I_H - V^* V > 0 \}.$$

If $V \neq 0$, the unbounded convex domains $\mathcal{M}_V$ are identical, by a simple rotation $X \to
\to iX$, with operator Siegel domains [14-18]. If $V = 0$, this set reduces to the open unit ball $\mathcal{B}_0 = \{ X \in \mathcal{B} : \| X \| < 1 \}$. As is known, the open unit balls $\mathcal{B}_0$ are bounded symme-
tric homogeneous domains [14-18]. For an isometry $V \in \mathcal{B}$, let $\mathcal{N}_V = \{ X \in \mathcal{B} : \Re V^* X - X^* (I_K - VV^*) X > 0 \}$.

Finite and infinite dimensional Siegel-Harris domains $\mathcal{M}_V$ and $\mathcal{N}_V$ and Cartan-Harris
bounded symmetric homogeneous domains $\mathcal{B}_0$ were treated of by several authors
(see e.g. [1, 6, 14-22, 24-26, 28, 29, 38, 39]).

The main objective of this paper is to define and characterize in $\mathcal{M}_V$ ($V$-a partial iso-
metry) and in $\mathfrak{M}_V$ (V-an isometry) infinite dimensional angular sets and solve the general problems concerning the existence of angular limits and angular derivatives for holomorphic maps $F: \mathfrak{M}_V \to \mathfrak{M}_V$ and $F: \mathfrak{N}_V \to \mathfrak{N}_V$ in such angular sets, respectively. Since $J^*$-algebras are natural generalizations of $C^*$-algebras, $B^*$-algebras, $JC^*$-algebras, ternary algebras, complex Hilbert spaces and others, therefore, in particular, various important results follow from this fact. Examples are given. The principal tool we use are general results of the Pick-Julia type for Siegel domains in $J^*$-algebras. This paper is a continuation of the studies in [44, 47, 49, 50].

2. MAIN RESULTS FOR $F: \mathfrak{M}_V \to \mathfrak{M}_V$

Before formulating our main results in $\mathfrak{M}_V$ in detail, we briefly review some background material.

For a partial isometry $V \in \mathfrak{B}$, let

\[(2.1) \quad \mathcal{M}_V = \{X \in \mathfrak{B}: \text{Re} V^* X > 0\}\]

where

\[(2.2) \quad \text{Re} V^* X = 2 \text{Re} V^* X - X^* (I_K - V V^*) X + I_H - V^* V.\]

For $X, Z \in \mathfrak{M}_V$, $V$ a non-zero partial isometry in $\mathfrak{B}$, let

\[(2.3) \quad P_{V, X, Z} = V^* X + Z^* V - Z^* (I_H - V V^*) X + I_H - V^* V.\]

It is evident that $P_{V, X, Z} = P_{V^*, Z, X}$ and $P_{V, X, X} = \text{Re} V^* X$.

Further, set

\[(2.4) \quad \mathfrak{P} = \{X \in \mathfrak{L}(H, H): \text{Re} X > 0\}.\]

Our first result of Pick-Julia type is

**Theorem 2.1.** Let $\mathfrak{B} \subset \mathfrak{L}(H, K)$ be a $J^*$-algebra containing a non-zero partial isometry $V$. If $F: \mathfrak{M}_V \to \mathfrak{P}$ is a holomorphic map such that $F(Z) = I_H$ for some $Z \in \mathfrak{M}_V$, then

\[(2.5) \quad \|F(X)\| \leq 4 \cdot \|\text{Re} V^* Z\|^{-1/2} P_{V, X, Z} (\text{Re} V^* X)^{-1/2}\]

for all $X \in \mathfrak{M}_V$. Here $\mathfrak{M}_V$, $\text{Re} V^* X$, $\mathfrak{P}$ and $P_{V, X, Z}$ are defined by (2.1), (2.2), (2.4) and (2.3), respectively.

If a $J^*$-algebra $\mathfrak{B} \subset \mathfrak{L}(H, K)$ contains an isometry $U$ and a non-zero partial isometry $V$, let, for $X \in \mathfrak{M}_V$,

\[(2.6) \quad A_{U, V}(X) = (I_H + U^* V) A_V^{-1/2} + (V^* - U^*) B_V^{-1/2} X\]

and

\[(2.7) \quad B_{U, V}(X) = (I_H - U^* V) A_V^{-1/2} + (V^* + U^*) B_V^{-1/2} X,\]

where

\[(2.8) \quad A_V = I_H + V^* V \quad \text{and} \quad B_V = I_K + V V^*.\]

Moreover, for $\alpha > 1$, let

\[(2.9) \quad D_{\alpha}(U, V) = \{X \in \mathfrak{B}: C(U, V; X) < (\alpha/2) \cdot C(V; X)\}\]
where

\begin{equation}
C(U, V; X) = \| [A_{U,V}(X)](I_H + V^* X)^{-1} A_{U,V}^{1/2} \|
\end{equation}

and

\begin{equation}
C(V; X) = \| A_{V}^{-1/2} (I_H + V^* X)(R^* V^* X)^{-1} (I_H + X^* V) A_{V}^{-1/2} \|^{-1}.
\end{equation}

We require an easy fact.

**Proposition 2.1.** If \( \alpha > 1 \), then \( D_{\alpha}(U, V) \subset \mathcal{M}_V \). Moreover, if \( X \in D_{\alpha}(U, V) \), then

\begin{equation}
\| B_{V}^{-1/2} (X - V)(I_H + V^* X)^{-1} A_{V}^{1/2} \| \to 1 
\end{equation}

or, equivalently, \( C(V; X) \to 0 \) if and only if

\begin{equation}
(X - V)(I_H + V^* X)^{-1} \to B_{V}^{1/2} U A_{V}^{-1/2}.
\end{equation}

If \( \alpha \leq 1 \), then \( D_{\alpha}(U, V) = \emptyset \).

For \( \alpha > 1 \), we call the sets \( D_{\alpha}(U, V) \subset \mathcal{M}_V \) angular sets determined by \( U \).

For \( X \in \mathcal{M}_V \), let

\begin{equation}
\mathcal{M}_{U,V}(X) = [B_{U,V}(X)] [A_{U,V}(X)]^{-1}.
\end{equation}

The next proposition will be most useful.

**Proposition 2.2.** For all \( X \in \mathcal{M}_V \), the operator \( \mathcal{M}_{U,V}(X) \) is invertible, i.e. \( [\mathcal{M}_{U,V}(X)]^{-1} \) exists, \( \mathcal{M}_{U,V}(X) \in \mathcal{D} \) for all \( X \in \mathcal{M}_V \) and

\begin{equation}
\text{Re} \mathcal{M}_{U,V}(X) = [A_{U,V}(X)]^{-1} \left\{ X^* B_{V}^{-1/2} (VV^* - UU^*) B_{V}^{-1/2} X + 2 \text{Re} X^* B_{V}^{-1/2} (V + UU^* V) A_{V}^{-1/2} + A_{V}^{-1/2} (I_H - V^* UU^* V) A_{V}^{-1/2} \right\} [A_{U,V}(X)]^{-1}.
\end{equation}

Moreover, for \( X \in \mathcal{M}_V \) and \( P \in \mathcal{B} \),

\begin{equation}
D([\mathcal{M}_{U,V}(X)]^{-1})(P) = -2A_{V}^{-1/2} (I_H + V^* X) [B_{U,V}(X)]^{-1} \cdot U^* B_{V}^{1/2} [I_K - X(I_H + V^* X)^{-1} V^*] P [B_{U,V}(X)]^{-1}
\end{equation}

and

\begin{equation}
D(\mathcal{M}_{U,V}(X))(P) = 2A_{V}^{-1/2} (I_H + V^* X) [A_{U,V}(X)]^{-1} \cdot U^* B_{V}^{1/2} [I_K - X(I_H + V^* X)^{-1} V^*] P [A_{U,V}(X)]^{-1}.
\end{equation}

Let us observe that, in particular, from (2.6)-(2.11) and (2.14)-(2.17) we get

\begin{equation}
D_{\alpha}(U, U) = \{ X \in \mathcal{B} : \| (I_H + U^* X)^{-1} \| < (\alpha/2) \| (I_H + U^* X)(R^* U^* X)^{-1} (I_H + X^* U) \|^{-1} \}
\end{equation}

for \( \alpha > 1 \), and that, for \( X \in \mathcal{M}_U \) and \( P \in \mathcal{B} \),

\begin{equation}
\mathcal{M}_{U,U}(X) = U^* X, \quad \text{Re} \mathcal{M}_{U,U} = \text{Re} U^* X,
\end{equation}

\begin{equation}
D([\mathcal{M}_{U,U}(X)]^{-1})(P) = - (I_H + U^* X)(U^* X)^{-1} (I_H + U^* X)^{-1} U^* P (U^* X)^{-1} = -(U^* X)^{-1} U^* P (U^* X)^{-1}
\end{equation}
and

\[(2.21) \quad D(\mathfrak{M}_U, U(X))(P) = U^* P\]

since \(U^* B_1^{1/2} = A_U^{1/2} U^*\) and \(U^* (I_K + XU^*)^{-1} = (I_H + U^* X)^{-1} U^*\). Moreover, (2.13) may be replaced by \((X - U)(I_H + U^* X)^{-1} \to U.\)

We are now able to formulate our main result.

**Theorem 2.2.** Let \(\mathcal{B} \subset \mathcal{L}(H, K)\) be a \(\dagger\)-algebra containing an isometry \(U\) and a non-zero partial isometry \(V\), and let \(F: \mathfrak{M}_V \to \mathfrak{M}_V\) be a map holomorphic in \(\mathfrak{M}_V.\)

(a) Suppose there is a Hermitian operator \(A \in \mathcal{L}(H, H)\) satisfying

\[(2.22) \quad \text{Re} (\mathfrak{M}_U, V \circ F)(X) > A^{1/2} [\text{Re} \mathfrak{M}_U, V(X)] A^{1/2}\]

for all \(X \in \mathfrak{M}_V.\) If \(D_\beta(U, V), \beta > 1,\) stands for an angular set such that, for any \(\varepsilon > 0,\) there exists a point \(Z \in D_\beta(U, V)\) for which the inequality

\[(2.23) \quad \| [\text{Re} \mathfrak{M}_U, V(Z)]^{-1/2} \cdot A^{-1/2} [\text{Re} \mathfrak{M}_U, V[F](Z)] A^{-1/2} [\text{Re} \mathfrak{M}_U, V(Z)]^{-1/2} - I_H \| < \varepsilon\]

holds, then, for any \(\alpha > 1,\) we have

\[(2.24) \quad \lim \| [\mathfrak{M}_U, V(X)]^{-1/2} \cdot A^{-1/2} [\mathfrak{M}_U, V[F](X)] A^{-1/2} [\mathfrak{M}_U, V(X)]^{-1/2} - I_H \| = 0,\]

\[(2.25) \quad \lim \| [\text{Re} \mathfrak{M}_U, V(X)]^{-1/2} \cdot A^{-1/2} [\text{Re} \mathfrak{M}_U, V[F](X)] A^{-1/2} [\text{Re} \mathfrak{M}_U, V(X)]^{-1/2} - I_H \| = 0,\]

\[(2.26) \quad \lim \| D\{ [\mathfrak{M}_U, V[F](X)] A^{1/2} [\mathfrak{M}_U, V(X)] A^{1/2}\}(U) \| = 0\]

and

\[(2.27) \quad \lim \| D\{ A^{1/2} [\mathfrak{M}_U, V[F](X)]^{-1} A^{1/2} - [\mathfrak{M}_U, V(X)]^{-1}\}(U) \| = 0\]

as \(C(V; X) \to 0, X \in D_\alpha(U, V).\)

(b) Suppose there is a Hermitian operator \(A \in \mathcal{L}(H, H)\) satisfying

\[(2.28) \quad \text{Re} (\mathfrak{M}_U, V \circ F)(X) = A^{1/2} [\text{Re} \mathfrak{M}_U, V(X)] A^{1/2}\]

for all \(X \in \mathfrak{M}_V.\) Then, for any \(\alpha > 1,\) assertion (2.24)-(2.27) holds as \(C(V; X) \to 0, X \in D_\alpha(U, V).\) Here \(\mathfrak{M}_V, \mathfrak{M}_U, V, D_\alpha(U, V)\) and \(C(V; X)\) are defined by (2.1), (2.14), (2.9) and (2.11), respectively.

3. Main results for \(F: \mathfrak{M}_V \to \mathfrak{M}_V\)

The statement of the results for \(\mathfrak{M}_V\) requires some definitions.

For an isometry \(V \in \mathcal{B},\) let

\[(3.1) \quad \mathfrak{M}_V = \{ X \in \mathcal{B}; \text{Re} V^* X - X^* (I_K - VV^*) X > 0 \}.\]
For $X, Z \in \mathcal{R}_V$, let
\[(3.2) \quad P_{V,X,Z} = 2V^*X + 2Z^*V - Z^*(I_K - VV^*)X.\]
It is evident that $P_{V,X,Z} = P_{V,X,Z}$. We can show

**Theorem 3.1.** Let $\mathcal{B} \subset \mathcal{L}(H, K)$ be a $J^*$-algebra containing an isometry $V$. If $F: \mathcal{R}_V \to \mathcal{P}$ is a holomorphic map such that $F(Z) = I_H$ for some $Z \in \mathcal{R}_V$, then
\[(3.3) \quad \|F(X)\| \leq 4 \cdot \|(P_{V,Z,Z})^{-1/2}P_{V,X,Z}(P_{V,X,X})^{-1/2}\|^2\]
for all $X \in \mathcal{R}_V$. Here $\mathcal{R}_V$, $\mathcal{P}$ and $P_{V,X,Z}$ are defined by (3.1), (2.4) and (3.2), respectively.

If a $J^*$-algebra $\mathcal{B} \subset \mathcal{L}(H, K)$ contains isometries $U$ and $V$, let, for $X \in \mathcal{R}_V$,
\[(3.4) \quad A_{U,V}(X) = I_H + U^*V + (V^* - U^*)X\]
and
\[(3.5) \quad B_{U,V}(X) = I_H - U^*V + (V^* + U^*)X.\]
Moreover, for $\alpha > 1$, let
\[(3.6) \quad D_\alpha(U, V) = \{X \in \mathcal{B}: C(U, V; X) < (\alpha/2) \cdot C(V; X)\}\]
where
\[(3.7) \quad C(U, V; X) = \|[A_{U,V}(X)](I_H + V^*X)^{-1}\|\]
and
\[(3.8) \quad C(V; X) = \|(I_H + V^*X)(P_{V,X,X})^{-1}(I_H + X^*V)^{-1}\|.\]

We shall need the following

**Proposition 3.1.** If $\alpha > 1$, then $D_\alpha(U, V) \subset \mathcal{R}_V$. Moreover, if $X \in D_\alpha(U, V)$,
then
\[(3.9) \quad \|(X - V)(I_H + V^*X)^{-1}\| \to 1 \quad \text{or, equivalently,} \quad C(V; X) \to 0\]
if and only if
\[(3.10) \quad (X - V)(I_H + V^*X)^{-1} \to U.\]
If $\alpha \leq 1$, then $D_\alpha(U, V) = \emptyset$.

For $\alpha > 1$, we call the sets $D_\alpha(U, V) \subset \mathcal{R}_V$ angular sets determined by $U$.

For $X \in \mathcal{R}_V$, let
\[(3.11) \quad \mathcal{R}_{U,V}(X) = [B_{U,V}(X)][A_{U,V}(X)]^{-1}.\]
We then have

**Proposition 3.2.** For all $X \in \mathcal{R}_V$, the operator $\mathcal{R}_{U,V}(X)$ is invertible, i.e. $[\mathcal{R}_{U,V}(X)]^{-1}$ exists, $\mathcal{R}_{U,V}(X) \in \mathcal{P}$ for all $X \in \mathcal{R}_V$ and
\[(3.12) \quad \text{Re} \mathcal{R}_{U,V}(X) = [A_{U,V}(X)^*]^{-1}\{X^*(VV^* - UU^*)X +
+ 2 \text{Re}X^*(V + UU^*V) + I_H - V^*UU^*V\}[A_{U,V}(X)]^{-1}.\]
Moreover, for $X \in \mathfrak{N}_V$ and $P \in \mathfrak{B}$,

\begin{equation}
D([\mathfrak{R}_{U,V}(X)]^{-1})(P) = -2(I_H + V^* X)[B_{U,V}(X)]^{-1} U^* B_V [I_K - X(I_H + V^* X)^{-1} V^* ] P[B_{U,V}(X)]^{-1}
\end{equation}

and

\begin{equation}
D(\mathfrak{R}_{U,V}(X))(P) = 2(I_H + V^* X)[A_{U,V}(X)]^{-1} U^* B_V [I_K - X(I_H + V^* X)^{-1} V^* ] P[A_{U,V}(X)]^{-1}.
\end{equation}

Let us observe that, in particular, from (3.4)-(3.8) and (3.11)-(3.14) we get

\begin{equation}
D_\alpha (U, U) = \{X \in \mathfrak{B} : 2\| (I_H + U^* X)^{-1} \| < \\
< (\alpha / 2)\| (I_H + U^* X)(P_{U,X,X})^{-1} (I_H + X^* U)\|^{-1} \}
\end{equation}

for $\alpha > 1$, and that, for $X \in \mathfrak{N}_U$ and $P \in \mathfrak{B}$,

\begin{equation}
\mathfrak{R}_{U,U}(X) = U^* X, \quad \text{Re} \mathfrak{R}_{U,U} = \text{Re} U^* X,
\end{equation}

\begin{equation}
D([\mathfrak{R}_{U,U}(X)]^{-1})(P) = -(U^* X)^{-1} U^* P(U^* X)^{-1}
\end{equation}

and

\begin{equation}
D(\mathfrak{R}_{U,U}(X))(P) = U^* P.
\end{equation}

The basic result of this section is

**Theorem 3.2.** Let $\mathfrak{B} \subset \mathfrak{L}(H, K)$ be a $\ast$-algebra containing isometries $U$ and $V$, and let $F: \mathfrak{N}_V \to \mathfrak{N}_V$ be a map holomorphic in $\mathfrak{N}_V$.

(a) Suppose there is a Hermitian operator $A \in \mathfrak{L}(H, H)$ satisfying

\begin{equation}
\text{Re} (\mathfrak{R}_{U,V} \circ F)(X) > A^{1/2} [\text{Re} \mathfrak{R}_{U,V}(X)] A^{1/2}
\end{equation}

for all $X \in \mathfrak{N}_V$. If $D_{\beta}(U, V), \beta > 1$, stands for an angular set such that, for any $\varepsilon > 0$, there exists a point $Z \in D_{\beta}(U, V)$ for which the inequality

\begin{equation}
\| [\text{Re} \mathfrak{R}_{U,V}(Z)]^{-1/2} \cdot A^{-1/2} [\text{Re} (\mathfrak{R}_{U,V} \circ F)(Z)] A^{-1/2} [\text{Re} \mathfrak{R}_{U,V}(Z)]^{-1/2} - I_H \| < \varepsilon
\end{equation}

holds, then, for any $\alpha > 1$, we have

\begin{equation}
\lim \|[\mathfrak{R}_{U,V}(X)]^{-1/2} A^{-1/2} [(\mathfrak{R}_{U,V} \circ F)(X)] A^{-1/2} [\mathfrak{R}_{U,V}(X)]^{-1/2} - I_H \| = 0,
\end{equation}

\begin{equation}
\lim \|[\text{Re} \mathfrak{R}_{U,V}(X)]^{-1/2} \cdot A^{-1/2} [\text{Re} (\mathfrak{R}_{U,V} \circ F)(X)] A^{-1/2} [\text{Re} \mathfrak{R}_{U,V}(X)]^{-1/2} - I_H \| = 0,
\end{equation}

\begin{equation}
\lim \|D\{ (\mathfrak{R}_{U,V} \circ F)(X) - A^{1/2} [\mathfrak{R}_{U,V}(X)] A^{1/2}\}(U)\| = 0
\end{equation}

and

\begin{equation}
\lim \|D\{ A^{1/2} [(\mathfrak{R}_{U,V} \circ F)(X)]^{-1} A^{1/2} - [\mathfrak{R}_{U,V}(X)]^{-1}\}(U)\| = 0
\end{equation}

as $C(V; X) \to 0$, $X \in D_{\alpha}(U, V)$.\]
(b) Suppose there is a Hermitian operator \( A \in \mathcal{L}(H, H) \) satisfying

\[
(3.25) \quad \text{Re}(\mathcal{R}_{U,V} \circ F)(X) = A^{1/2} [\text{Re} \mathcal{R}_{U,V}(X)] A^{1/2}
\]

for all \( X \in \mathcal{N}_V \). Then, for any \( \alpha > 1 \), assertion (3.21)-(3.24) holds as \( C(V; X) \to 0 \), \( X \in D_\alpha(U, V) \). Here \( \mathcal{N}_V \), \( \mathcal{R}_{U,V}(X) \), \( D_\alpha(U, V) \) and \( C(V; X) \) are defined by (3.1), (3.11), (3.6) and (3.8), respectively.

4. PROOF OF THEOREM 2.1

Let \( T_Y: \mathcal{B}_0 \to \mathcal{B}_0 \), \( Y \in \mathcal{B}_0 \), denote the Möbius biholomorphic map of the form (see [15, Theorem 2, p. 20])

\[
(4.1) \quad T_Y(X) = B_Y^{-1/2} (X - Y)(I_H - Y^* Y)^{-1/2} A_Y^{1/2}, \quad X \in \mathcal{B}_0,
\]

where

\[
(4.2) \quad A_Y = I_H - Y^* Y \quad \text{and} \quad B_Y = I_K - YY^* \quad \text{for} \quad Y \in \mathcal{B}_0.
\]

The biholomorphic map \( f_Y \) of \( \mathcal{B}_0 \) onto \( \mathcal{N}_V \) is defined by the formula (see [43, Theorem 5, p. 499])

\[
(4.3) \quad f_Y(X) = B_Y^{-1/2} (X + V)(I_H - V^* V)^{-1/2} A_Y^{1/2}, \quad X \in \mathcal{B}_0,
\]

and, moreover,

\[
(4.4) \quad f_Y^{-1}(Y) = B_Y^{-1/2} (Y - V)(I_H + V^* V)^{-1/2} A_Y^{1/2}, \quad Y \in \mathcal{N}_V,
\]

where \( A_Y = I_H + V^* V \) and \( B_Y = I_K + VV^* \).

Let \( \mathcal{D}_0 = \{ X \in \mathcal{L}(H, H): \|X\| < 1 \} \) and let \( f: \mathcal{D}_0 \to \mathcal{P} \) be a Cayley biholomorphic map of the form \( f(X) = (I_H + X)(I_H - X)^{-1} \), \( X \in \mathcal{D}_0 \).

Let \( R = f_Y^{-1}(Z) \). Using the Schwarz lemma to the holomorphic map \( g: \mathcal{B}_0 \to \mathcal{D}_0 \), \( g(0) = 0 \), defined by the formula \( g(Y) = (f^{-1} \circ f_Y \circ T_Y^{-1})(Y) \), \( Y \in \mathcal{B}_0 \), we obtain \( \|g(Y)\| \leq \|Y\| \), \( Y \in \mathcal{B}_0 \). In particular, for \( Y = (T_Y \circ f_Y^{-1})(X) \), \( X \in \mathcal{N}_V \), we get \( \|(f^{-1} \circ F)(X)\| \leq \|(T_Y \circ f_Y^{-1})(X)\| \), \( X \in \mathcal{N}_V \). Thus

\[
\{(f^{-1} \circ F)(X)\} \subseteq \{(f^{-1} \circ F)(X)\} \leq \gamma^2 I_H \quad \text{where} \quad \gamma = \|(T_Y \circ f_Y^{-1})(X)\|,
\]

and, in particular,

\[
\|F(X) F(X) - (1 + \gamma)(1 - \gamma)^{-1} F(X) F(X) - (1 + \gamma)(1 - \gamma)^{-1} F(X) + I_H \| \leq 0
\]

or, equivalently,

\[
\|F(X) - (1 + \gamma)(1 - \gamma)^{-1} I_H \| \leq 2\gamma^{1/2} (1 - \gamma)^{-1}
\]

because

\[
(f^{-1} \circ F)(X) = [F(X) - I_H][I_H + F(X)]^{-1}.
\]

In consequence, we have \( \|F(X)\| \leq (1 + \gamma)(1 - \gamma)^{-1} \). Hence it follows that \( \|F(X)\| \leq 4(1 - \|T_R(S)\|^2)^{-1} \), \( X \in \mathcal{N}_V \), \( S = f_Y^{-1}(X) \). This, together with the identity [45, formula (18), p. 247] \( (1 - \|T_R(S)\|^2)^{-1} = T(R, S) \) where

\[
(4.5) \quad T(R, S) = \|A_R^{-1/2} (I_H - R^* S) A_S^{-1} (I_H - S^* R) A_R^{-1/2}\|,
\]
is equivalent to

\[(4.6) \quad \|F(X)\| \leq 4 \cdot T(R, S), \quad S = f_V^{-1}(X), \quad R = f_V^{-1}(Z), \quad X \in \mathcal{M}_V.\]

Moreover, by (4.2) and (4.4),

\[
A_R = A_V^{1/2}(I_H + Z^* V)^{-1}(R E V^* Z)(I_H + V^* Z)^{-1}A_V^{1/2},
\]
\[
A_S = A_V^{1/2}(I_H + X^* V)^{-1}(R E V^* X)(I_H + V^* X)^{-1}A_V^{1/2},
\]
\[
I_H - R^* S = A_V^{1/2}(I_H + Z^* V)^{-1}P_{V, X, Z}(I_H + V^* X)^{-1}A_V^{1/2}.
\]

Consequently,

\[(4.7) \quad T(R, S) =
\]
\[
= \|W_Z^{-1}(R E V^* Z)^{-1/2}P_{V, X, Z}(R E V^* X)^{-1}P_{V, Z, X}(R E V^* Z)^{-1/2}(W_Z^{-1})*\|
\]
where \(W_Z\) is a unitary operator of the form

\[
W_Z = (R E V^* Z)^{-1/2}(I_H + Z^* V)A_V^{-1/2}.
\]

which, by (4.6) and (4.7), yields (2.5).

---

### 5. Proof of Proposition 2.1

For \(\alpha > 1\), we let

\[(5.1) \quad D_{\alpha}(U, V) = \{X \in \mathcal{B} : \|I_H - U^* f_V^{-1}(X)\| < (\alpha/2)(1 - \|f_V^{-1}(X)\|^2)\} \].

Of course, \(D_{\alpha}(U, V) \subset \mathcal{M}_V\) for all \(\alpha > 1\). When \(\alpha \leq 1\), this set is empty.

Now, let us observe that

\[(5.2) \quad I_H - U^* f_V^{-1}(X) =
\]
\[
= [(I_H + U^* V)A_V^{-1/2} + (V^* - U^*)B_V^{-1/2}X](I_H + V^* X)^{-1}A_V^{1/2};
\]

using the spectrum \(\sigma\), we show that

\[(5.3) \quad (1 - \|f_V^{-1}(X)\|^2)^{-1} = \sup \sigma \{[I_H - f_V^{-1}(X)^* f_V^{-1}(X)]^{-1}\} =
\]
\[
= \|A_V^{-1/2}(I_H + V^* X)(R E V^* X)^{-1}(I_H + X^* V)A_V^{-1/2}\|.\]

Consequently, (5.1) and (2.9) are identical.

Now, note that if \(X \in D_{\alpha}(U, V)\) and \(\|f_V^{-1}(X)\| \to 1\) or, by (5.3), equivalently \(C(V; X) \to 0\), then, by (5.1), \(C(U, V; X) \to 0\) or, equivalently, \(\|I_H - U^* f_V^{-1}(X)\| \to 0\), which implies that \(S = f_V^{-1}(X) \to U\) since \(U\) is an isometry. Consequently, (2.12) implies (2.13).
6. Proof of Proposition 2.2

We define a holomorphic map \( \mathcal{M}_{U, V}: \mathcal{M}_{V} \to \mathcal{L}(H, H) \) by the formula

\[
\mathcal{M}_{U, V}(X) = [I_H + U* f^{-1}_V(X)][I_H - U* f^{-1}_V(X)]^{-1}, \quad X \in \mathcal{M}_{V}.
\]

Let us observe that

\[
\text{Re} \mathcal{M}_{U, V}(X) = [I_H - f^{-1}_V(X)* U]^{-1} [I_H - f^{-1}_V(X) U U* f^{-1}_V(X)] [I_H - U* f^{-1}_V(X)]^{-1}, \quad X \in \mathcal{M}_{V}.
\]

Obviously, the operator \( \mathcal{M}_{U, V}(X) \) is invertible, i.e. \( [\mathcal{M}_{U, V}(X)]^{-1} \) exists and \( \mathcal{M}_{U, V}(X) \in \mathcal{P} \) for all \( X \in \mathcal{M}_{V} \).

Using (4.4) and (5.2), we get that (6.1) and (6.2) are identical with (2.14) and (2.15), respectively.

Finally, let us notice that

\[
D(\mathcal{M}_{U, V}(X))^{-1}(P) = -2[I_H + U* f^{-1}_V(X)]^{-1} U* D f^{-1}_V(X)(P)[I_H - U* f^{-1}_V(X)]^{-1},
\]

and, by [43, p. 510], we get

\[
D f^{-1}_V(X)(P) = B^{1/2}(I_K + XV*)^{-1} P(I_H + V* X)^{-1} A^{1/2}, \quad X \in \mathcal{M}_{V}, \quad P \in \mathcal{B}.
\]

Consequently, using (4.4) and the fact that \( (I_K + XV*)^{-1} = I_K - X(I_H + V* X)^{-1} V* \), we obtain (2.16) and (2.17).

7. Proof of Theorem 2.2

(a) Let \( \varepsilon > 0 \) be arbitrary and fixed. By (2.23), there exists \( Z \in D_\beta(U, V) \) such that

\[
||[\text{Re} \mathcal{M}_{U, V}(Z)]^{-1/2} A^{-1/2} [\text{Re} (\mathcal{M}_{U, V} \circ F)(Z)] A^{-1/2} [\text{Re} \mathcal{M}_{U, V}(Z)]^{-1/2} - I_H || < \varepsilon.
\]

We define maps \( E \) and \( G \), holomorphic in \( \mathcal{M}_{V} \), by the formulae

\[
E(X) = A^{-1/2} [((\mathcal{M}_{U, V} \circ F)(X)) A^{-1/2} - \mathcal{M}_{U, V}(X)]
\]

and

\[
G(X) = [\text{Re} E(Z)]^{-1/2} [E(X) - i \cdot \text{Im} E(Z)][\text{Re} E(Z)]^{-1/2},
\]

respectively. Let us observe that, by (2.22), \( \text{Re} E(X) > 0 \) and \( \text{Re} G(X) > 0 \) for all \( X \in \mathcal{M}_{V} \), and \( G(Z) = I_H \). Applying Theorem 2.1 to the map \( G \), we get, by (4.5) and (4.6),

\[
\|[G(X)]\| \leq 4 \cdot T(R, S) = 4 \|A^{-1/2}_R (I_H - R * S) A^{-1} (I_H - S * R) A^{-1/2}_R \|, \quad X \in \mathcal{M}_{V},
\]

where \( R = f^{-1}_V(Z) \), \( S = f^{-1}_V(X) \). Now, from (7.1) we obtain

\[
\|\mathcal{M}_{U, V}(X)\|^{-1/2} [E(X)] [\mathcal{M}_{U, V}(X)]^{-1/2} = \|\mathcal{M}_{U, V}(X)\|^{-1/2} A^{-1/2} [((\mathcal{M}_{U, V} \circ F)(X)) A^{-1/2} - \mathcal{M}_{U, V}(X)]^{-1/2} - I_H \| \leq \varepsilon.
\]
THE EXISTENCE OF ANGULAR DERIVATIVES OF HOLOMORPHIC MAPS ... 319

\[ \leq \left\| \mathfrak{M}_{U,V}(X)^{-1} \right\| \left\| \Re E(Z)^{1/2} G(X) [\Re E(Z)]^{1/2} + i \cdot \Im E(Z) \right\| \leq \]

\[ \leq \left\| \mathfrak{M}_{U,V}(X)^{-1} \right\| \left\{ \left\| \Re E(Z) \right\| \cdot \left\| G(X) \right\| + \left\| \Im E(Z) \right\| \right\} \leq \]

\[ \leq \left\| \mathfrak{M}_{U,V}(X)^{-1} \right\| \left\| \Re \mathfrak{M}_{U,V}(Z)^{-1/2} \left[ \Re E(Z) \right] \left[ \Re \mathfrak{M}_{U,V}(Z)^{-1/2} \right] \cdot \left\| \Re \mathfrak{M}_{U,V}(Z) \right\| \left\| G(X) \right\| + \left\| \mathfrak{M}_{U,V}(X)^{-1} \right\| \left\| \Im E(Z) \right\| . \]

Let \( \alpha > 1 \) be arbitrary and fixed. Since

\[ \left\| \mathfrak{M}_{U,V}(X)^{-1} \right\| \leq \left\| I_H - U^* S \right\| \left( I_H + U^* S \right)^{-1} \right\| , \]

by (6.2),

\[ \left\| \Re \mathfrak{M}_{U,V}(X) \right\| \leq \left\| (I_H - R^* U)^{-1} (I_H - R^* U U^* R) (I_H - U^* R)^{-1} \right\| \leq \]

\[ \leq \left\| (I_H - R^* U)^{-1} \right\|^2 \left\| I_H - R^* U + R^* U (I_H - U^* R) \right\| \leq \]

\[ \leq (1 - \left\| R \right\|^{-2} \left\| I_H - R^* U \right\| (1 + \left\| R^* U \right\|) < 2 (1 - \left\| R \right\|^{-2} \left\| I_H - R^* U \right\| , \]

\[ \left\| I_H - U^* S \right\| (1 - \left\| S \right\|^2)^{-1} < \alpha / 2 , \quad \left\| I_H - U^* R \right\| (1 - \left\| R \right\|^2)^{-1} < \beta / 2 \]

and, by (4.5) and (7.2),

\[ \left\| G(X) \right\| \leq 4 \cdot T(R, S) \leq 4 \left\| I_H - S^* R \right\|^2 \left( (1 - \left\| S \right\|^2) (1 - \left\| R \right\|^2) \right)^{-1} , \]

it follows that

\[ (7.3) \quad \left\| \mathfrak{M}_{U,V}(X) \right\|^{-1/2} \left( E(X) \right) \left[ \mathfrak{M}_{U,V}(X) \right]^{-1/2} \leq 2 \varepsilon \left\| \left( I_H + U^* S \right)^{-1} \right\| ; \]

\[ \cdot \left\| I_H - S^* R \right\|^2 \left( 1 - \left\| R \right\|^2 \right)^{-1} + \left\| I_H - U^* S \right\| \left( I_H + U^* S \right)^{-1} \left\| \Im E(Z) \right\| . \]

Consequently, since the right-hand side of inequality (7.3), by (2.12) and (2.13) (or see section 5), tends to

\[ \varepsilon \beta \left\| I_H - U^* R \right\|^2 (1 - \left\| R \right\|^2)^{-1} < \]

\[ < \varepsilon \beta \left\| I_H - U^* R \right\|^2 (1 - \left\| R \right\|^2)^{-2} (1 - \left\| R \right\|^2) < (1/2) \varepsilon \beta^3 \]

and \( \varepsilon > 0 \) can be arbitrarily small, therefore

\[ \lim \left\{ \left\| \mathfrak{M}_{U,V}(X) \right\|^{-1/2} A^{-1/2} \left( [\mathfrak{M}_{U,V} \circ F](X) \right) A^{-1/2} \left[ \mathfrak{M}_{U,V}(X) \right]^{-1/2} - I_H \right\} = 0 \]

as \( C(V; X) \to 0, X \in D_a(U, V) \), i.e. (2.24) holds.

Now, let us observe that

\[ \left\| \Re \mathfrak{M}_{U,V}(X) \right\|^{-1/2} \left[ \Re E(X) \right] \left[ \Re \mathfrak{M}_{U,V}(X) \right]^{-1/2} = \]

\[ = \left\| \Re \mathfrak{M}_{U,V}(X) \right\|^{-1/2} \left( \Re \mathfrak{M}_{U,V} \circ F)(X) \right) \left( \Re \mathfrak{M}_{U,V}(X) \right)^{-1/2} - I_H \right\| \leq \]

\[ \leq \left\| E(X) \right\| \left\| \left[ \Re \mathfrak{M}_{U,V}(X) \right]^{-1} \right\| \leq \]

\[ \leq \left\| \mathfrak{M}_{U,V}(X) \right\|^{-1/2} \left( E(X) \right) \left[ \mathfrak{M}_{U,V}(X) \right]^{-1/2} \left\| \mathfrak{M}_{U,V}(X) \right\| \left[ \Re \mathfrak{M}_{U,V}(X) \right]^{-1} \right\| . \]

But

\[ \left\| \mathfrak{M}_{U,V}(X) \right\| \leq \left\| I_H + U^* S \right\| (1 - \left\| S \right\|)^{-1} < 2 (1 - \left\| S \right\|)^{-1} \]

and

\[ \left\| \Re \mathfrak{M}_{U,V}(X) \right\|^{-1} \leq \left\| I_H - U^* S \right\|^2 (1 - \left\| S \right\|^2)^{-1} < (x^2 / 4)(1 - \left\| S \right\|^2) . \]
Thus
\[ \| [\text{Re} \mathcal{M}_U, \nu(X)]^{-1/2} [\text{Re} E(X)] [\text{Re} \mathcal{M}_U, \nu(X)]^{-1/2} \| < \alpha^2 \| [\mathcal{M}_U, \nu(X)]^{-1/2} [E(X)] [\mathcal{M}_U, \nu(X)]^{-1/2} \|. \]

Since, by (2.24), the right-hand side of the above inequality tends to zero, we have
\[ \lim \| [\text{Re} \mathcal{M}_U, \nu(X)]^{-1/2} A^{-1/2} [\text{Re} (\mathcal{M}_U, \nu \circ F)(X)] A^{-1/2} [\text{Re} \mathcal{M}_U, \nu(X)]^{-1/2} - I_H \| = 0 \]
as \( C(V; X) \to 0, X \in D_a(U, V) \), i.e. (2.25) holds.

Let \( 1 < \gamma < \alpha \). We shall need the following relation between \( D_a(U, V) \) and \( D_\gamma(U, V) \). Assume that
\[ (7.4) \quad 1 < \gamma < \alpha \quad \text{and} \quad \delta = (1/3)(1/\gamma - 1/\alpha) \]
and
\[ (7.5) \quad X \in D_\gamma(U, V), \quad \text{i.e.} \quad \| I_H - U* f_V^{-1}(X) \| < (\gamma/2)(1 - \| f_V^{-1}(X) \|^2) . \]
If
\[ (7.6) \quad |\lambda| \leq \delta \| I_H - U* f_V^{-1}(X) \| , \]
then
\[ (7.7) \quad f_V[f_V^{-1}(X) + \lambda U] \in D_a(U, V) , \]
i.e. \( \| I_H - U* f_V^{-1}(X) \| < (\alpha/2)(1 - \| f_V^{-1}(X) + \lambda U \|^2) . \)

Indeed, from (7.4) we have
\[ (7.8) \quad |\lambda|^2 < |\lambda| , \quad 2/\alpha < 2 , \quad (5\delta + 2/\alpha) < 2/\gamma \]
whenever \( |\lambda| \) is sufficiently small. From (7.5) we get
\[ (7.9) \quad \| f_V^{-1}(X) \|^2 + (2/\gamma) \| I_H - U* f_V^{-1}(X) \| < 1 . \]

Thus, using (7.8), (7.6) and (7.9), we obtain
\[ \| f_V^{-1}(X) + \lambda U \|^2 + (2/\alpha) \| I_H - U* f_V^{-1}(X) + \lambda U \| \leq \| f_V^{-1}(X) \|^2 + 3 |\lambda| + (2/\alpha) \| I_H - U* f_V^{-1}(X) \| < 1 . \]

This immediately yields (7.7).

Now, we prove (2.26) and (2.27). By Proposition 2.2 and the Cauchy integral formula [31, Proposition 2, p. 21],
\[ (7.10) \quad D \{ A^{1/2} [((\mathcal{M}_U, \nu \circ F)(X))]^{-1} A^{1/2} \mathcal{M}_U, \nu(X) - \mathcal{M}_U, \nu \circ F)(X) \} \]
\[ = D \{ A^{1/2} [((\mathcal{M}_U, \nu \circ F)(X))]^{-1} A^{1/2} - [\mathcal{M}_U, \nu(X)]^{-1} \} \]
\[ = \frac{1}{2\pi i} \int_{|\lambda| = r} [((\mathcal{M}_U, \nu \circ f_V)(S + \lambda U))]^{-1/2} \{ [((\mathcal{M}_U, \nu \circ f_V)(S + \lambda U))]^{1/2} A^{1/2} . \]
\[ • [(\mathcal{M}_{U, V} \circ f_V)(S + \lambda U)]^{-1} A^{1/2} \left( [(\mathcal{M}_{U, V} \circ f_V)(S + \lambda U)]^{1/2} - I_H \right) \cdot \]

\[ • [(\mathcal{M}_{U, V} \circ f_V)(S + \lambda U)]^{-1/2} \frac{\lambda^{-2} d\lambda}{2} \]

where \( S = f_V^{-1}(X) \), \( \lambda = r \cdot e^{it} \), \( r = r(X) = \frac{e}{||H_U - U^* S||} t \in [0; 2\pi] \) and \( f_V \) is defined by (4.3). But

\[ \left\| (\mathcal{M}_{U, V} \circ f_V)(S + \lambda U) \right\|^{-1} \left\| \lambda \right\|^{-1} \leq \left\| I_H - U^* S \right\| + |\lambda| \cdot \]

\[ \left\| I_H + U^* (S + \lambda U) \right\|^{-1} \left\| \lambda \right\|^{-1} = (\varepsilon^{-1} + 1) \left\| I_H + U^* (S + \lambda U) \right\|^{-1} \].

Since the right-hand side of the above inequality tends to \( 2^{-1}(\varepsilon^{-1} + 1) \), from (7.10) we get (2.26) and (2.27) by using (2.24), (7.4)-(7.7), (6.3)-(6.5), (2.16) and (2.17).

(b) If (2.28) holds for all \( X \in \mathcal{M}_V \), let \( \varepsilon > 0 \) be arbitrary and fixed and let \( \eta \) be such that \( 0 < \eta < \varepsilon \). Then

\[ A^{-1/2} \left[ \text{Re} \left( \mathcal{M}_{U, V} \circ f \right)(X) \right] A^{-1/2} + \eta \left[ \text{Re} \mathcal{M}_{U, V}(X) \right] > \text{Re} \mathcal{M}_{U, V}(X) \]

for all \( X \in \mathcal{M}_V \). Moreover, obviously, then there exists some \( Z \in D_\beta(U, V) \) for which the inequality

\[ \left\| \left[ \text{Re} \mathcal{M}_{U, V}(Z) \right]^{-1/2} A^{-1/2} \left[ \text{Re} \left( \mathcal{M}_{U, V} \circ f \right)(Z) \right] A^{-1/2} \left[ \text{Re} \mathcal{M}_{U, V}(Z) \right]^{-1/2} - I_H + \eta I_H \right\| = \eta < \varepsilon \]

holds. Now, we define maps \( E_\eta \) and \( G_\eta \), holomorphic in \( \mathcal{M}_V \), by the formulae

\[ E_\eta(X) = \left( X + V \right) \left( I_H - V^* X \right) \]

and

\[ G_\eta(X) = \left[ \text{Re} E_\eta(Z) \right]^{-1/2} \left[ E_\eta(X) - i \cdot \text{Im} E_\eta(Z) \right] \left[ \text{Re} E_\eta(Z) \right]^{-1/2} \]

respectively. Let us note that \( \text{Re} E_\eta(X) > 0 \) and \( \text{Re} G_\eta(X) > 0 \) for all \( X \in \mathcal{M}_V \), and that \( G_\eta(Z) = I_H \). Using analogous considerations as in part (a), we have, respectively, for \( R = f_V^{-1}(Z) \) and \( S = f_V^{-1}(X) \),

\[ \left\| \mathcal{M}_{U, V}(X) \right\|^{-1/2} E(X) \left[ \mathcal{M}_{U, V}(X) \right]^{-1/2} \]

\[ = \left\| \mathcal{M}_{U, V}(X) \right\|^{-1/2} A^{-1/2} \left[ \mathcal{M}_{U, V} \circ f \right](X) A^{-1/2} \left[ \mathcal{M}_{U, V}(X) \right]^{-1/2} - I_H \right\| \leq \]

\[ \leq \left\| \mathcal{M}_{U, V}(X) \right\|^{-1} \left\{ \left| \text{Re} E_\eta(Z) \right| \left\| G_\eta(X) \right\| + \left\| \text{Im} E_\eta(Z) \right\| \} + \eta \cdot \]

Thus, for any \( \alpha > 1 \), using analogous arguments as in part (a), we obtain

\[ \lim \left\| \mathcal{M}_{U, V}(X) \right\|^{-1/2} \left[ E(X) \right] \left[ \mathcal{M}_{U, V}(X) \right]^{-1/2} \leq (1/2) \varepsilon \alpha \beta^3 + \eta \]

as \( C(V; X) \to 0, X \in D_\alpha(U, V) \). This implies (2.24). Using arguments similar to those given in part (a) we prove that also (2.25), (2.26) and (2.27) hold as \( C(V; X) \to 0 \) in all angular sets \( D_\alpha(U, V), \alpha > 1 \).

8. Proof of Theorem 3.1

The biholomorphic map \( f_V \) of \( \mathcal{B}_0 \) onto \( \mathcal{M}_V \) is defined by the formula

\[ f_V(X) = \left( X + V \right) (I_H - V^* X)^{-1}, \quad X \in \mathcal{B}_0, \]
and, moreover,

\begin{equation}
(f_V^{-1}(Y) = (Y - V)(I_H + V^* Y)^{-1}, \quad Y \in \mathcal{R}_V.
\end{equation}

Thus, for \( R = f_V^{-1}(Z) \) and \( S = f_V^{-1}(X) \), \( z, X \in \mathcal{R}_V \), we obtain

\[
A_R = I_H - R^* R = (I_H + Z^* V)^{-1} P_{V,Z,Z}(I_H + V^* Z)^{-1},
\]

\[
A_S = I_H - S^* S = (I_H + X^* V)^{-1} P_{V,X,X}(I_H + V^* X)^{-1},
\]

\[
I_H - R^* S = (I_H + Z^* V)^{-1} P_{V,X,Z}(I_H + V^* X)^{-1}
\]

where \( P_{V,X,Z} \) is defined by (3.2) and, using analogous considerations as in section 4 where the maps \( f_V \) and \( f_V^{-1} \) are defined by (8.1) and (8.2), respectively, we get

\[
T(R, S) = \| A_R^{-1/2} (I_H - R^* S) A_S^{-1} (I_H - S^* R) A_R^{-1/2} \| = \| W_Z^{-1} (P_{V,Z,Z})^{-1/2} P_{V,X,Z} (P_{V,X,X})^{-1} P_{V,Z,Z} (P_{V,Z,Z})^{-1/2} (W_Z^{-1})^* \|
\]

where \( W_Z \) is a unitary operator of the form

\[
W_Z = (P_{V,Z,Z})^{-1/2} (I_H + Z^* V) \{ (I_H + Z^* V)^{-1} P_{V,Z,Z} (I_H + V^* Z)^{-1} \}^{1/2}.
\]

Since \( (W_Z^{-1})^* W_Z^{-1} = I_H \) we obtain

\[
T(R, S) = \| (P_{V,Z,Z})^{-1/2} P_{V,X,Z} (P_{V,X,X})^{-1/2} \|^2.
\]

This yields the desired requirement (3.3).

9. Proofs of Propositions 3.1 and 3.2

For \( f_V^{-1} \) defined by (8.2) and for \( \alpha > 1 \), let

\begin{equation}
D_\alpha(U, V) = \{ X \in \mathcal{B} : \| I_H - U^* f_V^{-1}(X) \| < (\alpha/2)(1 - \| f_V^{-1}(X) \|^2) \}.
\end{equation}

Then

\[
I_H - U^* f_V^{-1}(X) = [I_H + U^* V + (V^* - U^*) X](I_H + V^* X)^{-1}
\]

and, using the spectrum \( \sigma \), we show that

\[
(1 - \| f_V^{-1}(X) \|^2)^{-1} = \| (I_H + V^* X)(P_{V,X,X})^{-1}(I_H + X^* V) \|.
\]

Thus, (9.1) and (3.6) are identical.

Now, if \( X \in D_\alpha(U, V) \) and \( \| f_V^{-1}(X) \| \to 1 \), then, by (9.1), \( \| I_H - U^* f_V^{-1}(X) \| \to 0 \),

which implies that \( S = f_V^{-1}(X) \to U \) since \( U \) is an isometry. Thus (3.9) implies (3.10). The converse is obvious.

Moreover, if we define a holomorphic map \( \mathcal{R}_{U,V} : \mathcal{R}_V \to \mathcal{L}(H, H) \) by the formula

\begin{equation}
\mathcal{R}_{U,V}(X) = [I_H + U^* f_V^{-1}(X)][I_H - U^* f_V^{-1}(X)]^{-1}, \quad X \in \mathcal{R}_V,
\end{equation}

where \( f_V^{-1} \) is defined by (8.2), then

\begin{equation}
\text{Re} \mathcal{R}_{U,V}(X) = [I_H - f_V^{-1}(X)^* U]^{-1} [I_H - f_V^{-1}(X)^* UU^* f_V^{-1}(X)][I_H - U^* f_V^{-1}(X)]^{-1},
\end{equation}

where \( f_V^{-1} \) is defined by (8.2).
X \in \mathcal{H}_V$, the operator $\mathcal{H}_{U,V}(X)$ is invertible, i.e. $[\mathcal{H}_{U,V}(X)]^{-1}$ exists, $\mathcal{H}_{U,V}(X) \in \mathfrak{p}$ for all $X \in \mathcal{H}_V$.

\begin{equation}
D([\mathcal{H}_{U,V}(X)]^{-1})(P) = -2[I_H + U^* f_U^{-1}(X)]^{-1} U^* Df_U^{-1}(X)(P)[I_H + U^* f_U^{-1}(X)]^{-1},
\end{equation}

\begin{equation}
D(\mathcal{H}_{U,V}(X))(P) = 2[I_H - U^* f_U^{-1}(X)]^{-1} U^* Df_U^{-1}(X)(P)[I_H - U^* f_U^{-1}(X)]^{-1}
\end{equation}
and
\begin{equation}
Df_U^{-1}(X)(P) = B_V(I_K + XV^*)^{-1} P(I_H + V^* X)^{-1}
\end{equation}
for $X \in \mathcal{H}_V$ and $P \in \mathcal{B}$.

Formulae (9.2)-(9.6) imply (3.11)-(3.14) and, in particular, (3.15)-(3.18).

10. **Proof of Theorem 3.2**

Applying (9.2)-(9.4), (8.1), (8.2), the notations $R = f_U^{-1}(Z)$ and $S = f_V^{-1}(X)$, $Z, X \in \mathcal{H}_V$, conditions (3.19), (3.20) and (3.25) and using analogous argumentation as in section 7, we prove (3.21)-(3.24).

11. **Examples**

1. Let $\mathcal{B} \subset \mathfrak{L}(H, K)$ be a $J^*$-algebra containing an isometry $U$; let

$$\mathfrak{M}_U = \{ X \in \mathcal{B}: 2 \text{ Re } U^* X - X^* (I_K - UU^*) X > 0 \}$$

and let $F = f_U \circ f_U^{-1}: \mathfrak{M}_U \to \mathfrak{M}_U$ be a map holomorphic in $\mathfrak{M}_U$, where $f(X) = (X + U)/2$, $X \in \mathfrak{B}_0$, and $f_U$ and $f_U^{-1}$ are defined by (4.3) and (4.4), respectively. Then, for $A = 2I_H$, we have (when $X \in \mathfrak{M}_U$)

$$\text{Re } \mathfrak{M}_{U,U}(X) = [I_H - f_U^{-1}(X)^* U]^{-1} [I_H - f_U^{-1}(X)^* UU^* f_U^{-1}(X)] [I_H - U^* f_U^{-1}(X)]^{-1},$$

\begin{equation}
A^{-1/2} [\text{Re } (\mathfrak{M}_{U,U} \circ F)(X)] A^{-1/2} = [I_H - f_U^{-1}(X)^* U]^{-1} [I_H - f_U^{-1}(X)^* UU^* f_U^{-1}(X)] + (1/2)[I_H - f_U^{-1}(X)^* U][I_H - U^* f_U^{-1}(X)]^{-1} [I_H - U^* f_U^{-1}(X)]^{-1}
\end{equation}

and, consequently,

$$[\text{Re } \mathfrak{M}_{U,U}(X)]^{-1/2} A^{-1/2} [\text{Re } (\mathfrak{M}_{U,U} \circ F)(X)] A^{-1/2} [\text{Re } \mathfrak{M}_{U,U}(X)]^{-1/2} - I_H =$$

\begin{equation}
= (1/2) [\text{Re } \mathfrak{M}_{U,U}(X)]^{-1} =
\end{equation}

\begin{equation}
= -(1/2) \{ I_H - [I_H - U^* f_U^{-1}(X)]^{-1} - [I_H - f_U^{-1}(X)^* U]^{-1} \}^{-1}.
\end{equation}

Thus (2.24)-(2.27) holds in all angular sets $D_a(U, U), \alpha > 1$, defined by (2.18), for $A = 2I_H$ and when $f_U^{-1}(X) \to U$, $X \in D_a(U, U)$.

2. Let $\mathcal{B} \subset \mathfrak{L}(H, K)$ be a $J^*$-algebra containing an isometry $U$; let $a \in \Delta \setminus \{0\}$ be arbitrary and fixed, let $U_1 = a |a|^{-1} U$ and $U_2 = -a |a|^{-1} U$ and let $F_i = f_{U_i} \circ T_{U_i} \circ f_U^{-1}$ be biholomorphic maps of $\mathfrak{M}_{U_i}$ onto $\mathfrak{M}_{U_i}$, where (see (4.1), (4.2) and [48, Theo-
rem 2.1(c), p. 203]

\[ \mathfrak{M}_{U_i} = \{ X \in \mathfrak{B} : 2 \Re U_i^* X - X^* (I_K - U_i U_i^*) X > 0 \} , \]

\[ T_{aU}(X) = (I_K - |a|^2 UU^*)^{-1/2} (X - aU)(I_H - \bar{a} U^* X)^{-1} (1 - |a|^2)^{1/2}, \quad X \in \mathfrak{B}_0 , \]

and \( f_{U_i} \) and \( f_{U_i}^{-1} \) are defined by (4.3) and (4.4) for \( i = 1, 2 \), respectively.

Let

\[ A_1 = (1 - |a|)(1 + |a|)^{-1} I_H \quad \text{and} \quad A_2 = (1 + |a|)(1 - |a|)^{-1} I_H . \]

Since, for \( X \in \mathfrak{M}_{U_i} \),

\[ \Re \mathfrak{M}_{U_i, U_i}(X) = \]

\[ = [I_H - f_{U_i}^{-1}(X)* U_i]^{-1} [I_H - f_{U_i}^{-1}(X)* U_i U_i^* f_{U_i}^{-1}(X)][I_H - U_i^* f_{U_i}^{-1}(X)]^{-1} , \]

therefore

\[ A^{-1/2} [\Re (\mathfrak{M}_{U_i, U_i} \circ F_i)(X)] A^{-1/2} = \Re \mathfrak{M}_{U_i, U_i}(X) . \]

Moreover, for \( X \in \mathfrak{B}_0 , \)

\[ U_i^* DT_{aU}(X) U_i = \]

\[ = U_i^* (I_K - |a|^2 UU^*)^{1/2} (I_K - \bar{a} XU^*)^{-1} U_i (I_H - \bar{a} U^* X)^{-1} (1 - |a|^2)^{1/2} . \]

Consequently, \( F_i \) satisfies all the assumptions and assertions of Theorem 2.2(b) for \( A_i \) in all angular sets \( D_a(\U_i, \U_i) \), \( a > 1 \), defined by (3.15), for \( a = I_H \) when \( a = U_i \).

3. Let \( \mathfrak{B} \subset \mathcal{L}(H, K) \) be a \( J^* \)-algebra containing a unitary operator \( U \); let

\[ \mathfrak{R}_U = \{ X \in \mathfrak{B} : \Re U^* X > 0 \} \]

and let \( F = f_U \circ f_U^{-1} \) be a biholomorphic map of \( \mathfrak{R}_U \) into \( \mathfrak{M}_U \) where \( f_U \) and \( f_U^{-1} \) are defined by formulae (8.1) and (8.2), respectively, and (see [48, p. 206])

\[ f(X) = [aU + (2 - a)x][2 + a] I_H - a U^* X]^{-1} , \quad X \in \mathfrak{B}_0 , \]

\( a \in \mathbb{N} \) is arbitrary and fixed. Then, by (9.3), we have

\[ \Re \mathfrak{R}_{U_i, U_i}(X) = [I_H - f_{U_i}^{-1}(X)* U_i]^{-1} [I_H - f_{U_i}^{-1}(X)* U_i U_i^* f_{U_i}^{-1}(X)][I_H - U_i^* f_{U_i}^{-1}(X)]^{-1} \]

and

\[ \Re (\mathfrak{R}_{U_i, U_i} \circ F)(X) = [I_H - f_{U_i}^{-1}(X)* U_i]^{-1} \{ I_H - f_{U_i}^{-1}(X)* U_i U_i^* f_{U_i}^{-1}(X) + \]

\[ + (\Re a)[f_{U_i}^{-1}(X)* - U^*] [f_{U_i}^{-1}(X) - U] [I_H - U_i^* f_{U_i}^{-1}(X)]^{-1} . \]

Consequently, (3.19)-(3.24) holds in all angular sets \( D_a(U, U) \), \( a > 1 \), defined by (3.15), for \( A = I_H \) when \( f_{U_i}^{-1}(X) \to U \), \( X \in D_a(U, U) \).

12. REMARKS

1. Let \( K = \mathbb{C} \) and \( \nu = 1 \), i.e. \( \mathfrak{M}_1 = \mathbb{N} \).

(a) Then, for \( u = e^{i\mu} \neq 1, \mu \in \mathbb{R} \), by (2.9)-(2.11), we obtain

\[ D_a(u, 1) = \{ x \in \mathbb{C} : |1 + \bar{u} + (1 + \bar{u})x| < 2\alpha(\Re x)|1 + x|^{-1} \} . \]
Thus the condition

\[ x \in D_\alpha(u, 1) \quad \text{and} \quad (x - 1)(x + 1)^{-1} \to u \]

implies

\[ x \in D_\alpha(u, 1) \quad \text{and} \quad x \to (1 + u)(1 - u)^{-1} \in (\partial II) \cap (\partial D_\alpha(u, 1)). \]

(b) If \( u = 1 \), then, by (2.18),

\[ D_\alpha(1, 1) = \{ x \in \mathbb{C} : |1 - (x - 1)(x + 1)^{-1}| < (\alpha/2)[1 - |(x - 1)(x + 1)^{-1}|^2] \}. \]

Thus

\[ x \in D_\alpha(1, 1) \quad \text{and} \quad (x - 1)(x + 1)^{-1} \to 1 \quad \text{implies} \quad x \in D_\alpha(1, 1) \quad \text{and} \quad |x| \to \infty. \]

2. The following relations between \( D_\alpha(1, 1) \) and \( \Sigma_k \) hold:

(a) If \( x \in D_\alpha(1, 1) \), then \( x \in \Sigma_k \) for \( k = (\alpha^2 - 1)^{1/2} \). Indeed, then \( |1 + x| < \alpha(\text{Re} x) \) and, consequently,

\[ (1 + \text{Re} x)^2 + (\text{Im} x)^2 < (k^2 + 1)(\text{Re} x)^2 < [k^2 + (1 + \text{Re} x)^2(\text{Re} x)^{-2}](\text{Re} x)^2, \]

which implies that \( x \in \Sigma_k \).

(b) If \( x \in \Sigma_k \) and \( \text{Re} x > 1 \), then \( x \in D_\alpha(1, 1) \) for \( \alpha = (k^2 + 4)^{1/2} \). Indeed, then \( (\text{Im} x)^2 < k^2(\text{Re} x)^2 \) and, consequently,

\[ (1 + \text{Re} x)^2 + (\text{Im} x)^2 < (k^2 + 4)(\text{Re} x)^2. \]

3. If \( K = C = \ell(C, C) \), then sets (2.18) and (3.15) are identical and inequalities (2.5) and (3.3) are identical with the original Pick-Julia inequalities (see e.g. [1]).

13. SPECIAL CASE \( F: \mathcal{B}_0 \to \mathcal{B}_0 \)

We conclude this section with some other consequences of the arguments in sections 4-7 when \( V = 0 \).

Let \( \mathcal{B} \subset \ell(H, K) \) be a \( J^* \)-algebra containing an isometry \( U \).

For \( \alpha > 1 \), let

\[ D_\alpha(U) = \{ X \in \mathcal{B} : \|I_H - U^*X\| < (\alpha/2)(1 - \|X\|^2) \}. \]

Of course, \( D_\alpha(U) \subset \mathcal{B}_0 \) for all \( \alpha > 1 \). When \( \alpha \leq 1 \), this set is empty. We call \( D_\alpha(U) \), \( \alpha > 1 \), angular sets.

For \( Y \in \partial \mathcal{B}_0 \), we define a holomorphic map \( \mathcal{M}_Y: \mathcal{B}_0 \to \ell(H, H) \) by the formula

\[ \mathcal{M}_Y(X) = (I_H + Y^*X)(I_H - Y^*X)^{-1}, \quad X \in \mathcal{B}_0. \]

Let us observe that

\[ \text{Re} \mathcal{M}_Y(X) = (I_H - X^*Y)^{-1}(I_H - X^*YY^*X)(I_H - Y^*X)^{-1}, \quad X \in \mathcal{B}_0. \]

Obviously, the operator \( \mathcal{M}_Y(X) \) is invertible, i.e. \( \mathcal{M}_Y(X)^{-1} \) exists and \( \mathcal{M}_Y(X) \in \mathcal{B}_0 \) for all \( X \in \mathcal{B}_0 \) and \( Y \in \partial \mathcal{B}_0 \). Moreover, for \( X \in \mathcal{B}_0 \), \( P \in \mathcal{B} \) and \( Y \in \partial \mathcal{B}_0 \),

\[ D(\mathcal{M}_Y(X)^{-1})(P) = -2(I_H + Y^*X)^{-1}Y^*P(I_H + Y^*X)^{-1} \]
and
\[ D(\mathcal{M}_Y(X))(P) = 2(I_H - Y^*X)^{-1}Y^*P(I_H - Y^*X)^{-1}. \]

Using arguments similar to those given in sections 4-7 and in [50], one obtains the following result.

**Theorem 13.1.** Let \( \mathfrak{B} \subset \mathcal{L}(H, K) \) be a \( J^* \)-algebra containing an isometry \( U \), let \( F: \mathfrak{B}_0 \to \mathfrak{B}_0 \) be a map holomorphic in \( \mathfrak{B}_0 \) and let \( W \in \mathfrak{B}_0 \).

(a) Suppose there is a Hermitian operator \( A \in \mathcal{L}(H, H) \) satisfying
\[ A^{1/2} \left[ \text{Re} \left( \mathcal{M}_W \circ F \right)(X) \right] A^{1/2} > \text{Re} \mathcal{M}_U(X) \]
for all \( X \in \mathfrak{B}_0 \). If \( D_\beta(U), \beta > 1 \), stands for an angular set such that, for any \( \varepsilon > 0 \), there exists a point \( Z \in D_\beta(U) \) for which the inequality
\[ \| \left[ \text{Re} \mathcal{M}_U(Z) \right]^{-1/2} A^{1/2} \left[ \text{Re} \left( \mathcal{M}_W \circ F \right)(Z) \right] A^{1/2} \left[ \text{Re} \mathcal{M}_U(Z) \right]^{-1/2} - I_H \| < \varepsilon \]
holds, then, for any \( \alpha > 1 \), we have
\begin{align*}
(13.3) & \quad \lim \| \left[ \mathcal{M}_U(X) \right]^{-1/2} A^{-1/2} \left[ \left( \mathcal{M}_W \circ F \right)(X) \right]^{-1} A^{-1/2} \left[ \mathcal{M}_U(X) \right]^{-1/2} - I_H \| = 0, \\
(13.4) & \quad \lim \| \left[ \text{Re} \mathcal{M}_U(X) \right]^{-1/2} \cdot A^{-1/2} \left[ \text{Re} \left( \mathcal{M}_W \circ F \right)(X) \right]^{-1} A^{-1/2} \left[ \text{Re} \mathcal{M}_U(X) \right]^{-1/2} - I_H \| = 0,
\end{align*}
(13.5) \quad \lim \| D \left\{ A^{-1/2} \left[ \left( \mathcal{M}_W \circ F \right)(X) \right]^{-1} A^{-1/2} - \left[ \mathcal{M}_U(X) \right]^{-1} \right\}(U) \| = 0
and
(13.6) \quad \lim \| D \left\{ \left( \mathcal{M}_W \circ F \right)(X) - A^{-1/2} \left[ \mathcal{M}_U(X) \right] A^{-1/2} \right\}(U) \| = 0
as \( X \to U, X \in D_\alpha(U) \).

(b) Suppose there is a Hermitian operator \( A \in \mathcal{L}(H, H) \) satisfying
\[ A^{1/2} \left[ \text{Re} \left( \mathcal{M}_W \circ F \right)(X) \right] A^{1/2} = \text{Re} \mathcal{M}_U(X) \]
for all \( X \in \mathfrak{B}_0 \). Then, for any \( \alpha > 1 \), assertion (13.3)-(13.6) holds as \( X \to U, X \in D_\alpha(U) \). Here \( \mathfrak{M}_Y \) and \( D_\alpha(U) \) are defined by (13.2) and (13.1), respectively.

**Remark 13.1.** Examples which satisfy the assumptions of Theorem 13.1 are given in [50, section 3].

**References**


Institute of Mathematics
University of Łódź
Banacha 22 - 90 238 Łódź (Polonia)