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Emmanuele DiBenedetto, Vincenzo Vespri

Continuity for bounded solutions of multiphase Stefan problem

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Equazioni a derivate parziali. — *Continuity for bounded solutions of multiphase Stefan problem.* Nota di EMMANUELE DIBENEDETTO e VINCENZO VESPRI, presentata (*) dal Socio E. Magenes.

ABSTRACT. — We establish the continuity of bounded local solutions of the equation $\beta(u)_t = \Delta u$. Here β is any coercive maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, bounded for bounded values of its argument. The multiphase Stefan problem and the Buckley-Leverett model of two immiscible fluids in a porous medium give rise to such singular equations.

KEY WORDS: Singular parabolic equations; Regularity; Stefan problem; Maximal monotone graphs.

RIASSUNTO. — *Continuità per soluzioni limitate del problema di Stefan multifase.* In questa *Nota* si dimostra la continuità delle soluzioni locali limitate dell'equazione $\beta(u)_t = \Delta u$, dove β è un qualsiasi grafo massimale monotono e coercivo in $\mathbb{R} \times \mathbb{R}$, che si mantiene limitato per valori limitati del suo argomento. A questo contesto appartengono sia il problema di Stefan multifase che il modello di Buckley-Leverett di due fluidi immiscibili in un mezzo poroso.

1. INTRODUCTION

Consider the singular parabolic equation

$$\beta(u)_t - \Delta u = 0 \quad \text{in } \mathcal{D}'(\Omega_T); \qquad u \in L^2_{\text{loc}}(0, T; W^{1,2}_{\text{loc}}(\Omega)). \tag{1}$$

Here Ω is a domain in $\boldsymbol{R}^N$ and $\Omega_T \equiv \Omega \times (0, T)$, $T > 0$. In (1), β is any coercive maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ which remains bounded if its argument remains bounded. A typical example is

$$\beta(s) \equiv \begin{cases} 2 + s & \text{if } s > 1, \\ [1, 2] & \text{if } s = 1, \\ 1 + s & \text{if } s > 0, \\ [0, 1] & \text{if } s = 0, \\ s & \text{if } s < 0. \end{cases} \tag{2}$$

The equation (1), with such a choice of $\beta(\cdot)$ is taken as a model for multiple transitions of phases. More generally $\beta(\cdot)$ might exhibit several jumps (even infinitely many) or, might become «*vertical*» several times at any rate. We only require that β satisfies

$$w_1 - w_2 \geqslant \gamma_0 (u_1 - u_2), \qquad \forall w_i \in \beta(u_i), \qquad i = 1, 2,$$

for some positive constant γ_0. We say that $\beta(\cdot)$ is singular whenever it has a vertical tangent.

In the early '80s several authors investigated the local behaviour of solutions of (1) in the case when $\beta(\cdot)$ exhibits only one «*transition of phase*» (see for example [2, 4, 11, 12]). A summary of these contributions is that weak solutions are continuous, and a modulus of continuity can be computed quantitatively. In

(*) Nella seduta del 16 giugno 1994.

these investigations it is essential that $\beta(\cdot)$ be singular at only one value of its argument.

On the other hand graphs $\beta(\cdot)$ that are singular at multiple points, besides their intrinsic mathematical interest, arise naturally in phenomena of muyltiple transitions of phase. They also arise in the Buckley-Leverett model of two immiscible fluids in a porous medium (see [1, 3, 5, 8]). In such a case $\beta(u)$ is a function of the saturation and exhibits two singularities, say at $u = 0$ and $u = 1$. Near these points, $\beta(\cdot)$ could become vertical exponentially fast or could even exhibit a jump (connate water). For singular equations modelling immiscible fluids, some partial results appear in [5]. It is shown that the saturation is continuous provided that at least one of the two singularities is power-like.

The p.d.e. in (1) is meant weakly, and in the sense of inclusion of graphs.

The main result of this *Note* is that locally bounded weak solutions of (1) are locally continuous in Ω_T and that, in addition, their modulus of continuity can be estimated quantitatively. In what follows we refer to the set of numbers (γ_0, M, N) as the *data*, and for a constant C or γ, or a function $\omega(\cdot)$, we say

$$C \equiv C(\text{data}), \qquad \gamma \equiv \gamma(\text{data}), \qquad \omega(\cdot) \equiv \omega_{\text{data}}(\cdot),$$

if they can be determined a priori only in terms on the indicated quantities.

We assume in addition, that u can be constructed as the limit, in the weak topology of $L^2_{\text{loc}}(0, T; W^{1,2}_{\text{loc}}(\Omega))$ of a sequence of local smooth solutions to (1), for smooth $\beta(\cdot)$. This assumption in not restrictive, in view of the available existence theory (see for example [7, 9, 10]) and it is formulated only to justify some of the calculations. We stress however that our estimates, and the modulus of continuity of u depend only upon the data.

THEOREM. *Let u be a locally bounded weak solution of* (1). *Then u is continuous in Ω_T. Moreover, for every compact subset $\mathcal{K} \subset\subset \Omega_T$, there exists a continuous, non negative, increasing function*

$$s \to \omega_{\text{data}, \mathcal{K}}(s), \qquad \omega_{\text{data}, \mathcal{K}}(0) = 0,$$

that can be determined a priori only in terms of the data and the distance from $\mathcal{K}$ to the parabolic boundary of Ω_T, such that $|u(x_1, t_1) - u(x_2, t_2)| \leq \omega_{\text{data}, \mathcal{K}}(|x_1 - x_2| + |t_1 - t_2|^{1/2})$, for every pair of points $(x_i, t_i) \in \mathcal{K}$, $i = 1, 2$.

2. SKETCH OF THE PROOF

We will present the main points of the proof of the Theorem referring to [6] for detailed arguments. Let $P \in \Omega_T$ be fixed and assume, without loss of generality, that it coincides with the origin. For a positive number ρ, let K_ρ denote a cube of wedge 2ρ about the origin of $\boldsymbol{R}^N$ and denote by $Q_\rho \equiv K_\rho \times (-\rho^2, 0)$ the cylinder of «vertex» at the origin, height ρ^2, and cross section K_ρ. Assume that $Q_\rho \subset \Omega_T$ and set

$$\sup_{Q_\rho} u = \mu^+, \qquad \inf_{Q_\rho} u = \mu^-, \qquad \omega = \mu^+ - \mu^- = \operatorname*{osc}_{Q_\rho} u.$$

Defining

$$\omega_\infty(P) = \lim_{\rho \searrow 0} \sup \left(\sup_{Q_\rho} u - \inf_{Q_\rho} u \right),$$

the solution u of (1) is continuous in P if $\omega_\infty = 0$. Let $\delta \in (0, 1/4)$ be a positive parameter, and assume that there exist a time level $\tilde{t} \in (-\rho^2, -\delta^2\rho^2)$, such that

$$(3)^+ \qquad u(x, \tilde{t}) < \mu^+ - (1/4)\,\omega, \qquad \forall x \in K_{2\delta\rho}.$$

Then there exists a number $\xi \in (0, 1)$ such that

$$\sup_{Q_{\delta\rho}} u(x, t) < \mu^+ - \xi\omega, \quad \text{where} \quad Q_{\delta\rho} \equiv K_{\delta\rho} \times (-\delta^2\rho^2, 0).$$

Adding $\inf_{Q_{\delta\rho}} u$ to the left hand side and $-\mu^-$ to the right hand side of this inequality, gives

$$(4) \qquad \operatorname*{osc}_{Q_{\delta\rho}} u \leqslant (1 - \xi) \operatorname*{osc}_{Q_\rho} u.$$

Thus, going down from Q_ρ to the smaller cylinder $Q_{\delta\rho}$, the oscillation of u decreases by a factor $(1 - \xi)$. Analogously, if for some time level $\tilde{t} \in (-\rho^2, -\delta^2\rho^2)$ there holds

$$(3)^- \qquad u(x, \tilde{t}) > \mu^- + (1/4)\,\omega, \qquad \forall x \in K_{2\delta\rho}.$$

Then there exists a number $\xi \in (0, 1)$ such that $u(x, t) > \mu^- + \xi\omega$, $\forall (x, t) \in Q_{\delta\rho}$. This implies again (4) by a similar calculation. A key feature is that the number ξ depends upon δ but is independent of ω and ρ. Thus, due to the «initial conditions» $(3)^\pm$, solutions of (1) behave like solutions of the heat equation. Roughly speaking, the information $(3)^\pm$ on the status of the system at some given time suffices to control the singularity for all later times. To achieve such information we consider cylinders, coaxial with Q_ρ, «vertex» at $(0, \tilde{t})$, and congruent to $Q_{4\delta\rho}$, *i.e.*, $[(0, \tilde{t}) + Q_{4\delta\rho}] \equiv$ $\equiv K_{4\delta\rho} \times (\tilde{t} - 16\delta^2\rho^2, \tilde{t})$. As the time level $\tilde{t}$ ranges over

$$(5) \qquad [-(1 - 16\delta^2)\rho^2, -16\delta^2\rho^2],$$

the cylinders $[(0, \tilde{t}) + Q_{4\delta\rho}]$ move inside Q_ρ remaining coaxial with it. Suppose that for some $\tilde{t}$ in the range (7), the subset of $[(0, \tilde{t}) + Q_{4\delta\rho}]$ where u is close to μ^+ is small, *i.e.*

$$(6)^+ \qquad \operatorname{meas}\{(x, t) \in [(0, \tilde{t}) + Q_{4\delta\rho}] \,|\, u(x, t) > \mu^+ - (1/2)\,\omega\} \leqslant \nu |Q_{4\delta\rho}|.$$

Then $u(x, t) < \mu^+ - (1/4)\,\omega$, $\forall (x, t) \in [(0, \tilde{t}) + Q_{2\delta\rho}]$. Thus, in particular $(3)^+$ holds. Likewise, if the subset of $[(0, \tilde{t}) + Q_{4\delta\rho}]$ where u is close to μ^- is small, *i.e.*,

$$(6)^- \qquad \operatorname{meas}\{(x, t) \in [(0, \tilde{t}) + Q_{4\delta\rho}] \,|\, u(x, t) < \mu^- + (1/2)\,\omega\} < \nu |Q_{4\delta\rho}|,$$

then $u(x, t) > \mu^- + (1/4)\,\omega$, $\forall (x, t) \in [(0, \tilde{t}) + Q_{2\delta\rho}]$. Thus, in particular $(3)^-$ holds. The number ν can be determined a priori only in terms of the data and ω. The dependence of ν upon ω is due to the singularity of $\beta(\cdot)$.

The unfavorable case is when $(6)^\pm$ are both violated for every $\tilde{t}$ in the range (5). Consider any one of such cylinders and keep in mind that the parameter δ is to be chosen. The key observation here is that if $(6)^\pm$ are both violated for arbitrarily small δ, then near the axis of Q_ρ, at the time level $\tilde{t}$ there is a relatively large set where u is close to μ^+ and another relatively large set where the solution is close to μ^-. Since δ can be taken to be arbitrarily small, these two sets are arbitrarily close to each other. Therefore the space gradient Du must be large on a relatively large set. Since however $|Du| \in L^2(Q_\rho)$, this creates a contradiction. The technical implementation of this idea is in two stages. First we establish that there exist two cylinders $Q^i_{\delta^2\rho}$, $i = 1, 2$ contained in $[(0, \tilde{t}) + Q_{4\delta\rho}]$, such that within $Q^1_{\delta^2\rho}$ the solution u is above $(1/4)\mu^+$, and u is below $(1/4)\mu^-$ within $Q^2_{\delta^2\rho}$. Therefore for every pair of points $(x_i, t_i) \in Q^i_{\delta^2\rho}$, $i = 1, 2$, $(1/4)\omega \leq u(x_1, t_1) - u(x_2, t_2)$. Next, it is possible to locate the two cylinders $Q^i_{\delta^2\rho}$, $i = 1, 2$ near the lateral boundary of $[(0, \tilde{t}) + Q_{\delta\rho}]$ and by means of a contour integration we prove that

$$\delta^N \rho^N \omega \leq \gamma(\omega) \int_{\tilde{t} - \delta^2\rho^2}^{\tilde{t}} \int_{K_{\delta\rho} \setminus K_{\delta^2\rho}} |Du|^2 \, dx \, d\tau . \tag{7}$$

We observe that the number of disjoint cylinders of the type $[(0, \tilde{t}) + Q_{\delta\rho}]$ is of the order of δ^{-2}. Therefore adding (7) over such boxes, yields

$$\delta^{N-2} \rho^N \omega \leq \gamma(\omega) \int_{-\rho^2}^{0} \int_{\delta^2\rho < \|x\| < \delta\rho} |Du|^2 \, dx \, d\tau . \tag{8}$$

Since $\delta \in (0, 1)$ can be chosen to arbitrarily small, we conclude from this, that for all $n = 1, 2, \ldots,$

$$\delta^{n(N-2)} \rho^N \omega \leq \gamma(\omega) \int_{-\rho^2}^{0} \int_{\delta^{n+1}\rho < \|x\| < \delta^n\rho} |Du|^2 \, dx \, d\tau . \tag{8_n}$$

We also observe that for all $N \geq 1$,

$$\iint_{Q_\rho} |Du|^2 \, dx \, d\tau \leq \text{const}\, \rho^N . \tag{9}$$

3. The case $N = 2$

If $N = 2$, we add $(8)_n$ for $n = 1, 2, \ldots, n_0$. Taking into account (9), we obtain $n_0 \leq \leq \text{const}(\omega)$. It follows that at least one of $(3)^\pm$ must hold with δ replaced with δ^{n_0}. This implies the continuity Theorem for $N = 2$. As a consequence of the proof in the 2-dimensional case, the continuity remains valid for *bounded* solutions of general quasilinear versions of (1). Specifically, consider weak solutions of $\beta(u)_t - \operatorname{div} \boldsymbol{a}(x, t, u, Du) = b(x, t, u, Du)$, in Ω_T, where $\boldsymbol{a}\colon \Omega_T \times \boldsymbol{R}^3 \to \boldsymbol{R}^N$ and $b\colon \Omega_T \times \boldsymbol{R}^3 \to \boldsymbol{R}$ are only assumed to be measurable and satisfying the structure conditions

$$\begin{cases} \boldsymbol{a}(x, t, u, Du) \cdot Du \geq C_0 |Du|^2 - \varphi_0(x, t), \\ |\boldsymbol{a}(x, t, u, Du)| \leq C_1 |Du|^2 + \varphi_1(x, t), \\ |b(x, t, u, Du)| \leq C_2 |Du|^2 + \varphi_2(x, t), \end{cases} \tag{10}$$

for a.e. $(x,t)\in\Omega_T$. Here C_i, $i=0,1,2$ are given positive constants and φ_i, $i=0,1,2$, are given non negative functions defined in Ω_T and subject to the conditions

(11) $$\varphi_0,\quad \varphi_1^2,\quad \varphi_2\in L^q(\Omega_T),\quad q\in(1,\infty].$$

The same result holds whenever one has information that essentially reduce the number N of dimensions to 1 or 2. For example the continuity theorem remains valid for *radial* solutions of (1).

4. The case $N \geq 3$

Without loss of generality we may assume that $(4\delta)^{-1}$ is an integer m. We partition the original cube K_ρ, up to a set of measure zero, into m^N pairwise disjoint subcubes of wedge $8\delta\rho$ and denote by x_l their centres. Then we partition Q_ρ, up to a set of measure zero, into $m^N m^2$ pairwise disjoint cylinders. If we denote their «vertices» by (x_l, t_h), these boxes have the form

(12) $$[(x_l,t_h)+Q_{4\delta\rho}],\qquad l=1,2,\dots,m^N,\qquad h=1,2,\dots,m^2.$$

Moreover

$$\overline{Q}_\rho=\bigcup_{h=1}^{(4\delta)^{-2}}\ \bigcup_{l=1}^{(4\delta)^{-N}}[(x_l,t_h)+\overline{Q}_{4\delta\rho}].$$

If neither of $(6)^{\pm}$ holds for any of the boxes $[(x_l,t_h)+Q_{4\delta\rho}]$ then the analog of (7) must hold for each of them, *i.e.*,

$$\delta^N\rho^N\omega\leq\gamma(\omega)\int_{t_h-\delta^2\rho^2}^{t_h}\ \int_{\delta^2\rho<\|x-x_l\|<\delta\rho}|Du|^2\,dx\,d\tau.$$

We add these over $l=1,2,\dots,(4\delta)^{-N}$ and $h=1,2,\dots,(4\delta)^{-2}$ and take into account (9). This gives $\delta^{-2}\leq \text{const}\,(\omega)$. This is a contradiction for sufficiently small δ, depending upon ω. Thus at least one of $(6)^{\pm}$ must be verified for at least one of the boxes in (12). Assume that $(6)^-$ holds. Then

(13) $$u(x,t)\geq\mu^-+(1/4)\,\omega,\qquad\forall(x,t)\in[(x_l,t_h)+Q_{2\delta\rho}].$$

If such a box were coaxial with Q_ρ, *i.e.*, if $x_l\equiv 0$ then the proof could be concluded as before. The main difficulty is in showing that an estimate similar to (13) actually holds in a box coaxial with the original starting cylinder Q_ρ. This technical fact which we call «*sidewise expansion of positivity*», is established by introducing a comparison function v. The function v satisfies a suitably rescaled version of (1) in a cylindrical domain with annular cross section $\{\delta\rho<\|x-x_l\|<4\rho\}\times\{t_h, t_h+k\delta^2\rho^2\}$ for a sufficienty large k. The function v is prescribed to be μ^- on the parabolic boundary of such a domain except for $\|x-x_l\|=\delta\rho$, where we impose

(14) $$v(x,t)=\mu^-+(1/4)\,\omega,\qquad\text{for }\|x-x_l\|=\delta\rho.$$

The «*annular simmetry*» of v roughly speaking corresponds to a lowering of the space dimensions as indicated above. This permits us to establish that indeed v is continuous and bounded below in a sizeable part of its domain of definition. We then establish that $u \geqslant v$ thereby concluding the proof. Such an expansion of positivity is reminiscent of a Harnack-type estimate.

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E. DiBenedetto: Department of Mathematics
Northwestern University
Evanston, IL 60208 (U.S.A.)

V. Vespri: Dipartimento di Matematica
Università degli Studi di Pavia
Via Abbiategrasso, 209 - 27100 Pavia