Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Ugo Gianazza, Massimo Gobbino, Giuseppe Savarè

Evolution Problems and Minimizing Movements

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **5** (1994), n.4, p. 289–296.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1994_9_5_4_289_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1994.

Rend. Mat. Acc. Lincei
s. 9, v. 5:289-296 (1994)

Analisi matematica. — *Evolution Problems and Minimizing Movements.* Nota di
UGO GIANAZZA, MASSIMO GOBBINO e GIUSEPPE SAVARÈ, presentata (*) dal Socio E. De Giorgi.

ABSTRACT. — We recall the definition of Minimizing Movements, suggested by E. De Giorgi, and we consider some applications to evolution problems. With regards to ordinary differential equations, we prove in particular a generalization of maximal slope curves theory to arbitrary metric spaces. On the other hand we present a unifying framework in which some recent conjectures about partial differential equations can be treated and solved. At the end we consider some open problems.

KEY WORDS: Minimizing Movements; Variational evolution problems; Gradient flow; Maximal slope.

RIASSUNTO. — *Problemi di Evoluzione e Movimenti Minimizzanti.* Richiamiamo la definizione di Movimenti Minimizzanti, proposta da E. De Giorgi, e consideriamo alcune applicazioni a problemi d'evoluzione. Nel caso delle equazioni differenziali ordinarie si dimostra, in particolare, una generalizzazione a spazi metrici arbitrari della teoria delle linee di massima pendenza. Per le equazioni a derivate parziali è presentato un quadro teorico astratto in cui considerare e risolvere alcune congetture recentemente suggerite. Infine esaminiamo alcuni problemi aperti.

1. INTRODUCTION

When one considers the usual evolution equation of variational type

$$u'(t) + \nabla f[u(t)] = 0; \qquad u(0) = u_0 \tag{1.1}$$

where f is a regular function defined, for example, on a Hilbert space H, a possible approach to approximate the solution to (1.1) is to choose a time parameter $\lambda > 0$ and find a sequence $\{u_k^\lambda\}_{k \in N}$ which is recursively defined as follows

$$\begin{cases} u_0^\lambda = u_0\,; \\ u_{k+1}^\lambda \text{ realizes the minimum of } f^\lambda(v) = (\lambda/2)\|v - u_k^\lambda\|_H^2 + f(v)\,. \end{cases} \tag{1.2}$$

The values u_k^λ are strictly related to the time discretization of (1.1) by a backward Euler scheme of step $1/\lambda$; as $\lambda \to +\infty$ one must study the convergence in H of the piecewise constant functions $u^\lambda(t) = u_{[\lambda t]}^\lambda$ which assume the value u_k^λ on the interval $[k/\lambda, (k+1)/\lambda[$.

This kind of procedure is considered, for example, in [1] for maximal monotone operators relative to Hilbert spaces and has then been extended by Crandall and others to accretive ones in Banach spaces (see [2, 3]) and in [7-9, 12] to arbitrary metric spaces.

As a wide generalization of these methods and other similar ones, De Giorgi [5, 6] recently suggested the notion of *minimizing movement*, which applies to many problems in the calculus of variations, differential equations, geometric measure theory, etc.

In a word the minimizing movement is defined as the set of pointwise limits of se-

(*) Nella seduta del 16 giugno 1994.

quences of minimizers to an iterated variational problem for a proper functional. As typical examples the steepest descent method, the approximation of parabolic equations and the flow by mean curvature find here a natural framework in which they can be treated under a unifying point of view.

Needless to say, the link with Γ-convergence and penalization methods is clearly evident and can be traced back, for example, to [3, 4]. What is interesting is that the minimizing movement, if it exists, solves a «differential equation» in a sense that has to be specified each time and this equation is unique.

Let us finally give the exact definition. In the following $\overline{\boldsymbol{R}}$ will be the extended real line ($\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\}$) and S a topological space.

Definition 1.1. *Let* $F:]1, +\infty[\times \boldsymbol{Z} \times S \times S \to \overline{\boldsymbol{R}}$ *and* $u: \boldsymbol{R} \to S$; *we say that* u *is a* minimizing movement associated to F and S *and we write* $u \in \mathrm{MM}(F, S)$ *if there exists* $w:]1, +\infty[\times \boldsymbol{Z} \to S$ *such that for any* $t \in \boldsymbol{R}$

$$\lim_{\lambda \to +\infty} w(\lambda, [\lambda t]) = u(t) \tag{1.3}$$

and for any $\lambda \in]1, +\infty[$, $k \in \boldsymbol{Z}$

$$F(\lambda, k, w(\lambda, k+1), w(\lambda, k)) = \min_{s \in S} F(\lambda, k, s, w(\lambda, k)) . \tag{1.4}$$

Some examples considered in [6] suggest the possibility that $\mathrm{MM}(F, S) = \emptyset$ and motivate the following generalization.

Definition 1.2. *We say that a function* $u: \boldsymbol{R} \to S$ *is a* generalized minimizing movement associated to F and S *and we write* $u \in \mathrm{GMM}(F, S)$ *if there exists a sequence* $\{\lambda_i\}_{i \in \boldsymbol{N}}$ *such that* $\lim_{i \to +\infty} \lambda_i = +\infty$ *and a sequence* $\{w_i\}_{i \in \boldsymbol{N}}$ *of functions* $w_i: \boldsymbol{Z} \to S$ *such that*

$$\exists i_0 > 1: F(\lambda_i, k, w_i(k+1), w_i(k)) = \min_{s \in S} F(\lambda_i, k, s, w_i(k)) \quad \forall k \in \boldsymbol{Z}, \ \forall i > i_0 , \tag{1.5}$$

$$\lim_{i \to +\infty} w_i([\lambda_i t]) = u(t) \quad \forall t \in \boldsymbol{R} . \tag{1.6}$$

In the following we consider some problems proposed in [6] relative to evolutions problems in $\boldsymbol{R}^n$ and in infinite dimensional spaces. The proofs will be given in forthcoming papers [11, and M. Gobbino, work in preparation]. (To see a slightly different setting where minimizing movements also find a natural application, consider [10] where differential inclusions are studied).

2. Minimizing movements and ordinary differential equations

Let $S = \boldsymbol{R}^n$, let $x_0 \in \boldsymbol{R}^n$ and let $f: \boldsymbol{R}^n \to \boldsymbol{R}$ be a function. Let us consider the functional:

$$F(\lambda, k, x, y) = \begin{cases} |x - x_0|^2 & \text{if } k < 0 , \\ f(x) + \lambda |x - y|^2 & \text{if } k \geq 0 . \end{cases} \tag{2.1}$$

If $f \in \mathrm{Lip}(\boldsymbol{R}^n) \cap C^{1,1}(\boldsymbol{R}^n)$, then:

– $\mathrm{MM}(F, \boldsymbol{R}^n) = \mathrm{GMM}(F, \boldsymbol{R}^n) \neq \emptyset$;

– $u \in \mathrm{MM}(F, \boldsymbol{R}^n)$ if and only if u is the unique solution of the following problem:

$$\begin{cases} u(t) = x_0 & \forall t \leqslant 0\,, \\ 2\, du/dt = -\nabla f(u(t)) & \forall t > 0\,. \end{cases} \tag{2.2}$$

For gradient flow type equations in $\boldsymbol{R}^n$ De Giorgi (see [6]) pointed out that if we only assume $f \in C^1(\boldsymbol{R}^n)$, then in general not all the solutions of (2.2) are elements of $\mathrm{GMM}(F, \boldsymbol{R}^n)$. Neverthless every solution of (2.2) is a minimizing movement relative to a functional $\widetilde{F}$, obtained adding a suitable small perturbation to the original functional F. To be precise, let us give the following.

DEFINITION 2.1. *Let* S *and* F *be as in Definition* 1.1. *We say that the functional* $\widetilde{F}\colon]1, +\infty[\, \times \boldsymbol{Z} \times S \times S \to \overline{\boldsymbol{R}}$ *is a perturbation of* F, *and we write* $\widetilde{F} \in \mathcal{F}(F)$, *if*

$$\lim_{\lambda \to \infty} \lambda \cdot \sup_{x, y, k} |F(\lambda, k, x, y) - \widetilde{F}(\lambda, k, x, y)| = 0\,. \tag{2.3}$$

If F is the functional defined in (2.1), we will denote by $\mathcal{F}_0(F)$ the set of functionals $\widetilde{F} \in \mathcal{F}(F)$ such that $\widetilde{F}(\lambda, k, x, y) = |x - x_0|^2$ for $k < 0$.

THEOREM 2.2. *Let* F *be the functional defined in* (2.1) *with* $f \in C^1(\boldsymbol{R}^n)$. *Let us consider the following sets:*

$A := \{u \in C^0(\boldsymbol{R}, \boldsymbol{R}^n)\colon u$ is a solution of(2.2)$\}$, $\quad B := \cup\{\mathrm{GMM}(\widetilde{F}, \boldsymbol{R}^n)\colon \widetilde{F} \in \mathcal{F}_0(F)\}$.
Then $A = B$.

REMARK 2.3. Theorem 2.2 is true even if we replace GMM by MM in the definition of the set B. It is also possible to find $F^* \in \mathcal{F}_0(F)$ such that $A = B = \mathrm{GMM}(F^*, \boldsymbol{R}^n)$.

Let us now collect our results about minimizing movements in $\boldsymbol{R}^n$. All the examples presented in the first section of [6], and inclusion $A \subseteq B$ of Theorem 2.2, are particular cases of the following theorems.

THEOREM 2.4. *Let* (X, d) *be a locally compact metric space, and let* $x_0 \in X$. *Let* F *be a functional as in Definition* 1.1 *such that* $F(\lambda, k, x, y) = d(x, x_0)$ *for* $k < 0$. *Let* $M \in \boldsymbol{R}$ *be a costant such that for each* $y \in X$, $\lambda \in]1, +\infty[$, $k \geqslant 0$, *there exists* $x \in X$ *such that:*

$$d(x, y) \leqslant M/\lambda\,, \qquad F(\lambda, k, x, y) = \min_{s \in X} F(\lambda, k, s, y)\,.$$

Then:

$\mathrm{GMM}(F, X) \neq \emptyset$;

there exists $u \in \mathrm{GMM}(F, X)$ *such that* $u \in \mathrm{Lip}(\boldsymbol{R}, X)$ *and* $\mathrm{lip}(u) \leqslant M$.

THEOREM 2.5. *Let* $X = \boldsymbol{R}^n$, $x_0 \in X$, *and let* F *be a functional as in Definition* 1.1 *such that* $F(\lambda, k, x, y) = d(x, x_0)$ *for* $k < 0$. *Let us assume that there exist a constant* $M \in \boldsymbol{R}$, *a* *continuous function* $\Phi: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ *and a function* $\rho: \mathbf{R}^n \times \mathbf{R}^n \times]1, +\infty[\to \mathbf{R}^n$ *such that:*

$$F(\lambda, k, x, y) = \min_{s \in X} F(\lambda, k, s, y) \Rightarrow \begin{cases} x - y = \Phi(x, y)/\lambda + \rho(x, y, \lambda), \\ |x - y| \leq M/\lambda; \end{cases}$$

for each compact set $K \subseteq \mathbf{R}^n \times \mathbf{R}^n$ $\lim_{\lambda \to \infty} \lambda \cdot \sup_{x, y \in K} |\rho(x, y, \lambda)| = 0$.

Then:

$\mathrm{GMM}(F, \mathbf{R}^n) \neq \emptyset$;

$u \in \mathrm{GMM}(F, \mathbf{R}^n) \Rightarrow u$ *is a solution of the following problem:*

$$\begin{cases} u(t) = x_0 & \forall t \leq 0, \\ du/dt = \Phi(u(t), u(t)) & \forall t > 0. \end{cases} \tag{2.4}$$

THEOREM 2.6. *In the same hypotheses of Theorem 2.5, let us assume that the function* $y \to \Phi(y, y)$ *is locally lipschitz continuous. Then:*

$\mathrm{MM}(F, \mathbf{R}^n) = \mathrm{GMM}(F, \mathbf{R}^n) \neq \emptyset$;

$u \in \mathrm{MM}(F, \mathbf{R}^n) \Leftrightarrow u$ *is the unique solution of problem* (2.4).

REMARK 2.7. With only slight modifications, it is possible to include nonautonomous equations in the preceding theorems.

THEOREM 2.8. *Let* (X, d) *be a metric space,* $x_0 \in X$, $f: X \to \overline{\mathbf{R}}$. *Consider the functional*

$$F(\lambda, k, x, y) = \begin{cases} d^2(x, x_0) & \text{if } k < 0, \\ f(x) + \lambda d^2(x, y) & \text{if } k \geq 0. \end{cases}$$

Let us assume that:

f *is lower semicontinuous;*

$f(x_0) \in \mathbf{R}$;

for each $y \in X$, *and for any* $R > 0$, $c \in \mathbf{R}$, *the set* $B(y, R) \cap \{x \in X: f(x) \leq c\}$ *is relatively compact in* X;

there exists $y_0 \in X$ *such that:* $\inf_{x \in X} \dfrac{f(x)}{1 + d^2(x, y_0)} > -\infty$.

Then:

(*i*) $\mathrm{GMM}(F, X) \neq \emptyset$;

(*ii*) $u \in \mathrm{GMM}(F, X) \Rightarrow u$ *is locally* $1/2$-*Hölder continuous.*

The preceding theorem provides a generalization of gradient flow type equations to arbitrary metric spaces. Similar techniques were introduced in [7] in order to prove the existence of «maximal slope curves» for Φ-convex functions defined in Hilbert spaces. For the theory of «maximal slope curves» and other definitions of «evolution curves of variational type» the reader is referred to [7] and [12].

In the hypotheses of Theorem 2.8, it is in general not true that elements of $GMM(F, X)$ are maximal slope curves in the sense of [7] (see also the discussion at the end of section 4). Furthermore examples can be provided, in which the following two properties, trivial for maximal slope curves, do not hold:

P1. if $u \in GMM(F, X)$, then the function $f \circ u$ is non-increasing;

P2. «semigroup property»: combining a flow from $t = 0$ to $t = T_1$ with one from $t = T_1$ to $t = T_2$ yields a flow from $t = 0$ to $t = T_2$.

3. Minimizing movements and partial differential equations

In order to give a unifying setting to some conjectures about parabolic equations and minimizing movements suggested in [6], we introduce the following abstract framework (more details and the proofs will be given in [11]).

Precisely we consider

$$\text{(3.1)} \quad \begin{cases} V \text{ Hilbert space with norm } \|\cdot\|, \\ V' \text{ its dual, } \langle \cdot, \cdot \rangle \text{ the duality pairing}, \end{cases}$$

$$\text{(3.2)} \quad \begin{cases} \phi: V \to \boldsymbol{R} \cup \{+\infty\} \text{ proper, lower semicontinuous, convex}, \\ D(\phi) \text{ its domain, } \partial\phi \text{ its subdifferential, } \phi^* \text{ its conjugate}. \end{cases}$$

$$\text{(3.3)} \quad \begin{cases} a(\cdot,\cdot), b(\cdot,\cdot): V \times V \to \boldsymbol{R} \text{ continuous bilinear forms} \\ \text{to which the linear continuous operators } A, B: V \to V' \text{ are associated}. \end{cases}$$

Moreover, $a(\cdot,\cdot)$ is symmetric and the associated quadratic form is positive: $a(u, v) = a(v, u)$; $a(u, u) \geqslant 0$ $\forall u, v \in V$. We can then think to the following problem.

Problem P. *Let $u_0 \in V$ be given and consider*

$$\text{(3.4)} \quad F(\lambda, k, v, w) = \begin{cases} \|v - u_0\| & \text{if } k \leqslant 0, \\ (1/2)a(v, v) + b(v, w) + (1/\lambda)\phi[\lambda(v - w)] & \text{otherwise}. \end{cases}$$

We want to find conditions on a, b, ϕ, u_0 such that there exists $u: \boldsymbol{R} \to V$ with $u \in MM(F, V)$ and

$$\text{(3.5)} \quad \begin{cases} u \in C^0(\boldsymbol{R}, V) \cap AC_{\text{loc}}(]0, +\infty[; V), \\ u(t) \equiv u_0 \quad \forall t \leqslant 0, \\ 0 \in Au + Bu + \partial\phi(u') \quad \text{a.e. in }]0, +\infty[. \end{cases}$$

The first two results concern the symmetric case, *i.e.* $b \equiv 0$; when $a(\cdot,\cdot)$ is coercive on V we can apply the theory developped in [1] to our problem via a duality argument and we can prove the following

THEOREM 3.1. *If $a(\cdot,\cdot)$ is coercive on V and $-Au_0 \in \overline{D(\phi^*)}^{V'}$ Problem* P *admits a unique solution u that satisfies* (3.5) *and $u \in W^{1,+\infty}_{\mathrm{loc}}(]0,+\infty[;V)$. Moreover, if we define $\check{\phi}$ as the function $v \mapsto \phi(-v)$, u is the solution to*

$$\begin{cases} u(0) = u_0\,, \\ u' + \partial\check{\phi}^*(Au) \ni 0 \quad \text{a.e. in }]0,+\infty[\end{cases} \tag{3.6}$$

and belongs to $\mathrm{MM}(F^*, V)$, *where*

$$F^*(\lambda,k,v,w) = \begin{cases} \|v-u_0\| & \text{if } k \leqslant 0\,, \\ (\lambda/2)\,a(v-w,v-w) + \check{\phi}^*(Av) & \text{otherwise}\,. \end{cases} \tag{3.7}$$

When $a(\cdot,\cdot)$ is not coercive (which is the specific case involved by the conjectures we considered) we suppose that a *uniformly convex* Banach space W is given, which is compatible with V in the sense that

$$\begin{cases} V \text{ and } W \text{ are both continuously embedded in a Hausdorff topological space } \mathcal{T}, \\ V \cap W \text{ is dense in } V \text{ and in } W. \end{cases}$$

$\sqrt{a(u,u)} + \|u\|_W$ must be an equivalent norm for $V \cap W$, *i.e.*

$$\exists l, \alpha_l > 0\colon \sqrt{a(u,u)} + l\|u\|_W \geqslant \alpha_l \|u\|_V\,, \quad \forall u \in V \cap W \tag{3.8}$$

and $\phi(u)$ is of the form $\|u\|_W^\beta + \psi(u)$, $u \in V \cap W$ with

$$\beta \in]1,+\infty[,\ \ \psi\colon V \cap W \mapsto \boldsymbol{R} \cup \{+\infty\} \text{ proper, convex, l.s.c.} \tag{3.9}$$

THEOREM 3.2. *If* (3.8), (3.9) *hold true, then $\forall u_0 \in V$ Problem* P *has a unique solution u. Moreover, $\forall T > 0$ (if $0 \in \partial\psi(0)$ we can choose $T = +\infty$),*

$$\begin{cases} u \in W^{1,\beta}_{\mathrm{loc}}(]0,+\infty[;V)\,, \quad u' \in L^\beta(0,T;W)\,, \\ \phi^*(-Au)\,, \quad \psi(u') \in L^1(0,T) \end{cases}$$

and u satisfies (3.6) *too.*

Finally we consider the non symmetric case:

THEOREM 3.3. *If $b \not\equiv 0$, referring to the hypotheses of Theorem* 3.2, *we suppose that W is a Hilbert space which contains V, b can be extended to a continuous bilinear form on $V \times W \mapsto \boldsymbol{R}$ and $\beta = 2$. Then Problem* P *admits a unique solution u, which belongs also to $H^1(0,T;W)$ $\forall T > 0$.*

With regards to De Giorgi's conjectures, the first one can be obtained choosing $b \equiv 0$, $V = H^1(\boldsymbol{R}^n)$, $W = L^\beta(\boldsymbol{R}^n)$ with $\beta \in]1,2]$ and $\psi \equiv 0$; Theorem 3.2 ensures that u satisfies, in the sense specified above,

$$\begin{cases} \dfrac{\beta}{2}\left|\dfrac{\partial u}{\partial t}\right|^{\beta-2}\dfrac{\partial u}{\partial t} = \Delta u & \text{in } \boldsymbol{R}^n \times]0,+\infty[\,, \\ u = u_0 & \text{on } \boldsymbol{R}^n \times \{0\} \end{cases}$$

and $-\Delta u \in L^{\beta'}(\boldsymbol{R}^n \times]0,+\infty[)$, $\partial u/\partial t \in L^\beta(\boldsymbol{R}^n \times]0,+\infty[)$.

A further application of the same theorem to the case $b = 0$, $V = H^1(\boldsymbol{R}^n)$, $W = L^2(\boldsymbol{R}^n)$, $\psi = I_K$ with $K = \{u \geqslant 0\}$, gives a positive answer to the third conjecture; for a.e. $t > 0$, Δu will be a Radon measure with positive part in $L^2(\boldsymbol{R}^n)$ and u solves

$$\begin{cases} \partial u/\partial t = [\Delta u]^+, & t > 0, \\ u(0) = u_0 . \end{cases}$$

Finally, if $\psi \equiv 0$, $V = H^1(\boldsymbol{R}^n)$, $W = L^2(\boldsymbol{R}^n)$ and

$$b(u, v) = \sum_{i=1}^{n} \int_{\boldsymbol{R}^n} a_i \frac{\partial u}{\partial x_i} v \, dx$$

with $a_i \in L^\infty(\boldsymbol{R}^n)$ we have

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \displaystyle\sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i}, & t > 0, \\ u(0) = u_0 . \end{cases}$$

Maximal monotone operators and convexity theory associated to the ideas of abstract numerical analysis are basic tools of the proofs, which are not completely standard due to the lack of coerciveness for $a(\cdot, \cdot)$ and to the strong nonlinearity for ϕ.

4. Open problems

A number of open problems can be formulated. As to problems discussed in section 3 an interesting point is to determine the highest possible regularity of the solution $u \in \mathrm{MM}(F, V)$.

Another direction could consists in unifying the differential approaches used in sect. 2 and 3. Just to give an example, let us consider the functional

$$F(\lambda, k, v, w) = \begin{cases} \|v - u_0\|, & k \leqslant 0, \\ \displaystyle\int_{\boldsymbol{R}^n} [\,|\nabla_x v|^2 + \lambda|v - w|^2 - |\sin v|^{3/2} e^{-|x|^2}]\, dx & \text{otherwise}; \end{cases} \tag{4.1}$$

an open problem is the study of the perturbations that must be added to it in order to recover all the solutions of the Cauchy problem (but nothing more).

Moreover a functional like (4.1) could be further complicated in order to consider the associated variational inequality of evolution on a proper convex set K.

Unless special regularity is asked on the functional, the $\mathrm{GMM}(F, V)$ doesn't satisfy any kind of semigroup property (see sect. 2); how little regularity has to be asked, isn't precisely known and only few special counterexamples have been shown by now.

Finally, it would be interesting to see if all the machinery developped so far can be applied to geometric measure theory problems and deduce, for example, some results about mean curvature flow in low dimensions spaces.

References

[1] H. Brezis, *Opérateurs maximaux monotones et semigroups de contraction dans les espaces de Hilbert.* North-Holland Mathematics Studies, n. 5, Notas de Matematica, (50), Amsterdam-London 1973.

[2] M. G. Crandall, *An introduction to evolution governed by accretive operators.* Dynamical Systems - An International Symposium, Academic Press, 1976, 131-165.

[3] M. G. Crandall - J. M. Ligget, *Generation of semi-groups of nonlinear transformations on general Banach spaces.* Amer. J. Math., 93, 1971, 265-298.

[4] E. De Giorgi, *Some conjectures on flow by mean curvature.* In: M. L. Benevento - T. Bruno - C. Sbordone (eds.), *Methods of Real Analysis and PDEs.* Liguori, Napoli 1990.

[5] E. De Giorgi, *Movimenti Minimizzanti.* Proceedings of the Conference on *Aspetti e problemi della Matematica oggi*. Lecce 1992.

[6] E. De Giorgi, *New Problems on Minimizing Movements.* In: *Boundary Value Problems for PDEs and Applications.* Masson, 1993, 81-98.

[7] E. De Giorgi - A. Marino - M. Tosques, *Problemi di evoluzione in spazi metrici e curve di massima pendenza.* Atti Acc. Lincei Rend. fis., s. 8, v. 68, 1980, 180-187.

[8] E. De Giorgi - A. Marino - M. Tosques, *Funzioni* (p, q)*-convesse.* Atti Acc. Lincei Rend. fis., s. 8, v. 73, 1982, 6-14.

[9] M. Degiovanni - A. Marino - M. Tosques, *Evolution equations associated with* (p, q)*-convex functions and* (p, q)*-monotone operators.* Ric. Mat., XXXIII, 1, 1984, 81-112.

[10] Z. Denkowski - S. Migórski - S. Mortola, *Differential Inclusions and Minimizing Movements.* SNS, Pisa 1993, preprints di Matematica, 36.

[11] U. Gianazza - G. Savarè, *Some results on Minimizing Movements.* Istituto di Analisi Numerica del CNR, Pavia 1994, preprint n. 914.

[12] A. Marino - C. Saccon - M. Tosques, *Curves of maximal slope and parabolic variational inequalities on non-convex constraints.* Ann. Sc. Normale Sup., 4, XVI, 1989, 281-330.

U. Gianazza: Dipartimento di Matematica
Università degli Studi di Pavia
Via Abbiategrasso, 209 - 27100 Pavia

M. Gobbino: Scuola Normale Superiore
Piazza dei Cavalieri, 7 - 56126 Pisa

G. Savarè: Istituto di Analisi Numerica del C.N.R.
Via Abbiategrasso, 209 - 27100 Pavia