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The Hughes subgroup

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Teoria dei gruppi. — *The Hughes subgroup.* Nota di ROBERT BRYCE, presentata (*) dal Socio G. Zappa.

ABSTRACT. — Let G be a group and p a prime. The subgroup generated by the elements of order different from p is called the Hughes subgroup for exponent p. Hughes [3] made the following conjecture: if $H_p(G)$ is non-trivial, its index in G is at most p. There are many articles that treat this problem. In the present *Note* we examine those of Strauss and Szekeres [9], which treats the case $p = 3$ and G arbitrary, and that of Hogan and Kappe [2] concerning the case when G is metabelian, and p arbitrary. A common proof is given for the two cases and a possible lacuna in the first is filled.

KEY WORDS: Infinite groups; Hughes subgroup; Metabelian groups.

RIASSUNTO. — *Il sottogruppo di Hughes.* Sia G un gruppo e p un numero primo; si dice sottogruppo di Hughes relativo a p il sottogruppo $H_p(G)$ generato dagli elementi di G di ordine diverso da p. Hughes [3] fece la seguente congettura: se $H_p(G)$ non è banale, il suo indice in G è $\leqslant p$. Vi sono molti lavori relativi a questo problema. Nella presente *Nota* vengono presi in esame quello di Strauss e Szekeres [9] relativo al caso $p = 3$, G qualunque, e quello di Hogan e Kappe [2] concernente il caso G metabeliano, p qualunque. Si dà un procedimento unico per i due casi e si colma una possibile lacuna del primo lavoro.

1. INTRODUCTION AND PRELIMINARY RESULTS

For a prime p the *Hughes subgroup for exponent p* of a group G is that generated by the elements of G whose order is not p. It is usually denoted $H_p(G)$. Hughes [3] asked if the index of $H_p(G)$ in G is always 1, $|G|$ or p. Many results have subsequently been obtained on this problem. For example Hughes and Thompson [4] showed that the answer is positive for finite non-p-groups, whilst Wall [10] showed it to be negative for finite groups in general. For special classes of groups there are positive results. That of Strauss and Szekeres [9], giving a positive answer for $p = 3$, was the first (though Hughes [3] had already observed that for $p = 2$ his problem has an easy, positive answer). Subsequently Zappa [11], and Hogan and Kappe [2] gave positive answers for finite groups of class at most p, and finite metabelian p-groups, respectively. Macdonald [8] showed that the problem has a positive solution for finite groups of class at most $2p - 2$. He also observes that the result of Hogan and Kappe can be obtained easily from his result, and also directly from Zappa's result. Further comment will be made on this below. Khukhro [5] shows that for the primes 5, 7, 11 at least, the bound $2p - 2$ cannot be increased. More recent results of Khukhro [6] show that in finite groups the answer to Hughes' question is positive «almost always».

In discussion with Dott. V. Pannone about the Strauss-Szekeres proof we came upon what seems to be a lacuna. This article suggests a way of removing this while keeping the spirit of their proof. The method also proves a positive result for the problem in the case of metabelian groups.

(*) Nella seduta del 16 giugno 1994.

The following lemmas will be needed in what follows.

Lemma 1.1. *Let G be a group in which $H_p(G)$ is a proper subgroup. For each $x \notin H_p(G)$, $C_G(x)$ is a p-group.*

Proof. For, if y is an element of $C_G(x)$ which is not a p-element, then neither is xy a p-element. Hence y, $xy \in H_p(G)$ and therefore $x = (xy)y^{-1} \in H_p(G)$, a contradiction.

Corollary 1.2. *A nilpotent group with proper Hughes subgroup for exponent p is a p-group.*

Proof. If G is a nilpotent group with proper Hughes subgroup, but G is not a p-group, some factor group G with proper Hughes subgroup has elements of finite order co-prime to p, contradicting Lemma 1.1.

Lemma 1.3. *Let G be a group which is nilpotent of class at most 2, and let M be a ZG-module. Suppose that G has a normal subgroup H with the property that G/H is elementary abelian of order p^2. Suppose moreover that, regarding elements of G as endomorphisms of M, we have*

$$x^{p-1} + x^{p-2} + \ldots + x + 1 = 0, \qquad x \in G \setminus H. \tag{1.4}$$

Then

$$pM = 0 \tag{1.5}$$

and

$$y^{p-1} + y^{p-2} + \ldots + y + 1 = 0, \qquad y \in G. \tag{1.6}$$

Proof. Choose elements x, y for which $|\langle xH, yH\rangle| = p^2$. If $x_0 \in \langle H, x\rangle$ then the elements $x_0 y^i$ $(1 \leqslant j \leqslant p-1)$ are none of them in H. Write $w = [y, x_0]$. Then $(x_0 y)^i = x_0^i y^i w^{i(i-1)/2} = x_0^i f_i(y)$ where we regard x_0 as fixed. Note that $f_i(y^j) = f_i(y)^j$ $(1 \leqslant j \leqslant p-1)$, and that $f_i(y) \notin H$ $(1 \leqslant i \leqslant p-1)$. By hypothesis

$$0 = 1 + x_0 f_1(y) + x_0^2 f_2(y) + \ldots + x_0^{p-1} f_{p-1}(y). \tag{1.7}$$

In (1.7) we may replace y by y^j for each j in the range $(1, p-1)$. We get

$$0 = 1 + x_0 f_1(y)^j + x_0^2 f_2(y)^j + \ldots + x_0^{p-1} f_{p-1}(y)^j, \qquad 1 \leqslant j \leqslant p-1. \tag{1.8}$$

Add the $p-1$ equations (1.8) to give

$$0 = (p-1) + x_0(-1) + x_0^2(-1) + \ldots + x_0^{p-1}(-1). \tag{1.9}$$

When $x_0 = x$ this becomes $0 = p - 1 + (-1)(-1) = p$ as required by (1.5).

To prove (1.6) is similar: in the equations (1.8) choose $x_0 \in H$. On adding all these equations we get (1.9) again and from (1.5) we deduce that $0 = -1 - x_0 - x_0^2 - \ldots - x_0^{p-1}$ as required by (1.6).

The basic fact which underlies all results on the Hughes problem is contained in the following lemma.

LEMMA 1.10. *Let H be a normal subgroup of a group G. Every element of $G \setminus H$ has prime order p only if*

$$x^{p-1} + x^{p-2} + \dots + x + 1 = 0, \qquad x \in G \setminus H. \tag{1.11}$$

PROOF. For $x \notin H$ and $h \in H$, $xh \notin H$. Then $(xh)^p = x^p h^{x^{p-1}} h^{x^{p-2}} \dots h^x h$ whence (1.11).

Suppose now that G is a group with a normal subgroup H. The set $\mathcal{R}$ of all functions on H admits the following two binary operations. For ϕ and ψ in $\mathcal{R}$ define $\phi + \psi$ by: $h(\phi + \psi) = (h\phi)(h\psi)(h \in H)$. $\mathcal{R}$ is a group under this operation with identity element, which we denote by 0, the function $h \to 1$, and for every $\phi \in \mathcal{R}$ there is an inverse $-\phi$ defined by $h(-\phi) = (h\phi)^{-1}$. This operation is not, in general, commutative. The second operation on $\mathcal{R}$ is composition: $h(\phi\psi) = (h\phi)\psi$ $(h \in H)$. We note that multiplication distributes over addition from the left.

Now associate with each element x of G the automorphism which it induces by conjugation in H. We will denote this element of $\mathcal{R}$ also by x. The subset of $\mathcal{R}$ of such automorphisms we denote by Γ. Γ will be used in the sequel in this sense, that is, as the set of automorphisms induced on a normal subgroup H, without further comment.

The result we now prove is that, for the groups that arise in the Strauss-Szekeres result addition in $\mathcal{R}$, when restricted to Γ, is commutative.

LEMMA 1.12 (cf. [9]). *Let G be a group in which $H_3(G) \subseteq H \neq G$. For all $x, y \in \Gamma$, $x + y = y + x$.*

PROOF. Since all elements of Γ are multiplicatively invertible, and since left multiplication distributes over addition, it suffices to show that $1 + x = x + 1$ for all $x \in \Gamma$. From Lemma 1.10 we have

$$x^2 + x + 1 = 0, \qquad x \notin H. \tag{1.13}$$

In this equation we may replace x by its square, and this yields $x + x^2 + 1 = 0$. It then follows that $-1 = x^2 + x = x + x^2$ and hence, multiplying on the left by x^2, we get $x + 1 = 1 + x$.

We must show that this is true also for elements of H. With $x \in G \setminus H$ and $h \in H$ we have that $h^{-1}x \notin H$, and therefore $x + h = h(h^{-1}x + 1) = h(1 + h^{-1}x) = h + x$. In this equation we may replace x by x^2 to get $x^2 + h = h + x^2$. Since h commutes in addition with both x and x^2 it now follows from (1.13) that $h + 1 = 1 + h$. Therefore $g + 1 = 1 + g$ for all elements g of G. This completes the proof of Lemma 1.12.

The Strauss-Szekeres version of this lemma treats only elements x, y in the complement of H.

COROLLARY 1.14. *In a group G in which $H_3(G) \subseteq H \neq G$, the normal closure in G of every element of H is abelian. In particular G is soluble.*

Proof. The first statement follows from the last lemma, as a matter of definition. For the second, note that H and G/H both have nilpotency class at most 3, by a theorem of Levi [7].

2. Proofs

The aim of this section is to give a proof of the Strauss-Szekeres result [9], as far as possible in the spirit of the original. It is based on Lemma 1.3. We show also that Lemma 1.3 leads to a proof for metabelian groups. The discussion will be laid out so as to show how close these two results are.

Firstly let G be a soluble group, not of exponent p, for which $|G: H_p(G)| \geqslant p^2$. Let $H = H_p(G)$. Then G/H is soluble of exponent p and is therefore nilpotent by a result of Hall [1]. Hence we can find a pair of elements a, b of G for which $\langle aH, bH \rangle$ is elementary of order p^2. Define G_0 by $G_0/H = \langle aH, bH \rangle$. Note that all elements of $G_0 \setminus H$ are of order p. Also G_0 is not of exponent p. We may as well suppose, therefore, that G has a normal subgroup H whose index is p^2, and whose complement in G consists solely of elements of order p.

Now let $K = \langle a, b \rangle$. Note that $K \cap H$ is the normal closure in K of the commutator $[a, b]$. Suppose that $K \cap H$ is abelian, as it will be if either G is metabelian, or if $p = 3$ by Corollary 1.14. Then it is of exponent p. This is because: $|K: K \cap H| = p^2$; every element of $K \setminus K \cap H$ is of order p; and $K/K \cap H$ acts on $K \cap H$ so as to satisfy the hypotheses of Lemma 1.3, here using Lemma 1.10. Therefore K is of exponent p. That is

if either G is metabelian, or if $p = 3$, then K has exponent p.

Now let C be a subgroup of H chosen as follows. If G is metabelian put $C = G'$. If $p = 3$ let C be the normal closure of an arbitrary element of H. In either case C is abelian, here using Corollary 1.14 when $p = 3$. Note also that, in this latter case, K is of class at most 2, since it is a two generator group of exponent 3.

Now all elements of $KC \setminus (K \cap H)C$ are of order p. By Lemma 1.10 these elements act on C so as to satisfy (1.4). Hence C has exponent p. When $p = 3$, this means that H has exponent 3, and therefore so has G, a contradiction.

This concludes the proof of the Strauss-Szekeres result.

When G is metabelian we conclude from Lemma 1.3 that G has a law $[x, y, (p-1)z] = 1$. It is well known that the law $[x, y, z_1, z_2, \dots z_{p-1}] = 1$ can be deduced from this in a metabelian group. But this says that G has class at most p. Moreover, by Corollary 1.2, G/G' is a p-group, of exponent p in fact, being generated by elements of order p. Therefore G has exponent at most p^2, so is locally finite. Finally, let h be an arbitrary element of H. Then $L = \langle K, h \rangle$ is a finite subgroup of G, of class at most p, and with $H_p(L) \subseteq L \cap H$, which has index p^2 in L. By the result [11] of Zappa this means that h is of order p and hence H, and therefore G, is of exponent p, another contradiction. This concludes the proof in the metabelian result.

3. Final comments

1. The apparent difficulty with the proof in Strauss-Szekeres for the case $p = 3$ concerns their seeming assumption that multiplication commutes for elements of G in the structure Γ. I can see no reason why this should be so. The argument here isolates the calculation to the subgroup K, where multiplication almost commutes, because K is of class at most 2.

2. In none of the discussion above has finiteness been assumed. This does not necessarily mean that the result for metabelian groups is very much deeper then Hogan-Kappe [2]. It is easy to show, for example, that the locally finite groups in a subgroup closed class have the Hughes property provided the finite groups in the class do.

From this point of view we were able to rely on Zappa's result in proving that metabelian groups have the Hughes property, because we showed in the course of the proof above that, if a metabelian counter-example to the Hughes property exists, then there is one of exponent p^2 and class at most p, which means it is locally finite.

3. From Corollary 1.14 quick proofs of Strauss-Szekeres using all the machinery available, are easy to find: Lemma 1.3 is not needed. For example: if G is a counter-example to Hughes for exponent 3 then, as in the last section, there is one of the form KN where K is finite and $N = \langle h \rangle^K$ is abelian, by Corollary 1.14. For some normal subgroup S of G contained in N, G/S is a finite counter-example. We may as well assume that $S = 1$. By [4] (or by a straight-forward, direct argument based on the fact that G is soluble) G is a 3-group. Now $\gamma_5(G) \subseteq \gamma_4(G) \subseteq G^3$ (by [7]) $\subseteq H_3(G)$. But $G/\gamma_5(G)$ has class at most $4 = 2.3 - 2$ so, by [8], the exponent of $G/\gamma_5(G)$ is 3. Hence $G^3 \subseteq \subseteq \gamma_5(G) \subseteq \gamma_4(G) \subseteq G^3$. This means that $\gamma_4(G) = \gamma_5(G) = 1$ so G has class at most 3. This is a contradiction to Zappa [11].

4. I thank Professor Szekeres for his helpful comments in reply to my query about [9]. He tells me that originally he and Strauss independently submitted manuscripts on the Hughes problem for exponent three, and that these reached the editor of the Proceedings of the American Mathematical Society on the very same day! I owe to Rolf Brandl the observation that Strauss-Szekeres for finite 3-groups is a consequence of Macdonald's result.

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