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Aldo Bressan, Monica Motta

On control problems of minimum time for Lagrangian systems similar to a swing. II Application of convexity criteria to certain minimum time problems

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Meccanica. — *On control problems of minimum time for Lagrangian systems similar to a swing. II. Application of convexity criteria to certain minimum time problems.* Nota di

ALDO BRESSAN e MONICA MOTTA, presentata (*) dal Socio A. Bressan.

ABSTRACT. — This *Note* is the Part II of a previous *Note* with the same title. One refers to holonomic systems $\Sigma = \mathcal{A} \cup \mathcal{U}$ with two degrees of freedom, where the part $\mathcal{A}$ can schemetize a swing or a pair of skis and $\mathcal{U}$ schemetizes whom uses $\mathcal{A}$. The behaviour of $\mathcal{U}$ is characterized by a coordinate used as a control. Frictions and air resistance are neglected. One considers on Σ minimum time problems and one is interested in the existence of solutions. To this aim one determines a certain structural condition Γ which implies a well known convexity condition (briefly *WCC*) just ensuring the afore-mentioned existence. These proofs are based on the results of Part I. The condition Γ becomes equivalent to the *WCC* in both the cases of the swing or of the ski having constant curvature trajectory. An other equivalent structural condition is established in a simple case regarding the ski. The *WCC* fails to be verified, *e.g.*, for the simple pendulum of variable length. One observes that, also in the absence of the *WCC*, for certain initial and terminal data, the solution still exists.

KEY WORDS: Analytical mechanics; Lagrangian systems; Control theory.

RIASSUNTO. — *Su problemi di controllo di tempo minimo per sistemi Lagrangiani simili a un'altalena. II. Applicazione di criteri di convessità a certi problemi di tempo minimo.* Questa *Nota* è la Parte II di una precedente *Nota* dallo stesso titolo. Ci si riferisce a sistemi olonomi $\Sigma = \mathcal{A} \cup \mathcal{U}$ a due gradi di libertà, ove la parte $\mathcal{A}$ può schematizzare un'altalena o un paio di sci e $\mathcal{U}$ schematizza chi usa $\mathcal{A}$. Il comportamento di $\mathcal{U}$ è caratterizzato da una coordinata usata come controllo. Si trascurano l'attrito e la resistenza dell'aria. Si considerano su Σ problemi di tempo minimo e ci si interessa dell'esistenza di loro soluzioni. A tale scopo si determina una certa condizione strutturale Γ che implica una ben nota condizione di convessità (brevemente *WCC*) appunto assicurante la suddetta esistenza. In tali dimostrazioni ci si basa sui risultati della Parte I. La condizione Γ diviene equivalente alla *WCC* nel caso dell'altalena o dello sci avente traiettoria con curvatura costante. Un'altra condizione strutturale equivalente viene stabilita in un caso semplice riguardante lo sci. La *WCC* risulta non verificata, per es., per il pendolo semplice con lunghezza variabile. Si osserva che anche in assenza della *WCC*, per certi dati iniziali e finali, la soluzione esiste lo stesso.

4. ON $F(s, \mathcal{P})$'S CONVEXITY IN THE CASE $c \neq 0 = c' \neq y'$

This Part II is the continuation of Part I of the *Note* of the same title. Please refer to Part I for definitions, annotations and references (see Rend. Mat. Acc. Lincei, s. 9, vol. 5, 1994, 247-254).

Case $c \neq 0 = c' \neq y'$. By $(2.9)_2$ and $(\mathcal{B})$ below (3.4), the C^2-function $\eta = \eta(u)$ defined by $(3.4)_2$ can be inverted into a C^2-function $u = u(\eta)$ defined on the segment $G(s, \mathcal{P}, U) = \overline{Y}$ – see $(\mathcal{E})$ below (3.11). By introducing it in $(3.4)_1$ we obtain a C^2-function $z = z(\eta)$ defined on $\overline{Y}$. Since

$$(4.1)\qquad \beta_u = \Im c - \Im_u \xi, \quad \beta_{uu} = 2\Im_u c - \Im_{uu}\xi, \quad \gamma = 0, \quad \eta = 2mgy'\beta, \quad \sqrt{\mathcal{P}}z = \Im,$$

(*) Nella seduta del 12 febbraio 1994.

by $(3.2)_2$ we deduce [1]

$$(4.2)\qquad 4\sqrt{\mathcal{P}}(mgy')^2 z_{\eta\eta} = (\Im_{uu}\beta_u - \Im_u\beta_{uu})\beta_u^{-3} = (\Im_{uu}\Im - 2\Im_u^2)c(\Im c - \Im_u\xi)^{-3}.$$

Furthermore, by $(2.9)_2$, $c(\Im c - \Im_u \xi) > 0$ for both $c > 0$ and $c < 0$. On the other hand $F(s, \mathcal{P})$ is $z(\cdot)$'s epigraph. We conclude that

($\mathcal{H}$) *when* $c \neq 0 = c' \neq y'$, *the closed set* $\mathcal{F} = F(s, \mathcal{P})$ *is convex iff* $z_{\eta\eta}(\eta) \geqslant 0\ \forall \eta \in G(s, \mathcal{P}, U)$, *and hence iff we have the first of the relations*

$$(4.3)\qquad \Im_{uu}\Im \geqslant 2\Im_u^2, \quad \text{i.e.}\quad (I_C^{\mathfrak{a}} + I_C^{\mathfrak{u}})(I_C^{\mathfrak{u}})_{uu} \geqslant 2(I_C^{\mathfrak{u}})_u^2 \qquad \forall u \in U.$$

Let $I_C^{\mathfrak{a}}$ and $I^{\mathfrak{a}}$ [$I_C^{\mathfrak{u}}(u)$ and $I(u)$] be $\mathfrak{a}$'s [$\mathfrak{u}$'s] moments of inertia w.r.t. the axes parallel to z passing through C and through $\mathfrak{a}$'s [$\mathfrak{u}$'s] center of mass $P(s)$ [$P(s) + \sigma n$] respectively. Of course, denoting $\mathfrak{a}$'s [$\mathfrak{u}$'s] mass by m_1 [m_2] we have (for $c \neq 0$)

$$(4.4)\qquad mu = m_2\sigma, \quad I_C^{\mathfrak{a}} = I^{\mathfrak{a}} + m_1 r^2, \quad I_C^{\mathfrak{u}}(u) = I(u) + m_2(r-\sigma)^2, \quad (I_C^{\mathfrak{a}})_u = 0;$$

furthermore $\Im = c^2(I_C^{\mathfrak{a}} + I_C^{\mathfrak{u}})$ by $(2.8)_2$. Then $(4.3)_1$ is equivalent to $(4.3)_2$. To check the convexity condition (4.3) it is useful to write explicitly that

$$(4.5)\qquad \begin{cases} \Im = c^2 I^{\mathfrak{a}} + m_1 + c^2 I(u) + m_2(1-c\sigma)^2, \quad \Im_u = c^2 I_u(u) - 2m(1-c\sigma)c, \\ \Im_{uu} = c^2 I_{uu}(u) + 2c^2 m^2/m_2 \quad (d\sigma/du = m/m_2). \end{cases}$$

EXAMPLE 1 (*Simple pendulum of variable length*). $l = 1$ and $I(\cdot) \equiv 0$. One has $m = m_2$, $\sigma = u$, $I^{\mathfrak{a}} \equiv 0$, $\Im = m\xi^2$. Thus $(4.3)_1$ becomes $2mc^2\xi^2 \geqslant 8mc^2\xi^2\ \forall u \in U$. Hence $F(s, \mathcal{P})$ *fails to be convex for all* $c \neq 0 \neq y'(s)$.

EXAMPLE 2. (*a*) (*Natural or unnatural simplified revolutory swing*). $I(\cdot) \equiv 0$. In it, by (4.5), condition $(4.3)_1$ is equivalent to [2]

$$(4.6)\qquad c^2 I^{\mathfrak{a}} + m_1 \geqslant 3m_2(1-c\sigma)^2 \quad \forall \sigma \in [\sigma_1, \sigma_2], \quad \text{where } \sigma_i = mu_i/m_2 \ (i = 1, 2);$$

hence $(4.3)_1$ *holds iff*

$$(4.7)\qquad c^2 I^{\mathfrak{a}} + m_1 \geqslant 3m_2\mathcal{C}^2$$

where either $c < 0$ *and* $\mathcal{C} = 1 + |c|\sigma_2$, *or* $c > 0$ *and* $\mathcal{C}$ *is* $1 - c\sigma_2$ *or* $1 - c\sigma_1$ *according to whether or not* $\sigma_1 + \sigma_2 \geqslant 2r$, i.e. $m(u_1 + u_2) \geqslant 2m_2 r$ [3].

EXAMPLE 2. (*b*) (*Simplified skis-skier system when* l *is a circle arc*). $I(\cdot) \equiv 0$ again. Hence the conclusions involving (3.10)-(3.11) are still holding. Of course l is concave up [down] for $c > 0$ [$c < 0$].

In the examples above by «simplified systems» one refers to $\mathfrak{u}$ being reduced

[1] Indeed by (4.1) and $(3.2)_2$: $\sqrt{\mathcal{P}} z_{\eta\eta} = (\Im_{uu}\beta_u - \Im_u\beta_{uu})(2mgy')^{-2}\beta_u^{-3} = [\Im_{uu}(\Im c - \Im_u\xi) - \Im_u(2\Im_u c - \Im_{uu}\xi)](2mgy')^{-2}\beta_u^{-3} = (\Im_{uu}\Im - 2\Im_u^2)c(2mgy')^{-2}\beta_u^{-3}$. Hence (4.2) follows.

[2] Indeed, by $(4.5)_1$, (4.3) becomes $[c^2 I^{\mathfrak{a}} + m_1 + m_2(1-c\sigma)^2]2m^2c^2/m_2 \geqslant 8m^2(1-c\sigma)^2c^2$ $\forall u \in U$.

[3] Indeed the function $\sigma \mapsto (1-c\sigma)^2$ has a minimum for $\sigma = r$. Furthermore, for $c > 0$, $(1-c\sigma_2)^2 \geqslant (1-c\sigma_1)^2$ iff $c\sigma_2 - 1 \geqslant 1 - c\sigma_1$.

to a mass point. Furthermore the revolutory swing is called unnatural by footnote 1 of previous *Note.*

5. On $F(s, \mathcal{P})$'s convexity for $c \neq 0 \neq c'$

In this section we consider the case $c \neq 0 \neq c'$. First we note that, by $(3.3)_{2\text{-}4}$

$$(5.1)\qquad \frac{c}{2c'}\gamma_u = m\frac{\mathfrak{J}c + \mathfrak{J}_u\xi}{\mathfrak{J}^2}, \qquad \frac{\mathfrak{J}^3 c}{2mc'}\gamma_{uu} = \mathfrak{J}\mathfrak{J}_{uu} + 2(\mathfrak{J}c + \mathfrak{J}_u\xi)\mathfrak{J}_u .$$

Hence by $(4.1)_{1\text{-}2}$ (still holding) and (3.3-4)

$$(5.2)\qquad \begin{cases} \eta_u = \dfrac{2mc'}{\mathfrak{J}^2 c}(\mathfrak{J}c + \mathfrak{J}_u\xi)\mathcal{P} + 2mgy'(\mathfrak{J}c - \mathfrak{J}_u\xi), \\ \eta_{uu} = \dfrac{2mc'}{\mathfrak{J}^3 c}[\mathfrak{J}\mathfrak{J}_{uu} + 2(\mathfrak{J}c + \mathfrak{J}_u\xi)\mathfrak{J}_u]\mathcal{P} + 2mgy'(2\mathfrak{J}_u c - \mathfrak{J}_{uu}\xi). \end{cases}$$

Now having fixed $s \in \Delta$ and $\mathcal{P} > 0$, the alternative $(b)^{\pm}$ above (3.7) can always be used, as is observed below (3.10). In the following simple example, conditions $(3.7)^{\pm}$ are not always fulfilled and we have to deduce $F(s, \mathcal{P})$'s convexity by $(b)^{\pm}$.

We consider the special case where $\mathcal{U}$ can be regarded as a point: $I_G \equiv 0$ (*i.e.* $\mathfrak{J} = m\xi^2$ – see $(2.8)_1$); however we assume $c'cy' \neq 0$. Then either $(3.5)^+$ or $(3.5)^-$ holds and (5.2) together with $(3.4)_1$ yields

$$(5.3)\qquad \begin{cases} \eta_u = -2c'\mathcal{P}\xi^{-2} + 6m^2gcy'\xi^2, & z_u = -2mc\xi\mathcal{P}^{-1/2}, \\ \eta_{uu} = -4cc'\mathcal{P}\xi^{-3} - 12m^2gc^2y'\xi, & z_{uu} = 2mc^2\mathcal{P}^{-1/2}. \end{cases}$$

By (5.3), $(3.10)_1$ can be written as follows

$$(5.4)\qquad f(u) \doteq \eta_u(u)[2mc^2\mathcal{P}^{-1/2}(-2c'\mathcal{P}\xi^{-2} + 6m^2gcy'\xi^2) - 2mc\xi\mathcal{P}^{-1/2}\cdot$$
$$\cdot(4cc'\mathcal{P}\xi^{-3} + 12m^2gc^2y'\xi)] = -12mc^2\mathcal{P}^{-1/2}\xi^{-2}\eta_u(u)(c'\mathcal{P} + m^2gcy'\xi^4).$$

We set – see $(3.5)^{\pm}_{1,2}$

$$(5.5)\qquad \begin{cases} \xi' \doteq 1 - cu', & \xi'' \doteq 1 - cu'', \\ \mathcal{P}' \doteq m^2g|cy'(c')^{-1}|(\xi')^4, & \mathcal{P}'' \doteq m^2g|cy'(c')^{-1}|(\xi'')^4. \end{cases}$$

Now by a straightforward calculation one obtains that if either (*i*) both $c'cy' > 0$ and $\mathcal{P} \in (0, 3\mathcal{P}'] \cup [3\mathcal{P}'', +\infty)$, or (*ii*) $c'cy' < 0$, then

$$(5.6)\qquad \eta_u(u)cy' > 0 \quad \forall u \in (u_1, u_2);$$

otherwise, *i.e.* in case $(iii)^{\pm}$ $c'cy' > 0$ with $y' \lessgtr 0$ but $\mathcal{P} \in (3\mathcal{P}', 3\mathcal{P}'')$, we have that

$$(5.7)\qquad \eta_u(u) \neq 0 \quad \forall u \neq \bar{u}, \qquad \eta_u(u) > 0 \quad \forall u \gtrless \bar{u},$$

where $\bar{u} \in (u_1, u_2)$ is given by

$$(5.8)\qquad \bar{u} \doteq (1 - \sqrt[4]{c'\mathcal{P}/3m^2gcy'})c^{-1}.$$

Let us first assume (*i*) [(*ii*)]. Then in particular $\eta_u(u) \neq 0$ $\forall u \in (u_1, u_2)$ and by

(5.4) and (5.6), conditions $(3.7)_2^{\pm}$ are equivalent to

$$c'\mathcal{P} + m^2 gcy'\xi^4 \gtreqless 0 \quad \forall u \in U \qquad \text{if } cy' \gtrless 0\,. \tag{5.9}$$

Hence Theor. 3.1 yields that the set $F(s, \mathcal{P})$ is convex [iff $\mathcal{P} \geqslant \mathcal{P}''$].

In the case $(iii)^{\pm}$, none of conditions $(3.7)_1^{\pm}$ is verified and in order to test the convexity of $F(s, \mathcal{P})$ we have to check the validity of the alternative $(b)^{\pm}$ below $(3.7)^{\pm}$. The value $\bar{u}$ defined by (5.8) satisfies $(3.8)_1^{\pm}$ and by (5.7) and (5.9) it follows that $f(u) \geqslant 0$ $\forall u \in [u', \bar{u}]$, *i.e.* condition (3.10) of Theor. 3.1 is verified. Furthermore, by (5.7) condition (3.9) is equivalent to

$$\Delta \doteq \eta(u_1) - \eta(u_2) \begin{cases} \geqslant 0 & \text{if } c' > 0\,, \\ \leqslant 0 & \text{if } c' < 0\,. \end{cases} \tag{5.10}$$

Since $(5.10)_1$ is equivalent to

$$\Delta = 2[u_2 - u_1]\{c'\mathcal{P}(\xi')^{-1}(\xi'')^{-1} - m^2 gcy'[(\xi')^2 + \xi'\xi'' + (\xi'')^2]\}\,, \tag{5.11}$$

condition (5.10) – *i.e.* (3.9) – reads

$$\mathcal{P} \geqslant \overline{\mathcal{P}} \doteq m^2 gcy'(c')^{-1}[(\xi')^3\xi'' + (\xi')^2(\xi'')^2 + \xi'(\xi'')^3] \quad (\mathcal{P}' < \overline{\mathcal{P}} < \mathcal{P}'')\,, \tag{5.12}$$

so that, by $(b)^{\pm}$ in Theor. 3.1, the set $F(s, \mathcal{P})$ is convex iff $\mathcal{P} \geqslant \overline{\mathcal{P}}$.

We can summarize the above results as follows.

THEOREM 5.1. *Let $I_G \equiv 0$. Then in case $c'cy' < 0$ the set $F(s, \mathcal{P})$ is convex iff $\mathcal{P} \geqslant \mathcal{P}''$, while in case $c'cy' > 0$ it is convex iff $\mathcal{P} \geqslant \overline{\mathcal{P}}$.*

6. On the existence of the solution to the problem of minimum time for the swing. Preliminary considerations on Σ_1. Cases of existence without convexity

In order to treat the swings $\Sigma^{\pm}$, using $\theta = s/|r|$ instead of s, we refer to the version $(\widehat{\mathcal{P}})$ above (2.21) of problem (α). First let us note that for $\Sigma^{\pm}$ there are choices of $[\theta_0, \theta_1]$ for which the optimization problem $(\widehat{\mathcal{P}})$ has a solution even in the absence of the convexity condition.

THEOREM 6.1. *Assume that $(i)^+$ $c > 0 \equiv c'$ and $[\theta_0, \theta_1] \subset [0, \pi]$ or $(i)^-$ $c < 0 \equiv c'$ and $[\theta_0, \theta_1] \subset [-\pi, 0]$, and (ii) $\widehat{\mathcal{U}}^* \neq \emptyset$ – see $(2.23)_4$ –,* i.e. *some admissible process rendering the functional* (2.21) *meaningful exists. Then $\widehat{\mathcal{P}}^+(\cdot) = \widehat{\mathcal{P}}^+(\cdot, \mathcal{P}_0, \theta_0)$ defined below* (2.23) *solves problem $(\widehat{\mathcal{P}})$ above* (2.21).

REMARK 6.1. The above solution $\widehat{\mathcal{P}}^+(\cdot)$ – see the assertion below (2.18) – solves the problem expressed by (2.22)-(2.23) and

$$\widehat{\mathcal{P}}(\theta_1, u) \to \sup\,. \tag{6.1}$$

PROOF. Assume $\theta \in [\theta_0, \theta_1]$ and either hypothesis $(i)^+$ or $(i)^-$. Then by [6, sect. 10, pp. 173-175] we have that $\hat{u}^+(\theta) = u_2$ or $\hat{u}^+(\theta) = u_1$ respectively. Then, in case $(i)^{\pm}$ for $\hat{u}(\cdot) \in \widehat{\mathcal{U}}^*$, $(2.9)_2^{\pm}$ yields $\mathfrak{J}[\hat{u}(\theta)] \geqslant \mathfrak{J}[\hat{u}^+(\theta)]$. Furthermore (by definition) $\widehat{\mathcal{P}}(\theta, u) \leqslant \widehat{\mathcal{P}}^+(\theta)$. Hence, by (2.21), $T[\widehat{\mathcal{P}}(\cdot, \hat{u}), \hat{u}(\cdot)] \geqslant T[\widehat{\mathcal{P}}^+(\cdot), \hat{u}^+(\cdot)]$. □

Through Theor. 6.1 we have proved the following assertion on the skis-skier system – see Example 2(*b*) – with negligible friction and air resistance.

($\mathfrak{A}$) *In order to describe an ascending circle arc that (belongs to the skis' trajectory l and) is concave up [concave down] in the minimum time, the skier must stay in his most upright [bent down] position.*

Now, in connection with Σ_1 we set – see also the definitions below (2.23)

$$T_{\xi,\theta_0,\theta_1} = T_\xi \doteq \inf\{T[\widehat{\mathcal{P}}(\cdot,\xi,\widehat{u}),\widehat{u}(\cdot)]:\widehat{u}(\cdot)\in\widehat{\mathcal{U}}^*_{\xi,\theta_0}\} \quad \text{for } \widehat{\mathcal{U}}^*_{\xi,\theta_0}\neq\emptyset\,. \tag{6.2}$$

Since for every $\xi\in\boldsymbol{R}$, $\widehat{\mathcal{P}}^{\pm}(\cdot,\xi)$ solves the ODE (2.22) with $u=\widehat{u}^{\pm}(\theta)$, by the corresponding uniqueness theorem

$$\widehat{\mathcal{P}}^{\pm}(\theta,\xi_1)<\widehat{\mathcal{P}}^{\pm}(\theta,\xi_2)\quad\forall\theta\in[\theta_0,\theta_1],\qquad \widehat{\mathcal{U}}^*_{\xi_1,\theta_0}\subset\widehat{\mathcal{U}}^*_{\xi_2,\theta_0}\quad\text{when } \xi_1<\xi_2\,. \tag{6.3}$$

We shall use only a part of the following theorem, brief to prove, which states a property (having the opposite) of the Lipschitz type and possibly useful for other purposes.

THEOREM 6.2. *Assume* $0\leqslant\xi_1<\xi_2$ *and* $\widehat{\mathcal{U}}^*_{\xi_1}\neq\emptyset$, *problem* ($\widehat{\mathcal{P}}$) *above* (2.21) *being considered for* $\mathcal{P}_0=\xi_1$ *(and for* $\mathcal{P}_0=\xi_2$*). Then*

$$T_{\xi_1}-T_{\xi_2}\geqslant\min\Im(U)\int_{\theta_0}^{\theta^*}[\widehat{\mathcal{P}}^+(\theta,\xi_1)^{-1/2}-\widehat{\mathcal{P}}^-(\theta,\xi_2)^{-1/2}]\,d\theta>0\,, \tag{6.4}$$

θ^* *being* θ_1 *for* $S_{\widehat{u}^+}=\emptyset$ *and* $\inf S_{\widehat{u}^+}$ *otherwise, where*

$$S_{\widehat{u}^+}\doteq\{\theta\in[\theta_0,\theta_1]:\widehat{\mathcal{P}}^-(\theta,\xi_2)=\widehat{\mathcal{P}}^+(\theta,\xi_1)\}\,. \tag{6.5}$$

PROOF. Fix $\varepsilon>0$ arbitrarily. Then the first of the relations

$$\varepsilon+T_{\xi_1}-T_{\xi_2}\geqslant\int_{\theta_0}^{\theta_1}\Im[\widehat{u}_1(\theta)][\widehat{\mathcal{P}}(\theta,\xi_1,\widehat{u}_1)^{-1/2}-\widehat{\mathcal{P}}(\theta,\xi_2,\widehat{u}_1)^{-1/2}]\,d\theta\geqslant \tag{6.6}$$

$$\geqslant\min\Im(U)\int_{\theta_0}^{\theta^*}[\widehat{\mathcal{P}}^+(\theta,\xi_1)^{-1/2}-\widehat{\mathcal{P}}^-(\theta,\xi_2)^{-1/2}]\,d\theta$$

holds for some $\widehat{u}_1\in\widehat{\mathcal{U}}^*_{\xi_1}$ (and hence $\in\widehat{\mathcal{U}}^*_{\xi_2}$). Furthermore, θ^*'s definition yields

$$\widehat{\mathcal{P}}(\theta,\xi_1,\widehat{u}_1)\leqslant\widehat{\mathcal{P}}^+(\theta,\xi_1)\leqslant\widehat{\mathcal{P}}^-(\theta,\xi_2)\quad\forall\theta\in[\theta_0,\theta^*]\,. \tag{6.7}$$

Furthermore by $(6.3)_1$ the first integrand in (6.6) is positive. Then $(6.6)_2$ too holds. By ε's arbitrariness, (6.6) implies (6.4). □

Now assertion ($\mathfrak{A}$) above (6.2) can be strengthened as follows.

($\mathfrak{B}$) *If the skis-skier system* Σ *describes the whole trajectory l in the minimum time (under given initial conditions), then along any ascending circle arc belonging to l the skier must behave according to assertion* ($\mathfrak{A}$).

Indeed, let $[s_2,s_3]\subset[s_0,s_1]$ be an ascending circle arc and let $u\in\mathcal{U}^*$ be an optimal control for the minimum time problem. If $\xi_1\doteq\mathcal{P}(\bar{s},u)<\xi_2\doteq\mathcal{P}^+(\bar{s})$ for some $\bar{s}\in(s_2,s_3)$, then by Theor. 6.1 it follows that

$$\int_{s_2}^{\bar{s}} \Im[s, u(s)]\, \mathcal{P}(s, u)^{-1/2}\, ds \geq \int_{s_2}^{\bar{s}} \Im[s, u^+(s)]\, \mathcal{P}^+(s, \mathcal{P}(s_2, u), s_2)^{-1/2}\, ds,$$

while Theor. 6.2, which in particular states that the function $\xi \mapsto T_\xi[u(\cdot)]$ is strictly decreasing, implies that

$$\int_{\bar{s}}^{s_3} \Im[s, u(s)]\, \mathcal{P}(s, u)^{-1/2}\, ds > \int_{\bar{s}}^{s_3} \Im[s, u^+(s)]\, \mathcal{P}^+(s, \mathcal{P}(s_2, u), s_2)^{-1/2}\, ds.$$

Hence if we consider the control $\tilde{u} \in \hat{\mathcal{U}}^*$ that equals u on $[s_0, s_2] \cup [s_3, s_1]$ and u^+ on (s_2, s_3), one has

$$T[\mathcal{P}(\cdot, \tilde{u}), \tilde{u}(\cdot)] < T[\mathcal{P}(\cdot, u), u(\cdot)],$$

in contradiction to the optimality of u. Thus assertion $(\mathcal{B})$ is proved. □

7. Existence theorem for the skis-skier problem of minimum time

As a preliminary, we prove the following lemma.

Lemma 7.1. *Given* $\xi' \geq 0$ *and* $s', \bar{s} \in \Delta$, *let* $\{\mathcal{P}_r(\cdot)\}_{n \in N}$ *be a sequence with* $\mathcal{P}_r(\cdot) \doteq \mathcal{P}(\cdot, \xi', s', u_r)$ *and* $u_r \in \mathcal{U}$ $\forall r \in N$. *This implies (a) below.*

(*a*) *If* $(i)^{\pm}$ $\Delta^{\pm} \neq \emptyset$ *with* $\Delta^{\pm} \doteq \{s \gtrless \bar{s}\} \cap \Delta$ *and*

$$(7.1)^{\pm} \qquad \lim_{r \to \infty} \mathcal{P}_r(\bar{s}) = 0, \qquad y'(\bar{s}) \gtrless 0$$

hold, then there exist some $\tilde{s} \in \Delta^{\pm}$ *and* $\tilde{r} \in N$ *such that* $\mathcal{P}_r(\tilde{s}) < 0$ *for all* $r \geq \tilde{r}$.

(*b*) *If the sequence* $\{\mathcal{P}_r(\cdot)\}_{n \in N}$ *satisfies* $(7.1)_1^{\mp}$ *and* $\mathcal{P}_r(s) \geq 0$ $\forall s \in \Delta$ *and* $\forall r \in N$, *one has*

$$(7.2) \qquad y'(\bar{s}) = 0 \quad \text{if } \bar{s} \in (s_0, s_1); \qquad y'(\bar{s}) \leq 0 \quad \text{if } \bar{s} = s_0; \qquad y'(\bar{s}) \geq 0 \quad \text{if } \bar{s} = s_1.$$

(*c*) *Assume that* (*ii*) $u_r \in \mathcal{U}^*_{\xi', s'}$ $\forall r \in N$, *and for some* $\bar{t} > 0$

$$(7.3) \qquad \int_{s_0}^{s_1} \Im[s, u_r(s)]\, \mathcal{P}_r(s)^{-1/2}\, ds \leq \bar{t} \qquad \forall r \in N.$$

Then the sequence $\{\mathcal{P}_r(\cdot)\}_{r \in N}$ *satisfies*

$$(7.4) \qquad \inf_{r \in N} \mathcal{P}_r(\bar{s}) > 0 \text{ when } \bar{s} \in (s_0, s_1) \vee (\bar{s} = s_0 \wedge y'(s_0) \geq 0) \vee (\bar{s} = s_1 \wedge y'(s_1) \leq 0).$$

Proof. Assume $(i)^{\pm}$. Then for some $\eta > 0$

$$(7.5) \qquad M^{\pm} \doteq \min\{2mg\Im(s, u)\xi |y'(s)| : s \in I_\eta^{\pm}, u \in U\} > 0 \qquad \text{where } \emptyset \neq I_\eta^{\pm} \doteq [\bar{s} \pm \eta, \bar{s}) \subset \Delta.$$

For $\mathcal{P} = \mathcal{P}_r(s)$, $u = u_r$ and $s \in I_\eta^\pm$, $(2.12)_1$ yields the first of the relations

$$(7.6)\qquad \mathcal{P}_r(s) = \mathcal{P}_r(\bar{s}) \exp \int_{\bar{s}}^{s} (\mathfrak{I}_s/\mathfrak{I})[\sigma, u_r(\sigma)]\, d\sigma +$$

$$-2mg \int_{\bar{s}}^{s} \xi y'(\sigma)\, \mathfrak{I}[\sigma, u_r(\sigma)] \exp\left\{ \int_{\sigma}^{s} (\mathfrak{I}_s/\mathfrak{I})[\sigma', u_r(\sigma')]\, d\sigma' \right\} d\sigma \leqslant$$

$$\leqslant \mathcal{P}_r(\bar{s})\, e^{\operatorname{sign}(\mathcal{P}_r(\bar{s}))\, \mathfrak{M} |s - \bar{s}|} - M^{\pm} \int_{\min\{\bar{s}, s\}}^{\max\{\bar{s}, s\}} e^{-\mathfrak{M}|s - \sigma|}\, d\sigma =$$

$$= \mathcal{P}_r(\bar{s})\, e^{\operatorname{sign}(\mathcal{P}_r(\bar{s}))\, \mathfrak{M} |s - \bar{s}|} - \frac{M^{\pm}}{\mathfrak{M}} (1 - e^{-\mathfrak{M}|s - \bar{s}|}) < 0,$$

where $\operatorname{sign} \alpha \doteq \alpha / |\alpha|$ if $\alpha \in \boldsymbol{R} \backslash \{0\}$ and $\operatorname{sign} \alpha \doteq 0$ if $\alpha = 0$. Then, as it is easy to check, $(7.6)_{2\text{-}3}$ hold under the definition (4)

$$(7.7)\qquad \mathfrak{M} \doteq \max\{ |\mathfrak{I}_s/\mathfrak{I}|(\sigma, v) \colon \sigma \in \Delta, v \in U\} + 1.$$

Lastly, by $(7.1)_1^\mp$, for any fixed $s \in I_\eta^\pm$ some sufficiently large r also satisfies condition $(7.6)_4$. Therefore part (a) is proved.

If $\bar{s} \in (s_0, s_1)$ we have both of the conditions $(iii)^\pm$ ($\Delta^\pm \neq \emptyset$ and) $I_\eta^\pm \neq \emptyset$. Hence part (a) yields $(7.2)_1$. Otherwise, if $\bar{s} = s_0$ $[\bar{s} = s_1]$ only condition $(iii)^+$ $[(iii)^-]$ is satisfied so that part (a) yields $(7.2)_2$ $[(7.2)_3]$. Thus part (b) is also proved.

Assume condition (ii) above (7.3) and suppose the falsity of $(7.4)_1$. Then, for some $\bar{s} \in \Delta$ such that either $\bar{s} \in (s_0, s_1)$, or both $\bar{s} = s_0$ and $y'(s_0) \geqslant 0$, or else both $\bar{s} = s_1$ and $y'(s_1) \leqslant 0$, and for some subsequence of $\{\mathcal{P}_r(\cdot)\}_{r \in N}$, which we identify with $\{\mathcal{P}_r(\cdot)\}_{r \in N}$ for the sake of simplicity, $(7.1)_1^\pm$ is satisfied. Furthermore, (7.3) implies that $\mathcal{P}_r(s) \geqslant 0$ $\forall s \in \Delta$ and $\forall r \in \boldsymbol{N}$. Hence condition (7.2) holds, and together with $(2.12)_2$ it yields

$$(7.8)\qquad |-2mg\xi y'(s)\, \mathfrak{I}(s, u)| = |G(s, 0, v)| \leqslant \mathfrak{N}|\bar{s} - s| \quad \forall (s, v) \in \Delta \times U,$$

$$\text{where } \mathfrak{N} \doteq \max\{ |G_s(s, 0, v)| \colon (s, v) \in \Delta \times U\} + 1.$$

(4) Note that $\forall \alpha, \beta \in \boldsymbol{R}$ and $r \in \boldsymbol{N}$

$$\exp\{-\mathfrak{M}|\alpha - \beta|\} \leqslant \exp\left\{ \int_{\alpha}^{\beta} \mathfrak{I}_s/\mathfrak{I}[s, u_r(s)]\, ds \right\} \leqslant \exp\{\mathfrak{M}|\alpha - \beta|\}.$$

Hence, for $s \in I_\eta^\pm$, by $(i)^\pm$ and (7.5) one has

$$\int_{\bar{s}}^{s} 2mg\xi y'(\sigma)\, \mathfrak{I}[\sigma, u_r(\sigma)] \exp\left\{ \int_{\sigma}^{s} (\mathfrak{I}_s/\mathfrak{I})[\sigma', u_r(\sigma')]\, d\sigma' \right\} d\sigma = \int_{\min\{\bar{s}, s\}}^{\max\{\bar{s}, s\}} 2mg\xi |y'(\sigma)|\, \mathfrak{I}[\sigma, u_r(\sigma)] \cdot$$

$$\cdot \exp\left\{ \int_{\sigma}^{s} (\mathfrak{I}_s/\mathfrak{I})[\sigma', u_r(\sigma')]\, d\sigma' \right\} d\sigma \geqslant M^{\pm} \int_{\min\{\bar{s}, s\}}^{\max\{\bar{s}, s\}} \exp\{-\mathfrak{M}|\sigma - s|\}\, d\sigma.$$

Furthermore by $\mathcal{P}^{\pm}(\cdot, \xi', s')$'s definition above (2.18) with s' replaced by $\bar{s}$, for all $u \in \mathcal{U}^{*}_{\xi', \bar{s}}$ the first of the relations

$$(7.9) \qquad \mathcal{P}(s, \xi', \bar{s}, u) = \xi' \exp\left\{\int_{\bar{s}}^{s} (\mathfrak{I}_s/\mathfrak{I})[\sigma, u(\sigma)]\, d\sigma\right\} + $$

$$+ \int_{\bar{s}}^{s} G[\sigma, 0, u(\sigma)] \exp\left\{\int_{\sigma}^{s} (\mathfrak{I}_s/\mathfrak{I})[\tau, u(\tau)]\, d\tau\right\} d\sigma \leqslant$$

$$\leqslant \xi' e^{\mathcal{M}|s-\bar{s}|} + \left|\int_{\bar{s}}^{s} \mathcal{N}|\sigma - \bar{s}| e^{\mathcal{M}|s-\bar{s}|}\, d\sigma\right| = \xi' e^{\mathcal{M}|s-\bar{s}|} + \mathcal{N} f(|s-\bar{s}|)$$

holds $\forall s \in \Delta$ and $\forall \xi' \geqslant 0$. The second follows from (7.6) and $(7.8)_1$. We set $t = |s - \bar{s}|$ and

$$(7.10) \qquad f(t) \doteq \mathcal{M}^{-2}[e^{\mathcal{M}t} - 1 - \mathcal{M}t], \qquad g(t, \xi') \doteq \xi' e^{\mathcal{M}t} + \mathcal{N} f(t).$$

Then $(7.9)_3$ is easily checked.

By (7.10) $f(0) = 0 = f'(0)$ and $f''(0) = 1$ so that, for any $t'' > 0$, $\int_0^{t''} f(t)^{-1/2} dt = +\infty$. Hence for some $t' \in (0, t'')$, (iv) $(2\mathcal{N})^{-1/2} \int_{t'}^{t''} f(t)^{-1/2} dt > \bar{t}/\mathcal{M}^-$, where $\mathcal{M}^- \doteq \min\{\mathfrak{I}(s, u) : (s, u) \in \Delta \times U\}$ (> 0). In addition, by setting $\bar{\xi} \doteq \mathcal{N} f(t') e^{-\mathcal{M}t''}$ (> 0), $g(t, \bar{\xi}) \leqslant 2\mathcal{N} f(t)$ $\forall t \in [t', t'']$. Hence by (iv)

$$(7.11) \qquad \int_{t'}^{t''} g(t, \bar{\xi})^{-1/2} dt \geqslant (2\mathcal{N})^{-1/2} \int_{t'}^{t''} f(t)^{-1/2} dt > \bar{t}/\mathcal{M}^-.$$

By $(7.1)_1^{\mp}$ for some $\bar{r} \in \boldsymbol{N}$ (v) $0 \leqslant \mathcal{P}_r(\bar{s}) < \bar{\xi}$ $\forall r \geqslant \bar{r}$. Hence by (7.9)-(7-10)

$$(7.12) \qquad 0 \leqslant \mathcal{P}_r(s) = \mathcal{P}(s, \mathcal{P}_r(\bar{s}), \bar{s}, u_r) \leqslant g(|\bar{s} - s|, \bar{\xi}) \quad \forall s \in \Delta.$$

Then, by setting

$$(7.13) \qquad s' \doteq \begin{cases} \bar{s} + t', \\ \bar{s} - t'', \end{cases} \quad s'' \doteq \begin{cases} \bar{s} + t'' \\ \bar{s} - t' \end{cases} \quad \text{with } s_0 < s' < s'' < s_1,$$

the first of the relations

$$(7.14) \qquad \int_{s_0}^{s_1} \mathfrak{I}[\sigma, u_r(\sigma)]\, \mathcal{P}_r(\sigma)^{-1/2} d\sigma > \int_{s'}^{s''} \mathfrak{I}[\sigma, u_r(\sigma)]\, \mathcal{P}_r(\sigma)^{-1/2} d\sigma \geqslant$$

$$\geqslant \mathcal{M}^- \int_{s'}^{s''} \mathcal{P}_r(\sigma)^{-1/2} d\sigma \geqslant \mathcal{M}^- \int_{t'}^{t''} g(t, \bar{\xi})^{-1/2} dt > \bar{t}$$

holds. Furthermore the definition of $\mathcal{M}^-$ above (7.11) yields $(7.14)_2$, while $(7.14)_{3\text{-}4}$ follow from (7.11)-(7.12). Since (7.14) contrasts to assumption (7.3), we conclude that (7.4) holds. Thus part (c) holds and Lemma 7.1 is proved. □

THEOREM 7.1. *Let* $\{u_r\}_{r\in N}\subset \mathcal{U}^*$ *be a minimizing control sequence for problem* $(\mathcal{P})$ *above* (2.15) (*on* Σ_1). *Then* (*a*) *for any* $\varepsilon\in(0,(s_1-s_0)/2)$, *setting* $\mathcal{P}_r(\cdot)\doteq\mathcal{P}(\cdot,u_r)$ $(r\in N)$, *there are some* $\delta=\delta(\varepsilon)>0$ *and* $\bar r=\bar r(\varepsilon)\in N$ *such that in case* $y'(s_1)>0$ $[y'(s_1)\leqslant 0]$

$$\text{(7.15)}\qquad \begin{cases}\mathcal{P}_r(s)\geqslant\delta\quad\forall r\geqslant\bar r,\quad \forall s\in\Delta_\varepsilon\doteq[s_0^\varepsilon,s_1^\varepsilon]\quad \textit{where}\\ s_0^\varepsilon\doteq\begin{cases}s_0 & \textit{if } y'(s_0)\geqslant 0,\\ s_0+\varepsilon & \textit{if } y'(s_0)<0,\end{cases}\qquad s_1^\varepsilon\doteq\begin{cases}s_1 & \textit{if } y'(s_1)\leqslant 0,\\ s_1-\varepsilon & \textit{if } y'(s_1)>0;\end{cases}\end{cases}$$

(*b*) *under the above assumptions there is some* $\mathcal{P}(\cdot)\in AC(\Delta)$ *such that* $\mathcal{P}(\cdot)^{-1/2}\in\in\mathcal{L}^1(\Delta)$, *and moreover*

$$\text{(7.16)}\qquad \|\mathcal{P}(\cdot)-\mathcal{P}_r(\cdot)\|_0\doteq\sup\{|\mathcal{P}(s)-\mathcal{P}_r(s)|:s\in\Delta\}\to 0\quad \textit{as } r\to\infty\,,$$

up to passing to subsequences; and the following holds.

(*c*) *Assume further the convexity of the sets* $F(s,\mathcal{P})$ – *see* (3.1). *Then, for some* $u=u(\cdot)\in\mathcal{U}^*$, $\mathcal{P}(\cdot)=\mathcal{P}(\cdot,u)$ *and the process* $(\mathcal{P}(\cdot),u(\cdot))$ *solves problem* $(\mathcal{P})$ *above* (2.15).

PROOF. Suppose that part (*a*) fails. Then, for some $\varepsilon>0$, some sequence $s_r\in\Delta_\varepsilon$, some $\bar s\in\Delta_\varepsilon$, and some subsequence of $\{u_r\}_{r\in N}$ which is not restrictive to identify with $\{u_r\}_{r\in N}$, the first two of the equalities

$$\text{(7.17)}\qquad \lim_{r\to\infty}\mathcal{P}_r(s_r)=0\,,\quad \lim_{r\to\infty}s_r=\bar s\,,\quad \lim_{r\to\infty}\mathcal{P}_r(\bar s)=0\,,$$

hold. They imply the third because

$$\text{(7.18)}\qquad \mathcal{P}^\pm[\bar s,\mathcal{P}_r(s_r),s_r]\geqslant\mathcal{P}_r(\bar s)\quad\text{for } s_r\begin{cases}\leqslant\\ \geqslant\end{cases}\bar s;\quad \lim_{r\to\infty}\mathcal{P}^\pm[\bar s,\mathcal{P}_r(s_r),s_r]=0\,.$$

Since $\{u_r\}_{r\in N}$ is a minimizing sequence for problem $(\mathcal{P})$, for some $\bar t>0$ condition (7.3) is satisfied. Then by part (*c*) of Lemma 7.1 above (7.3), $(7.17)_3$ contradicts $(7.4)_1$ and the thesis in part (*a*) holds.

To prove part (*b*) we first note that if $y'(s_0)<0$, for some $\varepsilon_0\in(0,(s_1-s_0)/2)$ and $C_0>0$ one has (5)

$$\text{(7.19)}\qquad \mathcal{P}_r(s)\geqslant\mathcal{P}^-(s)\geqslant C_0[\mathcal{P}_0+(s-s_0)]\quad\forall s\in(s_0,s_0+\varepsilon_0]\,,\qquad(\mathcal{P}_0\geqslant 0, r\in N)\,.$$

Hence the restriction of $s\mapsto\mathcal{P}^-(s)^{-1/2}$ to $[s_0,s_0+\varepsilon_0]$ is in $\mathcal{L}^1$.

(5) Indeed, let $\varepsilon_0>0$ be such that $y'(s)<0$ $\forall s\in[s_0,s_0+\varepsilon_0]$. Then together with $(2.12)_1$ it yields the first two of the relations – see also (7.5) and ftn. (4) with $\bar s$ and η replaced by s_0 and ε_0 respectively

$$\mathcal{P}^-(s)=\mathcal{P}_0\exp\int_{s_0}^{s}(\mathfrak{J}_s/\mathfrak{J})[\sigma,u^-(\sigma)]\,d\sigma+\int_{s_0}^{s}2mg\mathfrak{J}[\sigma,u^-(\sigma)]\,\xi|y'(\sigma)|\cdot$$

$$\cdot\exp\left\{\int_{\sigma}^{s}(\mathfrak{J}_s/\mathfrak{J})[\sigma',u^-(\sigma')]\,d\sigma'\right\}d\sigma\geqslant\mathcal{P}_0e^{-\mathfrak{M}(s_1-s_0)}+M^+\int_{s_0}^{s}e^{-\mathfrak{M}(s_1-s_0)}d\sigma\geqslant C_0(\mathcal{P}_0+(s-s_0))\,;$$

the third holds under the definition $C_0\doteq\min\{e^{-\mathfrak{M}(s_1-s_0)},M^+e^{-\mathfrak{M}(s_1-s_0)}\}$. Condition $(7.21)_2$ can be proved in the same way.

Furthermore, by setting

$$l \doteq \inf_{r \in N} \mathcal{P}_r(s_1) \quad (\geqslant 0), \tag{7.20}$$

if $y'(s_1) > 0$, then there are some $\varepsilon_1 \in (0, (s_1 - s_0)/2)$ and $C_1 > 0$ such that

$$\mathcal{P}_r(s) \geqslant \mathcal{P}^+(s, l, s_1) \geqslant c_1[l + (s_1 - s)] \quad \forall s \in [s_1 - \varepsilon_1, s_1], \qquad (r \in N). \tag{7.21}$$

Hence also the restriction of $s \mapsto \mathcal{P}^+(s, l, s_1)^{-1/2}$ to $[s_1 - \varepsilon_1, s_1]$ is in $\mathcal{L}^1$.

By the regularity properties of the Cauchy problem $(2.12)_1 \cup (2.14)_2$, the sequence $\{\mathcal{P}_r(\cdot)\}_{r \in N}$ is uniformly bounded and uniformly Lipschitz continuous. Hence, by Ascoli-Arzelà's theorem, passing to subsequences, it is not restrictive to assume that (7.16) holds for some $\mathcal{P}(\cdot) \in AC(\Delta)$. Furthermore, by (7.15), (7.19), and (7.21) one can deduce that $\mathcal{P}(\cdot)^{-1/2}$ exists and is in $\mathcal{L}^1(\Delta)$. Thus thesis (*b*) holds.

To prove part (*c*) we fix any $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\})$. Then by (7.15) possibly with $\mathcal{P}_r(\cdot)$ replaced by $\mathcal{P}(\cdot)$, the R.H.S. of $(2.12)_1$ is regular enough on Δ_ε, to enable us to apply Filippov's theorem – see *e.g.* [16]. Since $F(s, \mathcal{P})$ – see (3.1) – is convex for $(s, \mathcal{P}) \in \Delta \times (0, +\infty)$, for some $u_0 \in \mathcal{U}^*$ the trajectory $\mathcal{P}(\cdot)$ solves on Δ_ε the ODE $(2.12)_1$ for $u = u_0$. By setting $\varepsilon_h = 2^{-h}\varepsilon$, calling u_h the analogue of u_0 for ε_h, and setting $u(s) = u_0(s)$ on Δ_ε and $u(s) = u_h(s)$ on $[s_0 + \varepsilon_h, s_1 - \varepsilon_h] \backslash [s_0 + \varepsilon_{h-1}, s_1 - \varepsilon_{h-1}]$ ($h \in N^*$), we easily see that the process $(\mathcal{P}(\cdot, u), u(\cdot))$ solves the original problem $(\mathcal{P})$ above (2.15). Thus thesis (*c*) too is proved. □

Remark 7.1. Part (*c*) of Lemma 7.1 and (7.19), (7.21) in Theor. 7.1 generalize the result below, still obtained in [6] in case Σ schemetizes a revolutory swing.

Given $\xi \geqslant 0$ *and* $s' \in \Delta$, *let* $u \in \mathcal{U}$. *Then* $u \in \mathcal{U}^*_{\xi, s'}$ – *see* (2.17) – *iff*

$$\begin{cases} \mathcal{P}(s, \xi, s', u) > 0 \quad \forall s \in (s_0, s_1), \\ \mathcal{P}(s_0, \xi, s', u) > 0 \vee y'(s_0) < 0, \quad \mathcal{P}(s_1, \xi, s', u) > 0 \vee y'(s_1) > 0. \end{cases} \tag{7.22}$$

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Dipartimento di Matematica Pura e Applicata
Università degli Studi di Padova
Via Belzoni, 7 - 35131 Padova