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Remarks on positive solutions to a semilinear Neumann problem

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Equazioni a derivate parziali. — *Remarks on positive solutions to a semilinear Neumann problem.* Nota di ANNA MARIA CANDELA e MONICA LAZZO, presentata (*) dal Corrisp. A. Ambrosetti.

ABSTRACT. — In this paper we study the influence of the domain topology on the multiplicity of solutions to a semilinear Neumann problem. In particular, we show that the number of positive solutions is stable under small perturbations of the domain.

KEY WORDS: Neumann problem; Variational methods; Multiple solutions.

RIASSUNTO. — *Osservazioni sull'esistenza di soluzioni positive di un problema di Neumann semilineare.* In questo lavoro studiamo l'influenza della topologia del dominio sul numero delle soluzioni di un problema di Neumann semilineare. In particolare, mostriamo che il numero delle soluzioni positive è stabile per piccole perturbazioni del dominio.

1. INTRODUCTION AND STATEMENT OF THE RESULT

In last years, there has been an increasing interest in studying non constant solutions of the Neumann problem

$$(P_d) \qquad \begin{cases} -d\Delta u + u = |u|^{p-2}u & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a bounded smooth domain in $\boldsymbol{R}^N$ $(N \geq 3)$, d is a positive constant, ν is the unit outer normal to $\partial\Omega$ and $2 < p < 2N/(N-2)$. This problem is a simpler version of the system proposed by Gierer and Meinhardt as a model of biological pattern formation, where d plays as a diffusion coefficient (cf. [7]).

An existence result for (P_d) is proved in [10, 12], where it is shown that it has at least a non constant solution for d sufficiently small, and it has no such solution for d large. Lately, motivated by similar results concerning a Dirichlet problem (for instance, see [1-3] and references therein), several authors have been studying the relations between the multiplicity of solutions to (P_d) and the topology of the boundary of the domain Ω (cf. [9, 11, 13]). Roughly speaking, in all these papers it is shown that the number of solutions to (P_d) is affected by the topological «richness» of $\partial\Omega$.

In this paper, inspired by [3], we aim to go further in this direction, investigating the stability of the number of solutions under perturbations of the domain Ω. More precisely, as an application of Theorem 1.1 below, we prove that there can be many solutions even in a topologically trivial domain Ω, provided Ω is obtained by adding a «small» set to a «rich» domain.

(*) Nella seduta del 12 marzo 1994.

To clarify what we mean by «small» set, we introduce a function μ (cf. [3]).

Definition 1.1. *Let* L, $\Omega \subset \boldsymbol{R}^N$ *be two bounded domains and*

$$V_{L,\Omega} = \left\{ u \in H^1(\Omega \cup L) \colon \int_L (u^+)^p dx = 1 \right\}$$

(as usual, $u^+ = \max\{u, 0\}$*). We define*

$$\mu(L,\Omega) = \begin{cases} \left(\inf\limits_{u \in V_{L,\Omega}} \int\limits_{\Omega \cup L} (|\nabla u|^2 + u^2)\, dx \right)^{-1} & \text{if } V_{L,\Omega} \neq \emptyset\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Remark. The function μ defined above is slightly different from the one introduced in [3]. In fact, as we deal with Neumann boundary conditions, we need to add the term u^2 into the integral in order that μ is well defined.

The following lemma states that small values of $\mu(L,\Omega)$ yield «small» L. From now on, we set $B_R = \{x \in \boldsymbol{R}^N \colon |x|_{\boldsymbol{R}^N} < R\}$ for any $R > 0$.

Lemma 1.2. *Let* $\Omega \subset \boldsymbol{R}^N$ *be a bounded domain and let* $R > 0$ *be such that* $\Omega \subset B_R$. *Let* $\mathcal{L} = \{L \subset B_R \colon |L| > 0\}$. *Then: for any* $L \in \mathcal{L}$ *there results* $\mu(L,\Omega) > 0$ *and*

$$\lim_{\substack{\mu(L,\Omega) \to 0 \\ L \in \mathcal{L}}} |L| = 0\,,$$

where $|\cdot|$ *is the Lebesgue measure in* $\boldsymbol{R}^N$.

Proof. For any $L \in \mathcal{L}$, let $\bar{u} \equiv |L|^{-1/p}$; plainly, $\bar{u} \in V_{L,\Omega}$, hence

$$\int\limits_{\Omega \cup L} (|\nabla \bar{u}|^2 + \bar{u}^2)\, dx = \frac{|\Omega \cup L|}{|L|^{2/p}} \Rightarrow \frac{|L|^{2/p}}{|\Omega \cup L|} \leqslant \mu(L,\Omega)\,;$$

therefore $L \subset B_R$ implies $|L|^{2/p} \leqslant |B_R|\, \mu(L,\Omega)$, which proves our claim. □

We recall some notation: for any Λ, $\Gamma \subset \boldsymbol{R}^N$, $\Lambda \subset \Gamma$, $\mathrm{cat}_\Gamma \Lambda$ is the Ljusternik-Schnirelman category of Λ in Γ, that is, the least number of closed and contractible sets in Γ which cover Λ; $\mathrm{cat}\, \Lambda$ is the Ljusternik-Schnirelman category of Λ in itself.

Now we can state the main result of this paper.

Theorem 1.3. *Let* $\Omega \subset \boldsymbol{R}^N$ $(N \geqslant 3)$ *be a bounded smooth domain and* $R > 0$ *be such that* $\Omega \subset B_R$. *For* $d > 0$ *sufficiently small there exists* $\mu^* > 0$ *such that, if* L *is a subset of* B_R, $\Omega \cup L$ *is smooth and* $\mu(L,\Omega) < \mu^*$, *then problem* (P_d) *has at least* $\mathrm{cat}\, \partial\Omega + 1$ *non constant solutions in* $\mathcal{D} = \Omega \cup L$.

Remark. By definition, $m(\emptyset, \Omega) = 0$; then Theorem 1.3 implies the following result, which includes [11, 13].

Corollary 1.4. *Let* $\Omega \subset \boldsymbol{R}^N$ $(N \geqslant 3)$ *be a bounded smooth domain. For* $d > 0$ *sufficiently small, problem* (P_d) *has at least* $\mathrm{cat}\, \partial\Omega + 1$ *non constant solutions in* Ω.

EXAMPLE. Assume Ω is the union of two disjoint balls B_1 and B_2 in $\boldsymbol{R}^N$; by Theorem 1.3, if d is small enough and L is a «thin» handle joining B_1 and B_2, then problem (P_d) has at least cat $\partial\Omega + 1 = 5$ non trivial solutions. We remark that, as $\Omega \cup L$ is contractible, multiplicity results in [11, 13] would provide cat $\partial(\Omega \cup L) = 2$ as a lower bound to the number of non constant solutions.

2. THE VARIATIONAL SETTING

Let $H^1(\mathcal{O})$ be the standard Sobolev space endowed with the norm

$$\|u\| = \left(\int_{\mathcal{O}} (|\nabla u|^2 + u^2)\, dx \right)^{1/2}.$$

We define the functional

$$E_{d,\mathcal{O}}(u) = \int_{\mathcal{O}} (d|\nabla u|^2 + u^2)\, dx\,, \qquad u \in H^1(\mathcal{O})$$

and the set

$$V(\mathcal{O}) = \left\{ u \in H^1(\mathcal{O}) \colon \int_{\mathcal{O}} (u^+)^p dx = 1 \right\}.$$

The following lemma can be easily proved.

LEMMA 2.1. *i)* $V(\mathcal{O})$ *is a* C^2 *manifold in* $H^1(\mathcal{O})$, i.e.: *for any* $u \in H^1(\mathcal{O})$ *such that* $g(u) = \int_{\mathcal{O}} (u^+)^p dx - 1 = 0$, *there results* $g'(u) \neq 0$;

ii) $E_{d,\mathcal{O}}$ *is a* C^2 *functional on* $V(\mathcal{O})$;

iii) $E_{d,\mathcal{O}}$ *satisfies* (PS) *condition on* $V(\mathcal{O})$, *that is: any sequence* $\{u_n\} \subset V(\mathcal{O})$, *such that* $\{E_{d,\mathcal{O}}(u_n)\}$ *is bounded and* $E'_{d,\mathcal{O}}(u_n)$ *goes to* 0 *in* $H^{-1}(\mathcal{O})$ *as* $n \to \infty$, *is relatively compact*;

iv) if $u \in H^1(\mathcal{O})$ *is a critical point of* $E_{d,\mathcal{O}}$ *constrained on* $V(\mathcal{O})$, *then the function* $v = ku$ *is a solution of* (P_d), *where* $k = (E_{d,\mathcal{O}}(u))^{1/(p-2)}$.

By Lemma 2.1, looking for solutions of (P_d) corresponds to looking for critical points of $E_{d,\mathcal{O}}$ on the manifold $V(\mathcal{O})$. To this aim, we will use an abstract citical point theorem which is obtained by simple changes in Theorem 3.1 of [3].

THEOREM 2.2. *Let* E *be a* C^2 *functional on a* C^2 *manifold* V *such that* E *is bounded from below and satisfies* (PS) *condition on* V. *Let* $\Lambda \subset \boldsymbol{R}^N$ *be bounded. Assume that there exist a closed set* Λ^+ *including* Λ, *a real* m *and two continuous maps*

$$\Phi \colon \Lambda \to \{u \in V \colon E(u) \leq m\}, \qquad \beta \colon \{u \in V \colon E(u) \leq m\} \to \Lambda^+$$

such that Λ^+ *is homotopically equivalent to* Λ *and* $\beta \circ \Phi$ *is homotopically equivalent to the embedding* $j \colon \Lambda \to \Lambda^+$. *Then*: E *has at least* cat Λ *critical points constrained on* V.

PROOF. We denote $E^m = \{u \in V \colon E(u) \leq m\}$. First of all we prove that $\operatorname{cat} E^m \geq \operatorname{cat} \Lambda$. Let $\operatorname{cat} E^m = n$, thus n is the least integer such that $E^m \subseteq A_1 \cup \ldots \cup A_n$, where each A_i is closed and contractible in E^m. For any $i = 1, \dots, n$, if we set $K_i = \Phi^{-1}(A_i) \subset \Lambda$, then K_i is closed and there results

$$\operatorname{cat}_{\Lambda^+} \Lambda \leq \sum_{i=1}^{n} \operatorname{cat}_{\Lambda^+} K_i \,. \tag{2.1}$$

We prove that for any $i = 1, \dots, n$ the set K_i is contractible in Λ^+. Since each A_i is contractible in E^m, there exist $H_i : [0, 1] \times A_i \to E^m$ and $\omega_i \in E^m$ such that

$$\begin{cases} H_i(0, u) = u & \text{for any } u \in A_i \,, \\ H_i(1, u) = w_i & \text{for any } u \in A_i \,. \end{cases}$$

By the hypothesis it follows that for any $t \in [0, 1]$ the map

$$G_i(t, \cdot) = \beta \circ H_i(t, \cdot) \circ \Phi : K_i \to \Lambda^+$$

is an homotopy between $\beta \circ \Phi$ and one point in Λ^+; moreover the imbedding map of K_i in Λ^+ is homotopically equivalent to $\beta \circ \Phi|_{K_i}$ in Λ^+, hence $\operatorname{cat}_{\Lambda^+} K_i \leq 1$ and, by (2.1), $\operatorname{cat} \Lambda = \operatorname{cat}_{\Lambda^+} \Lambda \leq n$.

As E is bounded from below and satisfies (PS) condition, by standard Ljusternik-Schnirelman arguments we deduce the existence of at least cat Λ distinct critical points of E in the sublevel E^m. □

3. The maps Φ and β

In this section we define two maps candidate to fulfil the requirements in Theorem 2.2.

The map β_{Ω}.

For $u \in V(\Omega)$, we define the mass center of u:

$$\beta_{\Omega}(u) = \int_{\Omega} (u^+)^p x \, dx \,;$$

plainly, β_{Ω} is a continuous map from $V(\Omega)$ to $\boldsymbol{R}^N$.

To introduce the second map Φ, we need some facts about solutions of problem (P_1) in $\boldsymbol{R}^N$; for the proofs, see [4-6, 8].

Proposition 3.1. *The equation: $-\Delta u + u = |u|^{p-2} u$ in $\boldsymbol{R}^N$ has (up to translations) a unique solution ω satisfying*

i) $\omega \in C^2(\boldsymbol{R}^N) \cap H^1(\boldsymbol{R}^N)$, $\omega > 0$ in $\boldsymbol{R}^N$;

ii) ω is radially symmetric and decreasing;

iii) ω and its first derivatives decay esponentially at infinity, i.e. *there exist $C, \lambda > 0$ such that $|D^k \omega(z)| \leq C e^{-\lambda |z|}$ for $z \in \boldsymbol{R}^N$ with $|k| \leq 1$.*

The map $\Phi_{d, \mathcal{O}}$.

Let $r > 0$ be such that the set $\partial\Omega^+ = \{x \in \boldsymbol{R}^N : \operatorname{dist}(x, \partial\Omega) \leq r\}$ is homotopically equivalent to $\partial\Omega$.

Let η be a smooth nonincreasing function defined on $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leq t \leq 1/2$, $\eta(t) = 0$ if $t \geq 1$ and η' is bounded.

For any $y \in \boldsymbol{R}^N$ and for $x \in \mathcal{O}$, we set

$$\phi_{d, \mathcal{O}}(y)(x) = \eta\left(\frac{|x-y|_{\boldsymbol{R}^N}}{r}\right)\omega\left(\frac{x-y}{\sqrt{d}}\right)$$

and

$$\Phi_{d, \mathcal{O}}(y)(x) = \frac{\phi_{d, \mathcal{O}}(y)(x)}{\|\phi_{d, D}(y)\|_{L^p(\mathcal{O})}}.$$

By construction, $\Phi_{d, \mathcal{O}}$ is a continuous map from $\boldsymbol{R}^N$ to $H^1(\mathcal{O})$.

From now on, we assume $\mathcal{O} = \Omega \cup L$, where Ω is a bounded smooth domain in $\boldsymbol{R}^N$ and L is a bounded set such that $\Omega \cup L$ is smooth. In order to prove some properties of $\beta_{\Omega \cup L}$ and $\Phi_{d, \Omega \cup L}$, we recall some results concerning the maps β_Ω and $\Phi_{d, \Omega}$. We denote

$$m(d, \Omega) = \inf_{u \in V(\Omega)} \int_\Omega (d|\nabla u|^2 + u^2)\, dx, \qquad \alpha = N\left(\frac{1}{2} - \frac{1}{p}\right). \tag{3.1}$$

Lemma 3.2. *There exist $\varepsilon_1 > 0$, $d_1 > 0$ such that, for any $d \in (0, d_1)$ and $u \in V(\Omega)$, if $E_{d, \Omega}(u) \leq m(d, \Omega) + \varepsilon_1 d^\alpha$, then $\beta_\Omega(u) \in \partial\Omega^+$.*

Proof. See [13, Proposition 2.3]. □

Lemma 3.3. *Uniformly for $y \in \partial\Omega$, there results*

$$\lim_{d \to 0} d^{-\alpha}(E_{d, \Omega}(\Phi_{d, \Omega}(y)) - m(d, \Omega)) = 0.$$

Proof. It follows by a straight combination of Propositions 2.1 and 2.2 in [13]. □

In the next propositions we will prove that, if $\mu(L, \Omega)$ is small enough, then the maps $\beta_{\Omega \cup L}$ and $\Phi_{d, \Omega \cup L}$ fulfil the assumptions of Theorem 2.2.

Proposition 3.4. *There exist $\varepsilon^* > 0$, $d_1 > 0$ such that for any $d \in (0, d_1)$ there exists $\mu_1 > 0$ such that, if $L \subset B_r$, $\Omega \cup L$ is smooth and $\mu(L, \Omega) < \mu_1$, then*

$$u \in V(\Omega \cup L), \qquad E_{d, \Omega \cup L}(u) \leq m(d, \Omega) + \varepsilon^* d^\alpha \Rightarrow \beta_{\Omega \cup L}(u) \in \partial\Omega^+. \tag{3.2}$$

Proof. Let $\varepsilon_1 > 0$, $d_1 > 0$ be as in Lemma 3.2. Let $\varepsilon^* < \varepsilon_1$, fix $\bar{d} \in (0, d_1)$ and define $m^*(\bar{d}) = m(\bar{d}, \Omega) + \varepsilon^* \bar{d}^\alpha$.

By contradiction, suppose that for any $n \in \mathbf{N}$ there exist $L_n \subset B_R$ and $u_n \in V(\Omega \cup L_n)$ such that $\Omega \cup L_n$ is smooth,

$$0 < \mu(L_n, \Omega) < 1/n\,, \tag{3.3}$$

$$E_{\bar{d}, \Omega \cup L_n}(u_n) \leq m^*(\bar{d})\,, \tag{3.4}$$

$$\beta_{\Omega \cup L_n}(u_n) \notin \partial\Omega^+\,. \tag{3.5}$$

Let $\hat{u}_n = u_n|_\Omega$; by (3.4), $(\hat{u}_n)$ is bounded in $H^1(\Omega)$, thus there exists $\hat{u} \in H^1(\Omega)$ such that (up to subsequences) $\hat{u}_n \to \hat{u}$ weakly in $H^1(\Omega)$ and $\hat{u}_n \to \hat{u}$ strongly in $L^p(\Omega)$. We claim that

$$\lim_{n \to \infty} \int_{L_n} (u_n^+)^p dx = 0\,. \tag{3.6}$$

Indeed, if (3.6) does not hold, up to a subsequence it is

$$\int_{L_n} (u_n^+)^p dx \geq \varepsilon > 0\,;$$

by (3.3) and Definition 1.1, if $d \leq 1$ (which is not restrictive to be assumed) there results

$$m^*(\bar{d}) \geq \int_{\Omega \cup L_n} (\bar{d}\,|\nabla u_n|^2 + u_n^2)\,dx \geq (\mu(L_n, \Omega))^{-1} \left(\int_{L_n} (u_n^+)^p dx \right)^{2/p} \geq n\varepsilon^{2/p}\,;$$

letting $n \to +\infty$ yields a contradiction, hence (3.6) is proved.

Moreover, by (3.6) and $L_n \subset B_R$ it follows

$$\lim_{n \to \infty} \int_{L_n} x(u_n^+)^p dx = 0\,. \tag{3.7}$$

By (3.6)

$$\int_{\Omega \cup L_n} (u_n^+)^p dx \to \int_\Omega (\hat{u}^+)^p dx\,,$$

hence $\hat{u} \in V(\Omega)$; by (3.7)

$$\lim_{n \to \infty} \beta_{\Omega \cup L_n}(u_n) = \beta_\Omega(\hat{u})\,,$$

therefore, by (3.5)

$$\beta_\Omega(\hat{u}) \notin \partial\Omega^+\,. \tag{3.8}$$

On the other hand, it is easy to see that $E_{\bar{d}, \Omega \cup L_n}(u_n) \geq E_{\bar{d}, \Omega}(\hat{u}) + o(1)$ (here $o(1) \to 0$ as $n \to +\infty$), whence, for n sufficiently large $E_{\bar{d}, \Omega}(\hat{u}) \leq E_{\bar{d}, \Omega \cup L_n}(u_n) + o(1) \leq$ $\leq m^*(\bar{d}) + o(1) \leq m(\bar{d}, \Omega) + \varepsilon_1 \bar{d}^\alpha$; then Lemma 3.2 implies $\beta_\Omega(\hat{u}) \in \partial\Omega^+$, which contradicts (3.8). $\square$

Proposition 3.5. *Let ε^* be as in Proposition 3.4. There exists $d_2 > 0$ such that for any $d \in (0, d_2)$ there exists $\mu_2 > 0$ such that, if $L \subset B_R$, $\Omega \cup L$ is smooth and $\mu(L, \Omega) < \mu_2$,* *then for any* $y \in \partial\Omega$ *there results*

$$E_{d,\Omega\cup L}(\Phi_{d,\Omega\cup L}(y)) \leq m(d,\Omega) + \varepsilon^* d^\alpha . \tag{3.9}$$

PROOF. By Lemma 3.3, there exists $d_2 > 0$ such that, for any $0 < d < d_2$ and for any $y \in \partial\Omega$, there results

$$E_{d,\Omega}(\Phi_{d,\Omega}(y)) \leq m(d,\Omega) + (\varepsilon^*/2) d^\alpha . \tag{3.10}$$

Next we fix $0 < \bar{d} < d_2$ and evaluate $E_{\bar{d},\Omega\cup L}(\Phi_{\bar{d},\Omega\cup L}(y))$. Without loss of generality, we can suppose that $|\Omega \cap L| = 0$. By definitions:

$$E_{\bar{d},\Omega\cup L}(\Phi_{\bar{d},\Omega\cup L}(y)) = \frac{\displaystyle\int_{\Omega\cup L} (\bar{d}|\nabla\phi_{\bar{d},\Omega\cup L}(y)|^2 + (\phi_{\bar{d},\Omega\cup L}(y))^2)\,dx}{\left(\displaystyle\int_{\Omega\cup L} (\phi_{\bar{d},\Omega\cup L}(y))^p\,dx\right)^{2/p}} =$$

$$= \frac{\displaystyle\int_{\Omega} (\bar{d}|\nabla\phi_{\bar{d},\Omega}(y)|^2 + (\phi_{\bar{d},\Omega}(y))^2)\,dx + \int_{L} (\bar{d}|\nabla\phi_{\bar{d},L}(y)|^2 + (\phi_{\bar{d},L}(y))^2)\,dx}{\left(\displaystyle\int_{\Omega} (\phi_{\bar{d},\Omega}(y))^p\,dx + \int_{L} (\phi_{\bar{d},L}(y))^p\,dx\right)^{2/p}} .$$

By simple computations and by taking into account the properties of η and ω, we obtain

$$\int_{L} (\bar{d}|\nabla\phi_{\bar{d},L}(y)|^2 + (\phi_{\bar{d},L}(y))^2)\,dx \leq c_1 |L|, \qquad \int_{L} (\phi_{\bar{d},L}(y))^p\,dx \leq c_2 |L|,$$

that is, the left-hand side terms above go to 0 uniformly in $y \in \partial\Omega$ as $|L|$ goes to 0. This yields that $E_{\bar{d},\Omega\cup L}(\Phi_{\bar{d},\Omega\cup L}(y))$ tends to $E_{\bar{d},\Omega}(\Phi_{\bar{d},\Omega}(y))$ as $|L| \to 0$ uniformly in $y \in \partial\Omega$. Then by Lemma 1.2 there exists $\mu_2 > 0$ such that for any $L \subset B_R$ and $\mu(L,\Omega) < \mu_2$, there results

$$|E_{\bar{d},\Omega\cup L}(\Phi_{\bar{d},\Omega\cup L}(y)) - E_{\bar{d},\Omega}(\Phi_{\bar{d},\Omega}(y))| < (\varepsilon^*/2)\bar{d}^\alpha .$$

A simple combination with (3.10) gives (3.9). □

4. PROOF OF THE MAIN THEOREM

We divide the proof into three steps.

STEP 1. *Existence of* cat $\partial\Omega$ *solutions.*

Let Ω be as in Theorem 1.3. Let ε^*, d_1 and d_2 be as in Proposition 3.4 and Proposition 3.5, let $d^* = \min\{d_1, d_2\}$ and fix $d \in (0, d^*)$. Then there exist μ_1 and μ_2 such that, taken $\mu^* = \min\{\mu_1, \mu_2\}$, if $L \subset B_R$ is such that $\Omega \cup L$ is smooth and

$\mu(L, \Omega) \leq \mu^*$, then (3.2) and (3.9) hold. Thus, considered the sublevel $V^{m^*(d)} = \{u \in V(\Omega \cup L): E_{d, \Omega \cup L}(u) \leq m^*(d)\}$, where $m^*(d) = m(d, \Omega) + \varepsilon^* d^\alpha$, there results

$$\Phi_{d, \Omega \cup L}(\partial\Omega) \subset V^{m^*(d)}, \qquad \beta_{\Omega \cup L}(V^{m^*(d)}) \subset \partial\Omega^+ .$$

By construction, $\partial\Omega^+$ is homotopically equivalent to $\partial\Omega$ and $\beta_{\Omega \cup L} \circ \Phi_{d, \Omega \cup L}$ is homotopically equivalent to the imbedding $j: \partial\Omega \to \partial\Omega^+$. Then, by Lemma 2.1 and Theorem 2.2, it follows that $E_{d, \Omega \cup L}$ has at least cat $\partial\Omega$ critical points in $V(\Omega \cup L)$, whose energy is less than $m^*(d)$.

STEP 2. *Existence of one more solution.*

We claim that there exists a constant $M^*(d) > m^*(d)$ such that $\Phi_{d, \Omega \cup L}(\partial\Omega)$ is contractible in $V^{M^*(d)}$. If this claim holds, as $\Phi_{d, \Omega \cup L}(\partial\Omega)$ in not contractible in $V^{m^*(d)}$ (note that $\partial\Omega$ is not contractible in itself and $\beta_{\Omega \cup L} \circ \Phi_{d, \Omega \cup L} \sim j$), then $V^{M^*(d)}$ cannot be retracted into $V^{m^*(d)}$. At this point, (PS) condition yields the existence of at least one more critical level between $m^*(d)$ and $M^*(d)$.

Now we turn to find $M^*(d)$. It is not restrictive to assume that $0 \in \Omega$ and that r, fixed in Section 3, is so small that $B_r \subset\subset \{x \in \Omega: \operatorname{dist}(x, \partial\Omega) > r\}$. Let

$$S(d, r) = \min \left\{ \int_{B_r} (d |\nabla u|^2 + |u|^2)\, dx : u \in H_0^1(B_r), \int_{B_r} (u^+)^p \, dx = 1 \right\}.$$

It is well known that $S(d, r)$ is achieved in a positive function, radially symmetric around the origin; let u^* be a such a minimizer: obviously, $u^* \notin \Phi_{d, \Omega \cup L}$. Define $F: [0, 1] \times \Phi_{d, \Omega \cup L}(\partial\Omega) \to V(\Omega \cup L)$ by setting

$$F(t, u) = (tu^* + (1-t)u) / \|(tu^* + (1-t)u)\|_p$$

(as $u^* \geq 0$, $tu^* + (1-t)u$ is positive and not identically zero in $\Omega \cup L$ for any (t, u)). Let

$$M^*(d) = \max \{ E_{d, \Omega \cup L}(F(t, u)) : t \in [0, 1], u \in \Phi_{d, \Omega \cup L}(\partial\Omega) \}. \tag{4.1}$$

It is easy to see that F is an homotopy between $\Phi_{d, \Omega \cup L}(\partial\Omega)$ and u^* in $V(\Omega \cup L)$; therefore $\Phi_{d, \Omega \cup L}(\partial\Omega)$ is contractible in

$$V^{M^*(d)} = \{ u \in V(\Omega \cup L): E_{d, \Omega \cup L}(u) \leq M^*(d) \}.$$

This proves the claim.

STEP 3. *Nontriviality of solutions.*

Observe that the solutions found in Steps 1 and 2 lie in the sublevel $V^{M^*(d)}$; we aim to prove that, for d small, $M^*(d)$ is less than the critical level $|\Omega \cup L|^{1-2/p}$, corresponding to the constant solution.

To this purpose, we need some estimates. By combining Lemma 2.1 in [1] and a simple rescaling argument, there results

$$S(d, r) = d^\alpha [m(1, \boldsymbol{R}^N) + o(1)] \quad \text{as } d \to 0 \tag{4.2}$$

($m(1, \boldsymbol{R}^N)$ is defined as in (3.1)). Moreover, it is possible to prove (*e.g.* cf. [10, 13]) that $m(d, \Omega) \leqslant C_0 d^\alpha$, where $C_0 > 0$ depends only on Ω and p. This implies that it is not restrictive to choose d^* in such a way that for any $d \in (0, d^*)$ there results

$$S(d, r) + m^*(d) = S(d, r) + m(d, \Omega) + \varepsilon d^\alpha < |\Omega|^{1-2/p}. \tag{4.3}$$

Let $0 \leqslant t \leqslant 1$ and $u \in \Phi_{d, \Omega \cup L}(\partial\Omega)$; then there exists $y \in \partial\Omega$ such that $u = \Phi_{d, \Omega \cup L}(y)$. We remark that $\operatorname{supp}(u) \subset \overline{B_r(y)}$, $\operatorname{supp}(u^*) \subset B_r$, and, by our choise of r, the supports of u and u^* are disjoint. This implies

$$\int_{\Omega \cup L} (d|\nabla(tu^* + (1-t)u)|^2 + (u^* + (1-t)u)^2)\,dx =$$

$$= t^2 \int_{B_r} (d|\nabla u^*|^2 + (u^*)^2)\,dx + (1-t)^2 \int_{B_r(y) \cap (\Omega \cup L)} (d|\nabla u|^2 + u^2)\,dx =$$

$$= t^2 E_{d, \Omega \cup L}(u^*) + (1-t)^2 E_{d, \Omega \cup L}(u) \leqslant t^2 S(d, r) + (1-t)^2 m^*(d)$$

and

$$\int_{\Omega \cup L} (tu^* + (1-t)u)^p\,dx = t^p \int_{B_r} (u^*)^p dx + (1-t)^p \int_{B_r(y) \cap (\Omega \cup L)} u^p dx = t^p + (1-t)^p.$$

Then

$$E_{d, \Omega \cup L}(F(t, u)) \leqslant (t^2 S(d, r) + (1-t)^2 m^*(d)) / (t^p + (1-t)^p)^{2/p} \leqslant S(d, r) + m^*(d);$$

along with (4.1) and (4.3) this inequality implies

$$M^*(d) \leqslant S(d, r) + m^*(d) < |\Omega|^{1-2/p} \leqslant |\Omega \cup L|^{1-2/p}. \qquad \square$$

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References

[1] V. Benci - G. Cerami, *The effect of the domain topology on the number of solutions of nonlinear elliptic problems.* Arch. Rat. Mech. Anal., 114, 1991, 79-93.

[2] V. Benci - G. Cerami, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology.* To appear.

[3] V. Benci - G. Cerami - D. Passaseo, *On the number of the positive solutions of some nonlinear elliptic problems.* A tribute in honour of G. Prodi, Scuola Norm. Sup. Pisa, 1991, 93-107.

[4] H. Berestycki - T. Gallouet - O. Kavian, *Equations de champs scalaires euclidiens nonlinéaires dans le plan.* C. R. Acad. Sc. Paris, Série I Math., 297, 1983, 307-310.

[5] H. Berestycki - P. L. Lions, *Nonlinear scalar field equations, I-Existence of a ground state.* Arch. Rat. Mech. Anal., 82, 1983, 313-375.

[6] S. Coleman - V. Glaser - A. Martin, *Action minima among solutions to a class of Euclidean scalar field equations.* Comm. Math. Phys., 58, 1978, 211-221.

[7] A. Gierer - H. Meinhardt, *A theory of biological pattern formation.* Kybernetik (Berline), 12, 1972.

[8] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\boldsymbol{R}^N$. Arch. Rat. Mech. Anal., 105, 1989, 243-266.

[9] M. Lazzo, *Morse theory and multiple positive solutions to a Neumann problem.* Ann. Mat. Pura e Appl., to appear.

[10] C. S. Lin - W. M. Ni - I. Takagi, *Large amplitude stationary solutions to a chemotaxis system.* Jour. Diff. Eq., 72, 1988, 1-27.

[11] G. Mancini - R. Musina, *The role of the boundary in some semilinear Neumann problems.* Rend. Sem. Mat. Padova, 88, 1992, 127-138.

[12] W. M. Ni - I. Takagi, *On the shape of least-energy solutions to a semilinear Neumann problem.* Comm. Pure Appl. Math., 45, 1991, 819-851.

[13] Z. Q. Wang, *On the existence of multiple, single-peaked solutions for a semilinear Neumann problem.* Arch. Rat. Mech. Anal., 120, 1992, 375-399.

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