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Two examples of fattening for the curvature flow with a driving force

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Calcolo delle variazioni. — *Two examples of fattening for the curvature flow with a driving force.* Nota di GIOVANNI BELLETTINI e MAURIZIO PAOLINI, presentata (*) dal Socio E. De Giorgi.

ABSTRACT. — We provide two examples of a regular curve evolving by curvature with a forcing term, which degenerates in a set having an interior part after a finite time.

KEY WORDS: Nonlinear partial differential equations of parabolic type; Mean curvature flow; Viscosity solutions.

RIASSUNTO. — *Due esempi di «rigonfiamento» per il moto secondo la curvatura con termine forzante.* Vengono dati due esempi di una curva regolare che evolve secondo la curvatura con un termine forzante, e dopo un certo tempo perde regolarità e degenera in un insieme con parte interna.

0. INTRODUCTION

In this *Note* we construct two examples of a Lipschitz function $u_0 : \boldsymbol{R}^2 \to \boldsymbol{R}$ whose level curves evolve in time, in the generalized viscosity sense considered in [6, 7, 13, 16], according to the law

$$V(x,t) = \kappa(x,t) + g(t), \qquad x = (x_1, x_2) \in \boldsymbol{R}^2, \qquad t \in [0,T], \tag{0.1}$$

where V is the normal velocity and κ is the curvature, with the property that there exists an initial level curve $\{u_0 = \lambda^\star\}$ of u verifying

$$\begin{cases} u_0 \text{ is of class } \mathcal{C}^\infty \text{ in a neighbourhood of } \{u_0 = \lambda^\star\}, \\ \{u_0 = \lambda^\star\} \text{ is compact and } |\nabla_x u_0| > 0 \text{ on } \{u_0 = \lambda^\star\}, \end{cases} \tag{0.2}$$

which develops, after a positive finite time, a full two-dimensional interior part. The function g is called driving force, or forcing term; the two examples are given with $g(t) = 1 - t$ and $g(t) \equiv 1$, respectively.

Such a behaviour is usually called *fattening*, and can be interpreted as a phenomenon of nonuniqueness of the level-set flow; in this case the convergence results of the scaled reaction-diffusion equation to the generalized motion by mean curvature are quite delicate (see [12, 20, 24]).

If $g = 0$ it is known that all smooth compact level curves of u_0 never develop an interior. This is a consequence of the results of Grayson [17] on the evolution of curves and the fact that the generalized evolution agrees with the classical motion by mean curvature so long as the latter exists [13]. Evans and Spruck [13] showed an example of a compact nonsmooth level curve that instantly develops an interior for $g = 0$, and afterwards Ilmanen established nonuniqueness of the level-set flow for noncompact smooth curves [19]. In three dimensions Soner and Souganidis [26] proved that the evolving torus does not develope interior (see also Huisken [18] for the evolution of a dumbbell

(*) Nella seduta del 12 febbraio 1994.

shaped region). This fact is in agreement with the numerical simulations of Paolini and Verdi [23].

The aim of the present paper is to discuss two examples in which, starting from level sets which are union of two disjoint circles, we have development of fattening.

The outline of the paper is the following. In section 1 we recall some notation and known results. In section 2 we show an example of fattening by taking a linearly time-dependent driving force $g(t) = 1 - t$ for $t \in [0, 1[$. Finally, in section 3 we show an example when $g(t) \equiv 1$.

1. Notation and preliminary results

For any $x \in \boldsymbol{R}^2$ and any $\rho > 0$ we denote by $B_\rho(x)$ the open ball centered at x with radius ρ. Given a smooth function u: $\boldsymbol{R}^2 \times [0, T] \to \boldsymbol{R}$ we denote by $\partial_t u$, ∇u, div, and $\nabla^2 u$ the time derivative of u, the gradient of u, the divergence, and the Hessian of u in the two space variables, respectively.

We denote by S^2 the space of the real 2×2 symmetric matrices. If $\gamma \in \boldsymbol{R}$, and K is an open subset of $\boldsymbol{R}^m$, we indicate by $C_\gamma(K)$ the space of all continuous functions u such that $u - \gamma$ is compactly supported in K.

Three different proposals have been suggested to treat the mean curvature evolution of surfaces even past singularities: the varifold approach [4], the asymptotic limits of the scaled Allen-Cahn equation [5, 9, 11, 3], and the viscosity sense. In the present paper we shall interpret the evolution according to the latter approach, which has been developed by Evans and Spruck [13-15], and, independently, Chen, Giga, and Goto [6] (see also Soner [24]). In [16] Giga, Goto, Ishii, and Sato study a more general class of geometric motions, including the mean curvature evolution with a driving force (for the driven motion of fronts see also [1]). The idea of viscosity solution to parabolic equations goes back to Crandall and P.-L. Lions [8], P.-L. Lions [22], and Jensen [21] (see also [7] and the references therein).

We will choose the conventions that the curvature is negative for convex sets and the velocity is positive for expanding curves.

For the sake of completeness we recall the definition of viscosity solution; this definition is a particular case of the definitions given in [7, 16]. Let $g \in \mathcal{C}([0, T])$, and let F: $[0, T] \times \boldsymbol{R}^2 \setminus \{0\} \times S^2 \to \boldsymbol{R}$ be defined as

$$F(t, p, X) = -\operatorname{trace}\left(\left(\mathrm{Id} - \frac{p \otimes p}{|p|^2}\right) X\right) + |p|\, g(t).$$

Let $F_\star$ (respectively $F^\star$) be the lower (respectively upper) semicontinuous envelope of F.

Let $u_0 \in \mathcal{C}_\gamma(\boldsymbol{R}^2)$. We say that a function $u \in \mathcal{C}(\boldsymbol{R}^2 \times [0, T])$ is a viscosity subsolution (respectively a viscosity supersolution) of

$$u_t + F(t, \nabla u, \nabla^2 u) = u_t - |\nabla u| \operatorname{div}(\nabla u / |\nabla u|) + |\nabla u|\, g = 0 \tag{1.1}$$

with initial condition

$$u(x,\,0) = u_0(x) \tag{1.2}$$

if whenever $(x_0, t_0) \in \boldsymbol{R}^2 \times [0, T]$ and

$$u(x,t) \leqslant u(x_0,t_0) + p\cdot(x-x_0) + q(t-t_0) + (1/2)(x-x_0)^T R(x-x_0) + \\ + o(|x-x_0|^2 + |t-t_0|) \text{ as } (x,t)\to(x_0,t_0), \qquad t\in[0,T]$$

(respectively $u(x,t) \geqslant u(x_0,t_0)+p\cdot(x-x_0)+q(t-t_0)+(1/2)(x-x_0)^T R(x-x_0)+ + o(|x-x_0|^2 + |t-t_0|)$ as $(x,t)\to(x_0,t_0)$) for some $p\in\boldsymbol{R}^2$, $q\in\boldsymbol{R}$, $R\in S^2$, then $q + F_\star(t,p,X) \leqslant 0$ (respectively $q + F^\star(t,p,X) \geqslant 0$). A function $u \in \mathcal{C}(\boldsymbol{R}^2 \times \times [0,T]) \cap L^\infty(\boldsymbol{R}^2 \times [0,T])$ is a viscosity solution of (1.1)-(1.2) provided u is both a viscosity subsolution and a viscosity supersolution.

One can prove [16] comparison results for (1.1)-(1.2), that (1.1)-(1.2) admits a unique viscosity solution, and such a solution belongs to $\mathcal{C}_\gamma(\boldsymbol{R}^2\times[0,T])$.

Following the notation of [10], given a function $u_0 \in \mathcal{C}_\gamma(\boldsymbol{R}^2)$, for any $\lambda\in\boldsymbol{R}$ and any $t\in[0,T]$ we define

$$\begin{cases} MCM(\{u_0=\lambda\},g)(t) = \{x\in\boldsymbol{R}^2 : u(x,t)=\lambda\}, \\ MCM(\{u_0<\lambda\},g)(t) = \{x\in\boldsymbol{R}^2 : u(x,t)<\lambda\}, \end{cases} \tag{1.3}$$

where u is the unique viscosity solution of (1.1)-(1.2). The set $MCM(\{u_0=\lambda\},g)(t)$ is then the generalized evolution of $\{u_0=\lambda\}$ according to equation (1.1)-(1.2).

2. Example 1: $g(t) = 1 - t$

We start with the case in which the forcing term g is a positive linearly time-dependent function.

Theorem 2.1. *Let $g(t) = 1-t$ for $t\in[0,1[$. Then there exist a function $u_0 \in \in \mathrm{Lip}(\boldsymbol{R}^2)\cap L^\infty(\boldsymbol{R}^2)$, a real number $\lambda^\star$, a ball B centered at the origin and a time interval $[\alpha,\beta]\subseteq]0,1[$ such that $\{u_0=\lambda^\star\}$ verifies (0.2) and $B\subseteq MCM(\{u_0=\lambda^\star\},g)(t)$ for any $t\in[\alpha,\beta]$. Precisely, u_0 is of the form*

$$u_0(x) = \min(\mathrm{dist}(x,E) - \mathrm{dist}(x,\boldsymbol{R}^2\backslash E) + R^\star,\, 2R^\star), \tag{2.1}$$

with $E = B_{R^\star}(-R^\star,0)\cup B_{R^\star}(R^\star,0)$, where $R^\star$ is a suitable positive real number, with $R^\star > \lambda^\star$.

Proof. Let $\lambda\in]3,5[$. Consider in $\boldsymbol{R}^2$ a circle of radius $R_\lambda(t)$ which evolves in time $t\in[0,1[$ according to (0.1), with $R_\lambda(0)=\lambda$. Then $R_\lambda(t)$ is the solution of the Cauchy problem

$$\begin{cases} \dot R_\lambda(t) = -1/R_\lambda(t) + g(t) \quad \forall t\in]0,1[, \\ R_\lambda(0) = \lambda. \end{cases} \tag{2.2}$$

As $R_\lambda(0)=\lambda>3$ we have $\dot R_\lambda(0) = -\lambda^{-1}+1 > 2/3$, and

$$1 < \lambda - 2 < R_\lambda(t) < \lambda + 2 \quad \forall t\in[0,1[. \tag{2.3}$$

Indeed, assume by contradiction that there exists $\tau\in]0,1[$ such that $R_\lambda(\tau) = \lambda - 2$

and $R_\lambda(t) > \lambda - 2$ for any $t \in [0, \tau[$. Let $s \in [0, \tau[$ be such that $\dot{R}_\lambda(s)\tau = R_\lambda(\tau) - \lambda = -2$; we have $|\dot{R}_\lambda(s)| = 2\tau^{-1} > 2$. But $|\dot{R}_\lambda(s)| \leq (R_\lambda(s))^{-1} + g(s) < (\lambda - 2)^{-1} + 1 < 2$, contradiction. The right inequality of (2.3) is a consequence of $\dot{R}_\lambda < 2$.

Since by (2.3) we have $R_\lambda^3 \leq 7^3$ on $[0, 1[$, there exists a constant $c > 0$ independent of λ such that $\ddot{R}_\lambda = \dot{R}_\lambda / R_\lambda^2 - 1 = -1/R_\lambda^3 + g/R_\lambda^2 - 1 < -c + g/R_\lambda^2 - 1 \leq -c$ on $[0, 1[$. Therefore R_λ is strictly concave on $[0, 1[$, so that for any $\lambda \in]3, 5[$ there exists a unique $t_\lambda \in]0, 1[$ such that $\dot{R}_\lambda(t_\lambda) = 0$ (solve $R_\lambda = 1/g$).

The following comparison result is a consequence of the comparison results for the geometric motion of two initially disjoint concentric circles. Let $\lambda_1, \lambda_2 \in]3, 5[$; then

$$\lambda_1 < \lambda_2 \Rightarrow R_{\lambda_1}(t) < R_{\lambda_2}(t) \quad \forall t \in [0, 1[. \tag{2.4}$$

Set $\lambda^\star = 4$, $t_{\lambda^\star} = t^\star$, and let $R^\star = R_{\lambda^\star}(t^\star) > R_{\lambda^\star}(0) = \lambda^\star$. Let $E = B_{R^\star}(-R^\star, 0) \cup B_{R^\star}(R^\star, 0)$, and let u_0 be defined as in (2.1). Then $u_0 \in \mathrm{Lip}(\boldsymbol{R}^2) \cap L^\infty(\boldsymbol{R}^2)$, and, as $\lambda^\star < R^\star$, we have that $\{u_0 = \lambda^\star\}$ verifies (0.2).

For any $t \in [0, 1[$ define the sets

$$\Sigma_\lambda^\pm(t) = \{x \in \boldsymbol{R}^2 : |x - (\pm R^\star, 0)| = R_\lambda(t)\},$$

$$I_\lambda^\pm(t) = \{x \in \boldsymbol{R}^2 : |x - (\pm R^\star, 0)| < R_\lambda(t)\}.$$

Step 1. Let us prove that $\Sigma_\lambda^-(t) \cap \Sigma_\lambda^+(t) = \emptyset$ for any $\lambda \in]3, \lambda^\star[$ and any $t \in [0, 1[$.

Let $\lambda \in]3, \lambda^\star[$; if by contradiction there exists $\tau \in]0, 1[$ such that $R_\lambda(\tau) = R^\star$, we have $R_\lambda(\tau) = R_{\lambda^\star}(t_{\lambda^\star})$. As $\lambda = R_\lambda(0) < R_{\lambda^\star}(0) = \lambda^\star$, and since $\dot{R}_{\lambda^\star}(t_{\lambda^\star}) = 0$, $\ddot{R}_{\lambda^\star}(t) \leq -c$ for any $t \in [0, 1[$, we necessarily find a point $s \in [0, 1[$ such that $R_\lambda(s) = R_{\lambda^\star}(s)$, which contradicts (2.4). This proves *step 1*. Therefore $\Sigma_\lambda^-(t)$ and $\Sigma_\lambda^+(t)$ evolve independently, and $MCM(\{u_0 = \lambda\}, g)(t) = \Sigma_\lambda^-(t) \cup \Sigma_\lambda^+(t)$ for any $\lambda \in]3, \lambda^\star[$ and any $t \in [0, 1[$.

Step 2. Let us prove that there exist a time interval $[\alpha, \beta] \subset]t^\star, 1[$ and a ball B centered at the origin, which are independent of $\lambda \in]\lambda^\star, 5[$, such that

$$B \subset MCM(\{u_0 < \lambda\}, g)(t) \quad \forall \lambda \in]\lambda^\star, 5[\quad \forall t \in [\alpha, \beta]. \tag{2.5}$$

Let $\lambda \in]\lambda^\star, 5[$ and let $\lambda^\star < \mu < \lambda$. Since $I_\mu^-(0) \subseteq \{u_0 < \mu\}$ and both sets evolve according to the same geometric law, a comparison argument gives $I_\mu^-(t) \subseteq MCM(\{u_0 < \mu\}, g)(t)$ for any $t \in [0, 1[$. The same argument applies to $I_\mu^+(t)$, and hence $I_\mu^-(t) \cup I_\mu^+(t) \subseteq MCM(\{u_0 < \mu\}, g)(t)$ for any $t \in [0, 1[$.

As $\lambda^\star < \mu$, by (2.4) we have $R^\star < R_\mu(t^\star)$. Therefore $I_\mu^-(t^\star) \cup I_\mu^+(t^\star)$ is the union of two open balls both containing the origin. Let us consider an open smooth connected dumbbell shaped set $D(t^\star)$, with $I_{\lambda^\star}^-(t^\star) \cup I_{\lambda^\star}^+(t^\star) \subset\subset D(t^\star) \subset\subset I_\mu^-(t^\star) \cup I_\mu^+(t^\star) \subseteq MCM(\{u_0 < \mu\}, g)(t^\star)$, and which is symmetric with respect to the two coordinate axes. Denote by $D(t)$ the evolution of $D(t^\star)$ by curvature (without forcing term) for any $t \in [t^\star, 1[$. We shall prove that after some time the set $D(t)$ contains a ball B centered at the origin which is independent of λ (see (2.6)).

Denote by $C^{\pm}(t)$ the evolution by curvature (without forcing term) of $\Sigma^{\pm}_{\lambda^{\star}}(t^{\star})$ for any $t \in [t^{\star}, 1[$. Let also $C(t)$ be the evolution by curvature (without forcing term) of the circle $\partial B_{R^{\star}}(0, R^{\star})$ for any $t \in [t^{\star}, 1[$. Let $t^{\star} < \beta < 1$ be such that $C(\beta) \cap \cap C^{+}(\beta) \neq \emptyset$, and set $\alpha = (t^{\star} + \beta)/2$. Finally, choose a ball B centered at the origin which does not intersect $C^{\pm}(t) \cup C(t)$ for any $t \in [\alpha, \beta]$. We stress that α, β, and B do not depend on $\lambda \in]\lambda^{\star}, 5[$.

We can assume that the set $C(t^{\star}) \cap \partial D(t^{\star})$ consists of two points, that we shall denote by $p^{\pm}(t^{\star})$, the superscript corresponding to the sign of the x_1-coordinate. Let $\{p^{-}(t), p^{+}(t)\} = C(t) \cap \partial D(t)$ be the continuous evolution of $p^{-}(t^{\star})$ and $p^{+}(t^{\star})$; these two points exist and are distinct for some time after $t^{\star}$. In addition they exist for all $t \in [t^{\star}, \beta]$. Indeed, in view of a result of Angenent (see [2, Th. 3.2]) such two points disappear only when they meet, and this cannot happen before $t = \beta$, since they are forced to lie outside $C^{\pm}(t)$. We conclude that the piece of $\partial D(t)$ (with counter-clockwise orientation) between $p^{-}(t)$ and $p^{+}(t)$ must lie inside $C(t)$ for any $t \in [t^{\star}, \beta]$. Therefore, as $B \cap (C^{\pm}(t) \cup C(t)) = \emptyset$ for any $t \in [\alpha, \beta]$, we have

$$B \subset D(t) \quad \forall t \in [\alpha, \beta]. \tag{2.6}$$

Since $g(t) > 0$ for any $t \in [t^{\star}, \beta]$, the solution of (1.1) is a subsolution to the mean curvature equation without forcing term. Therefore using a comparison argument [6, 13] we get that $D(t) \subset MCM(\{u_0 < \lambda\}, g)(t)$ for any $t \in [t^{\star}, \beta]$. Hence by (2.6) we deduce (2.5).

Conclusions. Let α, β, B be as in *step 2*. Then by (2.5) we have

$$\begin{aligned} (2.7)\qquad & MCM(\{u_0 < \lambda^{\star}\}, g)(t) \cup MCM(\{u_0 = \lambda^{\star}\}, g)(t) = \\ & = \bigcap_{\lambda \in]\lambda^{\star}, 5[} MCM(\{u_0 < \lambda\}, g)(t) \supseteq B \qquad \forall t \in [\alpha, \beta]. \end{aligned}$$

Moreover, possibly choosing a smaller B, recalling that $\alpha > t^{\star}$, from *step 1* we have

$$\begin{aligned} (2.8)\qquad & B \cap MCM(\{u_0 < \lambda^{\star}\}, g)(t) = \\ & = B \cap \Big(\bigcup_{\lambda \in]3, \lambda^{\star}[} MCM(\{u_0 < \lambda\}, g)(t) \Big) = \emptyset \qquad \forall t \in [\alpha, \beta]. \end{aligned}$$

By (2.7) and (2.8) it follows that $B \subseteq MCM(\{u_0 = \lambda^{\star}\}, g)(t)$ for any $t \in [\alpha, \beta]$, and this concludes the proof of Theorem 2.1. □

3. Example 2: $g \equiv 1$

In this section we discuss the example when the forcing term g is a positive constant. For simplicity we choose $g \equiv 1$, and we use some notation of section 2.

THEOREM 3.1. *Let $g \equiv 1$. Then there exist a function $u_0 \in \mathrm{Lip}(\boldsymbol{R}^2) \cap L^{\infty}(\boldsymbol{R}^2)$, a ball B centered at the origin and a time interval $[\alpha, \beta] \subseteq [0, T]$ such that $\{u_0 = 0\}$ verifies (0.2)* *and* $B \subseteq MCM(\{u_0 = 0\}, g)(t)$ *for any* $t \in [\alpha, \beta]$. *Precisely,* u_0 *is of the form*

$$u_0(x) = \min(\operatorname{dist}(x, F) - \operatorname{dist}(x, \boldsymbol{R}^2 \setminus F), 1), \tag{3.1}$$

with $F = B_{R_{\bar\mu}(0)}(-R^\star, 0) \cup B_{R_{\bar\lambda}(0)}(r^\star, 0)$, *where* $R_{\bar\mu}(0)$, $R_{\bar\lambda}(0)$, $R^\star$, $r^\star$ *are suitable positive real numbers, with* $R_{\bar\mu}(0) + R_{\bar\lambda}(0) < R^\star + r^\star$, *and* $r^\star < R^\star$.

Proof. Let $\lambda > 0$. Consider in $\boldsymbol{R}^2$ a circle of radius $R_\lambda(t)$ which evolves according to (0.1), with $R_\lambda(0) = \lambda$. Then

$$\begin{cases} \dot R_\lambda(t) = -1/R_\lambda(t) + 1 & \forall t \in]0, t^\lambda], \\ R_\lambda(0) = \lambda, \end{cases}$$

where t^λ denotes the extinction time. One can verify that if $0 < \lambda < 1$ then $t^\lambda \in$ $\in]0, +\infty[$, and R_λ is a nonnegative concave strictly decreasing function on $[0, t^\lambda]$ such that $R_\lambda(t^\lambda) = 0$. If $\lambda = 1$ then $R_\lambda \equiv 1$, so that there is no extinction time (we take $t^\lambda = +\infty$ and replace $]0, t^\lambda]$ by $]0, +\infty[$), and if $\lambda > 1$ then R_λ is a positive convex strictly increasing function on $[0, +\infty[$ such that $\lim_{t \to +\infty} \dot R_\lambda(t) = 1$ (and again $t^\lambda = +\infty$).

Fix $0 < \bar\lambda < 1$ sufficiently close to 1, in such a way that $\dot R_{\bar\lambda}(0) = -1/2$, $\dot R_{\bar\lambda}(\tau) < -1$, for a suitable $\tau \in]0, T[$, where $T = t^{\bar\lambda}$. We have $\ddot R_{\bar\lambda} \leqslant -\sigma$ on $[0, \tau]$ for a suitable $\sigma > 0$. Choose $\bar\mu > 1$ large enough in such a way that $\dot R_{\bar\mu}(0) \geqslant 3/4$, and $\ddot R_{\bar\mu} < \sigma$ on $[0, \tau]$. Setting $f = R_{\bar\lambda} + R_{\bar\mu}$, we have $\dot f(0) > 0$, $\dot f(\tau) < 0$, and $\ddot f < 0$ on $[0, \tau]$. Hence f has a unique strict local maximum $t^\star \in]0, \tau[$ on $[0, \tau]$.

Set $R^\star = R_{\bar\mu}(t^\star)$, $r^\star = R_{\bar\lambda}(t^\star)$, and $F = B_{R_{\bar\mu}(0)}(-R^\star, 0) \cup B_{R_{\bar\lambda}(0)}(r^\star, 0)$. Observe that F is the union of two disjoint balls. Finally, define u_0 as in (3.1), and let u be the viscosity solution of (1.1) with initial datum (3.1). For any $\lambda \in]-R_{\bar\mu}(0), 1[$ and any $t \in [0, T]$ define the (possibly empty) sets

$$\Sigma_\lambda^-(t) = \{x \in \boldsymbol{R}^2 : |x + (R^\star, 0)| = R_{\lambda + \bar\mu}(t)\},$$

$$\Sigma_\lambda^+(t) = \{x \in \boldsymbol{R}^2 : |x - (r^\star, 0)| = R_{\lambda + \bar\lambda}(t)\},$$

$$I_\lambda^-(t) = \{x \in \boldsymbol{R}^2 : |x + (R^\star, 0)| < R_{\lambda + \bar\mu}(t)\},$$

$$I_\lambda^+(t) = \{x \in \boldsymbol{R}^2 : |x - (r^\star, 0)| < R_{\lambda + \bar\lambda}(t)\}.$$

Using comparisons arguments one can prove that $\Sigma_\lambda^-(t) \cap \Sigma_\lambda^+(t) = \emptyset$ for any $\lambda \in]-R_{\bar\mu}(0), 0[$ and any $t \in [0, \tau]$, so that $MCM(\{u_0 = 0\}, g)(t) = \Sigma_\lambda^-(t) \cup$ $\cup \Sigma_\lambda^+(t)$.

Now let us repeat the proof of *step 2* in section 2 with some slight modification: replace $]\lambda^\star, 1[$ by $]0, 1[$ and $]t^\star, 1[$ by $]t^\star, \tau]$. Fix $0 < \mu < \lambda < 1$; we then have $I_\mu^-(t) \cup I_\mu^+(t) \subseteq MCM(\{u_0 < \mu\}, g)(t)$ for any $t \in [0, \tau]$. Moreover $I_\mu^-(t^\star) \cup I_\mu^+(t^\star)$ is the union of two open balls both containing the origin. Let $D(t^\star)$ be the set chosen as in *step 2*; the only difference is that now $D(t^\star)$ can not be taken symmetric with respect to the x_2-axis. Denote by $C(t)$ the evolution by curvature of $\partial B_{r^\star}(0, r^\star)$. Repeating the arguments of *step 2* we get the assertion. $\square$

If we compare the two examples, one can observe that if the forcing term depends on time then the fattening can be obtained by starting from a level curve which is union of two disjoint circles having the same radius; on the other hand, if the forcing term is a positive constant, we need to start from a level curve which is union of two disjoint circles having different radii.

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