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Fixed points of multivalued contractions with nonclosed, nonconvex values

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Matematica. — *Fixed points of multivalued contractions with nonclosed, nonconvex values.* Nota di Salvatore A. Marano, presentata (*) dal Corrisp. R. Conti.

Abstract. — For a class of multivalued contractions with nonclosed, nonconvex values, the set of all fixed points is proved to be nonempty and arcwise connected. Two applications are then developed. In particular, one of them is concerned with some properties of the set of all classical trajectories corresponding to continuous controls for a given nonlinear control system.

Key words: Multivalued contraction; Fixed point; Arcwise connectedness; Nonlinear control system.

Riassunto. — *Punti fissi di contrazioni multivoche con valori non chiusi e non convessi.* Si studia una classe di contrazioni multivoche con valori non necessariamente chiusi né convessi e si dimostra che l'insieme dei punti fissi non è vuoto ed è connesso per archi. Del risultato si fanno due applicazioni una delle quali riguarda la struttura dell'insieme delle traiettorie in senso classico corrispondenti a controlli continui di un sistema di controllo non lineare.

Introduction

Let E be a complete metric space and let Γ be a multivalued contraction from E into itself, with nonempty values. If $\Gamma(x)$ is closed for all $x \in E$, Corollary 3 of [7] ensures that the set $Fix(\Gamma)$ of all fixed points of Γ is nonempty. Since, contrary to the singlevalued case, $Fix(\Gamma)$ may have many elements, it is of interest to perform a qualitative study of it, for instance, from a topological point of view.

In this framework, some years ago, B. Ricceri established the following result (see [13, Théorème 1]).

Theorem A. *Let E be a Banach space, let X be a nonempty, convex, closed subset of E and let Γ be a multivalued contraction from X into itself, with convex, closed values. Then the set $Fix(\Gamma)$ is a retract of E; consequently, it is arcwise connected.*

Later on, several papers have been devoted to possible extensions and applications of Theorem A [4, 9, 11, 12, 15]. For instance, if $X = L^1(T)$ for some measure space T, the basic assumption

$$(a_1) \qquad \Gamma(x) \text{ is convex and closed for all } x \in X$$

may be replaced by

$$(a_2) \qquad \Gamma(x) \text{ is bounded, closed and decomposable for all } x \in X$$

and a satisfactory theory, including applications to multivalued differential equations, developed (see [4]).

To the best of our knowledge, there are not other significant theorems concerning topological properties of the set $Fix(\Gamma)$.

(*) Nella seduta del 12 marzo 1994.

In the present paper we consider a multivalued contraction Γ of the form

$$\Gamma(x) = \Psi(\Phi(x)), \qquad x \in X,$$

where Φ and Ψ are multifunctions satisfying the assumptions of Theorem A. Obviously, in this case, condition (a_1) may be not at all verified. Nevertheless, we prove that the set $Fix(\Gamma)$ is nonempty and arcwise connected (really, more sophisticated results are established; see Theorems 2.1 and 2.2).

Next, we present two applications. The first of them (Theorem 3.1) deals with the arcwise connectedness of the solution set to a nonlinear equation of the type $w \in G(x) + F(x)$, where w is a given element of X, F is a multifunction satisfying hypotheses like those of Theorem A, and G is a convex process.

The second application (Theorem 3.2) exhibits some properties of the set $S(\lambda)$ of all trajectories $x \in C^1([a,b], \boldsymbol{R}^n)$ corresponding to controls $u \in C^0([a,b], \boldsymbol{R}^m)$ for the nonlinear control process $x' = f(t, x, u(t))$, with control constraint $u(t) \in U(t,x)$ and initial condition $x(a) = \lambda$. In particular, the arcwise connectedness in $C^1([a,b], \boldsymbol{R}^n)$ of the sets $S(\lambda)$ and $\bigcup_{\lambda \in \boldsymbol{R}^n} S(\lambda)$ is achieved.

For measurable controls and Carathéodory's trajectories, results of this kind are already known [5, 16]. Moreover, continuous controls have been previously employed to study the controllability of various nonlinear control systems by many authors (see, for instance, [1, 8] and the references given therein).

1. Basic definitions and preliminary results

Let (E, d) be a metric space. For every $z \in E$ and every nonempty set $X \subseteq E$, we define $d(z, X) = \inf_{x \in X} d(z, x)$. If X and Z are two nonempty subsets of E, we define $d^\star(X, Z) = \sup_{x \in X} d(x, Z)$ and $d_H(X, Z) = \max\{d^\star(X, Z), d^\star(Z, X)\}$. A simple computation shows that the following proposition is true.

Proposition 1.1. *Let (E_1, d_1) and (E_2, d_2) be two metric spaces and let $E = E_1 \times E_2$, equipped with the metric*

$$d((x', y'), (x'', y'')) = \max\{d_1(x', x''), d_2(y', y'')\}, \qquad (x', y'), (x'', y'') \in E.$$

Then, for every pair of nonempty sets $X' \times Y'$, $X'' \times Y'' \subseteq E$, one has

$$d_H(X' \times Y', X'' \times Y'') \leq \max\{d_{1_H}(X', X''), d_{2_H}(Y', Y'')\}.$$

Let E_1 and E_2 be two nonempty sets. The symbol $\Phi\colon E_1 \to 2^{E_2}$ means that Φ is a multifunction from E_1 into E_2, namely a function from E_1 into the family of all subsets of E_2. The range of Φ is the set $\Phi(E_1) = \bigcup_{x \in E_1} \Phi(x)$. When $\Phi(E_1) = E_2$, we say that the multifunction Φ is surjective. The graph of Φ, denoted by $gr(\Phi)$, is the set $\{(x, y) \in E_1 \times E_2 : y \in \Phi(x)\}$. If $E_1 = E_2$, we write $Fix(\Phi)$ for $\{x \in E_1 : x \in \Phi(x)\}$. A function $\varphi\colon E_1 \to E_2$ such that $\varphi(x) \in \Phi(x)$ for all $x \in E_1$ is said to be a selection of Φ. For every set $Y \subseteq E_2$, we define $\Phi^-(Y) = \{x \in E_1 : \Phi(x) \cap Y \neq \emptyset\}$. If E_1 and E_2 are two topological spaces and, for any open set $Y \subseteq E_2$, the set $\Phi^-(Y)$ is open in E_1, we say that the multifunction Φ is lower semicontinuous. When (E_1, d_1) and (E_2, d_2) are two metric spaces and there is a real number $L \geqslant 0$ so that $d_{2_H}(\Phi(x'), \Phi(x'')) \leqslant L d_1(x', x'')$ for all x', $x'' \in E_1$, we say that Φ satisfies a Lipschitz condition with constant L. If $L < 1$, the multifunction Φ is said to be a multivalued contraction. It is a simple matter to see that any multifunction verifying a Lipschitz condition is lower semicontinuous.

Now, let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two real normed spaces. The multifunction Φ is said to be a convex process if, for every $x', x'' \in E_1$ and every $\alpha, \beta \in [0, +\infty[$, one has $\alpha\Phi(x') + \beta\Phi(x'') \subseteq \Phi(\alpha x' + \beta x'')$. If Φ is a surjective convex process, d_1 is the metric induced by $\|\cdot\|_1$ and θ_1 is the zero vector of E_1, we define $L_\Phi = \sup\{d_1(\theta_1, \Phi^-(y)) : y \in E_2, \|y\|_2 \leqslant 1\}$. When E_1 and E_2 are Banach spaces and the set $gr(\Phi)$ is closed in $E_1 \times E_2$, the Corollary p. 131 of [14] guarantees that $L_\Phi < +\infty$.

Given a positive integer n, we write $(\boldsymbol{R}^n, |\cdot|_n)$ for the real Euclidean n-space and δ_n for the metric induced by $|\cdot|_n$. If I is a compact real interval, the symbol $C^0(I, \boldsymbol{R}^n)$ is used to denote the space of all continuous functions $u: I \to \boldsymbol{R}^n$, equipped with the norm $\|u\|_{C^0(I, \boldsymbol{R}^n)} = \max\limits_{t \in I} |u(t)|_n$. Moreover, $C^1(I, \boldsymbol{R}^n)$ stands for the space of all $v \in C^0(I, \boldsymbol{R}^n)$, which are continuously differentiable in I. The norm in this space is defined by $\|v\|_{C^1(I, \boldsymbol{R}^n)} = \|v\|_{C^0(I, \boldsymbol{R}^n)} + \|v'\|_{C^0(I, \boldsymbol{R}^n)}$, where v' is the derivative of v.

Proposition 5 in [2, p. 44] combined with Theorem 3.2″ of [10] yields the following

PROPOSITION 1.2. *Let I be a compact real interval and let $\Phi: I \to 2^{\boldsymbol{R}^n}$ be a lower semicontinuous multifunction, with nonempty, convex, closed values. Suppose $\psi: I \to \boldsymbol{R}^n$ is a continuous function and $\beta: I \to [0, +\infty[$ is a lower semicontinuous function satisfying $\delta_n(\psi(t), \Phi(t)) \leqslant \beta(t)$ for all $t \in I$. Then, for every $\varepsilon > 0$ there is a continuous selection φ of Φ such that $|\psi(t) - \varphi(t)|_n \leqslant \beta(t) + \varepsilon$ for all $t \in I$.*

Let E be a topological space and let X be a nonempty subspace of E. We say that X is a retract of E if there exists a continuous function $r: E \to X$ such that $r(x) = x$ for all $x \in X$. The space X is said to be an absolute extensor for paracompact spaces if, for every paracompact space Λ, every closed subset Λ_0 of Λ and every continuous function $\varphi_0: \Lambda_0 \to X$, there is a continuous function $\varphi: \Lambda \to X$ such that $\varphi(\lambda) = \varphi_0(\lambda)$ for all $\lambda \in \Lambda_0$. The following proposition establishes a close connection between the concepts just defined.

PROPOSITION 1.3. *Let E be a Banach space and let X be a nonempty subspace of E. Then X is a retract of E if and only if it is an absolute extensor for paracompact spaces and is closed.*

The proof is easily performed by using Example 1.3★ and Theorem 3.2″ of [10]; so we omit it.

As a simple consequence of the preceding proposition, we obtain that every continuous image of a retract of a Banach space is an arcwise connected space. It is also possible to prove [6] that any arcwise connected space is a continuous image of an absolute retract, according to [3, p. 85].

Finally, we observe that Proposition 1.3, together with Theorem 1 of [13], produces Theorem A of Introduction.

2. Main result

In this and in the following section, Λ denotes a paracompact space, $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ are two Banach spaces, d_i $(i = 1, 2)$ stands for the metric induced by $\|\cdot\|_i$, X is a nonempty, convex, closed subset of E_1 and Y is a nonempty, convex, closed subset of E_2.

We are in a position now to establish the main result of this paper.

Theorem 2.1. *Suppose $\Phi: \Lambda \times X \to 2^Y$ and $\Psi: \Lambda \times X \times Y \to 2^X$ are two nonempty, convex, closed-valued multifunctions, with the following properties:*

(a_1) *The multifunction $\lambda \to \Phi(\lambda, x)$ is lower semicontinuous for every $x \in X$.*

(a_2) *There is a continuous function $L: \Lambda \to [0, 1[$ such that $d_{2_H}(\Phi(\lambda, x'), \Phi(\lambda, x'')) \leq \leq L(\lambda)\|x' - x''\|_1$ for all $\lambda \in \Lambda$, $x', x'' \in X$.*

(a_3) *The multifunction $\lambda \to \Psi(\lambda, x, y)$ is lower semicontinuous for every $(x, y) \in X \times Y$.*

(a_4) *There is a continuous function $M: \Lambda \to [0, 1[$ such that*
$$d_{1_H}(\Psi(\lambda, x', y'), \Psi(\lambda, x'', y'')) \leq M(\lambda) \max\{\|x' - x''\|_1, \|y' - y''\|_2\}$$
for all $\lambda \in \Lambda$, (x', y'), $(x'', y'') \in X \times Y$.

For every $(\lambda, x) \in \Lambda \times X$, we define $\Gamma(\lambda, x) = \Psi(\lambda, x, \Phi(\lambda, x))$. Then

(i_1) *The set $Fix(\Gamma(\lambda, \cdot))$ is nonempty and arcwise connected for all $\lambda \in \Lambda$.*

(i_2) *For every $\lambda_1, \ldots, \lambda_p \in \Lambda$ and every $x_i \in Fix(\Gamma(\lambda_i, \cdot))$, $i = 1, \ldots, p$, there is a continuous function $\gamma: \Lambda \to X$ such that $\gamma(\lambda_i) = x_i$ for each $i = 1, \ldots, p$, and $\gamma(\lambda) \in Fix(\Gamma(\lambda, \cdot))$ for all $\lambda \in \Lambda$.*

Proof. Fix $\lambda \in \Lambda$ and set, for every $(x, y) \in X \times Y$, $\Sigma(\lambda, x, y) = \Psi(\lambda, x, y) \times \times \Phi(\lambda, x)$. Owing to Proposition 1.1 and assumptions (a_2) and (a_4), one has
$$d_H(\Sigma(\lambda, x', y'), \Sigma(\lambda, x'', y'')) \leq$$
$$\leq \max\{d_{1_H}(\Psi(\lambda, x', y'), \Psi(\lambda, x'', y'')), d_{2_H}(\Phi(\lambda, x'), \Phi(\lambda, x''))\} \leq$$
$$\leq \max\{L(\lambda), M(\lambda)\} \cdot \max\{\|x' - x''\|_1, \|y' - y''\|_2\}$$
for all (x', y'), $(x'', y'') \in X \times Y$, so that the multifunction $(x, y) \to \Sigma(\lambda, x, y)$ is a multivalued contraction on $X \times Y$, with nonempty, convex, closed values. Therefore, by Theorem A, the set $Fix(\Sigma(\lambda, \cdot))$ is nonempty and arcwise connected in $E_1 \times E_2$. If $p_1: E_1 \times E_2 \to E_1$ denotes the projection onto the first coordinate, a simple computation shows that $Fix(\Gamma(\lambda, \cdot)) = p_1(Fix(\Sigma(\lambda, \cdot)))$. Thus, assertion (i_1) follows immediately from the continuity of p_1.

Let us prove assertion (i_2). Pick $\lambda_1, \ldots, \lambda_p \in \Lambda$ and, for each $i = 1, \ldots, p$, choose $x_i \in Fix(\Gamma(\lambda_i, \cdot))$. Then, there are $y_1, \ldots, y_p \in Y$ satisfying $(x_i, y_i) \in Fix(\Sigma(\lambda_i, \cdot))$, $i = = 1, \ldots, p$. We already know that, for any $\lambda \in \Lambda$, the multifunction $(x, y) \to \Sigma(\lambda, x, y)$ is a multivalued contraction on $X \times Y$, with constant $\max\{L(\lambda), M(\lambda)\}$ and nonempty, convex, closed values. Furthermore, because of assumptions (a_1) and (a_3), for any

$(x, y) \in X \times Y$, the multifunction $\lambda \to \Sigma(\lambda, x, y)$ is lower semicontinuous. By [11, Theorem 3.3], this yields a continuous mapping $\sigma: \Lambda \to X \times Y$ such that $\sigma(\lambda_i) = (x_i, y_i)$ for each $i = 1, \dots, p$, and $\sigma(\lambda) \in Fix(\Sigma(\lambda, \cdot))$ for all $\lambda \in \Lambda$. We define $\gamma(\lambda) = p_1(\sigma(\lambda))$, $\lambda \in \Lambda$. Since γ is a continuous function and one has $\gamma(\lambda_i) = x_i$ for every $i = 1, \dots, p$, $\gamma(\lambda) \in Fix(\Gamma(\lambda, \cdot))$ for all $\lambda \in \Lambda$, the proof is complete. △

The hypotheses of Theorem 2.1 do not imply that the set $Fix(\Gamma(\lambda, \cdot))$, $\lambda \in \Lambda$, is a retract of E_1, as the following example shows.

EXAMPLE 2.1. Set $(E_1, \|\cdot\|_1) = (\mathbf{R}^2, |\cdot|_2)$ and $(E_2, \|\cdot\|_2) = (\mathbf{R}, |\cdot|_1)$. Moreover, choose $X = \{x \in E_1 : \|x\|_1 \leq 2^{-1}\}$, $Y = [0, 2\pi]$, $\Phi(\lambda, x) = Y$ for all $(\lambda, x) \in \Lambda \times X$, $\Psi(\lambda, x, y) = \{(2^{-1}\cos y, 2^{-1}\sin y)\}$ for all $(\lambda, x, y) \in \Lambda \times X \times Y$. It is a simple matter to see that the multifunctions Φ and Ψ satisfy all the assumptions of Theorem 2.1. Hence, by conclusion (i_1), the set $\{x \in X : x \in \Psi(\lambda, x, \Phi(\lambda, x))\}$ is nonempty and arcwise connected. Nevertheless, since one has

$$\{x \in X : x \in \Psi(\lambda, x, \Phi(\lambda, x))\} = \{x \in E_1 : \|x\|_1 = 2^{-1}\},$$

it is not a retract of E_1 (see, for instance, Proposition 3.9 in [3, p. 12]).

We conclude this section with Theorem 2.2 below, which is a version of Theorem 2.1 interesting enough to be stated explicitly.

THEOREM 2.2. *Let $\Phi: X \to 2^Y$ and $\Psi: Y \to 2^X$ be two nonempty, convex, closed-valued multifunctions, satisfying a Lipschitz condition with constants L and M respectively. For every $x \in X$, we define $\Gamma(x) = \Psi(\Phi(x))$. If $LM < 1$, then the set $Fix(\Gamma)$ is nonempty and arcwise connected.*

PROOF. Obviously, we may suppose $M > 0$. Choose $k \in]L, M^{-1}[$ and associate to each $y \in E_2$ the norm (equivalent to the previous one) $\|y\|_k = k^{-1}\|y\|_2$. A simple computation ensures that

$$d_{k_H}(\Phi(x'), \Phi(x'')) \leq Lk^{-1}\|x' - x''\|_1 \quad \text{and} \quad d_{1_H}(\Psi(y'), \Psi(y'')) \leq Mk\|y' - y''\|_k$$

for all $x', x'' \in X$, $y', y'' \in Y$, where d_k denotes the metric induced by $\|\cdot\|_k$. Hence, the multifunctions Φ and Ψ are now multivalued contractions with constants Lk^{-1} and Mk respectively, and nonempty, convex, closed values. So, the same arguments used in the proof of Theorem 2.1 yield the desired conclusion. △

REMARK 2.1. Of course, it is also possible to formulate versions of Theorem 2.1 where the composition of a finite number q, $q > 2$, of multifunctions is considered. They are not in any way more difficult to prove than the special case $q = 2$.

3. SOME APPLICATIONS

In this section we present two applications of Theorem 2.1. The first of them deals with the arcwise connectedness of the solution set to a nonlinear equation.

THEOREM 3.1. *Let E_1 and E_2 be over the real number field. Suppose $F: \Lambda \times E_1 \to 2^{E_2}$ is a nonempty, convex, closed-valued multifunction, having the following properties:*

(b_1) *The multifunction $\lambda \to F(\lambda, x)$ is lower semicontinuous for all $x \in E_1$.*

(b_2) *There is a continuous function $L: \Lambda \to [0, 1[$ so that $d_{2_H}(F(\lambda, x'), F(\lambda, x'')) \leq \leq L(\lambda)\|x' - x''\|_1$ for every $\lambda \in \Lambda$, x', $x'' \in E_1$.*

Moreover, let $G: E_1 \to 2^{E_2}$ be a nonempty-valued, surjective, convex process, such that

(b_3) *the set $gr(G)$ is closed in $E_1 \times E_2$ and $L_G < 1$.*

Then

(j_1) *The set $S(\lambda, w) = \{x \in E_1 : w \in G(x) + F(\lambda, x)\}$ is nonempty and arcwise connected for all $\lambda \in \Lambda$, $w \in E_2$.*

(j_2) *For every $\lambda_1, \ldots, \lambda_p \in \Lambda$, every $w \in E_2$ and every $x_i \in S(\lambda_i, w)$, $i = 1, \ldots, p$, there is a continuous function $s: \Lambda \to E_1$ such that $s(\lambda_i) = x_i$ for each $i = 1, \ldots, p$, and $s(\lambda) \in \in S(\lambda, w)$ for all $\lambda \in \Lambda$.*

PROOF. Fix $w \in E_2$. For every $\lambda \in \Lambda$, $x \in E_1$, $y \in E_2$, we define

$$\Phi(\lambda, x) = w - F(\lambda, x), \qquad \Psi(\lambda, x, y) = G^-(y), \qquad \Gamma(\lambda, x) = \Psi(\lambda, x, \Phi(\lambda, x)).$$

Since one has $S(\lambda, w) = Fix(\Gamma(\lambda, \cdot))$, $\lambda \in \Lambda$, to accomplish the proof it is sufficient to verify that all the hypotheses of Theorem 2.1 are fulfilled. Of course, the multifunction Φ has nonempty, convex, closed values. Moreover, conditions (a_1) and (a_2) are a simple consequence of (b_1) and (b_2) respectively. Due to the assumptions, the multifunction Ψ has nonempty, convex, closed values and, by Theorem 6 of [14] and (b_3), it is a multivalued contraction from E_2 into E_1, with constant L_G. Therefore, the conditions (a_3) and (a_4) of Theorem 2.1 are verified too. $\triangle$

The hypotheses of the preceding theorem do not guarantee that the set $S(\lambda, w)$, $\lambda \in \Lambda$, $w \in E_2$, is a retract of E_1. This may be concluded from the following

EXAMPLE 3.1. Suppose $(E_2, \|\cdot\|_2)$ is a nonreflexive Banach space and define $B_2 = = \{y \in E_2 : \|y\|_2 \leq 1\}$. If $(E_1, \|\cdot\|_1) = (\mathbf{R}, |\cdot|_1)$ and $\psi: E_2 \to E_1$ is a continuous linear functional such that $\|\psi\| = \sup_{y \in B_2} |\psi(y)|_1 < 1$ and $|\psi(y)|_1 < \|\psi\|$ for all $y \in B_2$, we set $F(\lambda, x) = B_2$ and $G(x) = \psi^{-1}(x)$ for all $\lambda \in \Lambda$, $x \in E_1$. A straightforward argument ensures that the multifunctions F and G satisfy all the conditions of Theorem 3.1. Nevertheless, since one has

$$\{x \in E_1 : \theta_2 \in G(x) + F(\lambda, x)\} = \{x \in E_1 : G(x) \cap B_2 \neq \emptyset\} =]-\|\psi\|, \|\psi\|[$$

(θ_2 denotes the zero vector of E_2), the set $S(\lambda, \theta_2)$ is not a retract of E_1.

Using Theorem 2.2 in place of Theorem 2.1, it is possible to establish the following result, which can be regarded as a multivalued version of Théorème 4 in [13].

THEOREM 3.2. *Let E_1 and E_2 be over the real number field. Let $F: E_1 \to 2^{E_2}$ be a nonempty, convex, closed-valued multifunction, satisfying a Lipschitz condition with constant L. Let $G: E_1 \to 2^{E_2}$ be a nonempty-valued, surjective, convex process, such that the set $gr(G)$ is closed in $E_1 \times E_2$. If $L \cdot L_G < 1$, then, for any $w \in E_2$, the set $\{x \in E_1 : w \in G(x) + F(x)\}$ is nonempty and arcwise connected.*

The second application we wish to emphasize, is concerned with some properties of the set of all classical trajectories corresponding to continuous controls for a given nonlinear control system.

We denote by m, n two positive integers and by I the compact real interval $[a, b]$.

THEOREM 3.3. *Let $f: I \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$ be a continuous function and let $U: I \times \boldsymbol{R}^n \to 2^{\boldsymbol{R}^m}$ be a nonempty, convex, closed-valued multifunction. Suppose the following conditions hold:*

(c_1) *There are $M > 0$ and $\mu \in [0, 1[$ such that $|f(t, x', y') - f(t, x'', y'')|_n \leq M|x' - x''|_n + \mu|y' - y''|_m$ for all $t \in I$, $x', x'' \in \boldsymbol{R}^n$, $y', y'' \in \boldsymbol{R}^m$.*

(c_2) *The multifunction $t \to U(t, x)$ is lower semicontinuous for every $x \in \boldsymbol{R}^n$.*

(c_3) *There exists $L > 0$ so that $\delta_{m_H}(U(t, x'), U(t, x'')) \leq L|x' - x''|_n$ for all $t \in I$, x', $x'' \in \boldsymbol{R}^n$.*

Then

(k_1) *For every $\lambda \in \boldsymbol{R}^n$, the set*

$$S(\lambda) = \{x \in C^1(I, \boldsymbol{R}^n): x(a) = \lambda \text{ and there is } u \in C^0(I, \boldsymbol{R}^m) \text{ such that } x'(t) = f(t, x(t), u(t)), u(t) \in U(t, x(t)) \text{ for all } t \in I\}$$

is nonempty and arcwise connected in $C^1(I, \boldsymbol{R}^n)$.

(k_2) *For any $\lambda_1, \ldots, \lambda_p \in \boldsymbol{R}^n$ and any $x_i \in S(\lambda_i)$, $i = 1, \ldots, p$, there exists a continuous function s: $\boldsymbol{R}^n \to C^1(I, \boldsymbol{R}^n)$ satisfying $s(\lambda_i) = x_i$ for each $i = 1, \ldots, p$, and $s(\lambda) \in S(\lambda)$ for all $\lambda \in \boldsymbol{R}^n$.*

(k_3) *The set $S = \bigcup_{\lambda \in \boldsymbol{R}^n} S(\lambda)$ is arcwise connected in $C^1(I, \boldsymbol{R}^n)$.*

(k_4) *If there is a convex compact set $K \subseteq \boldsymbol{R}^n$ such that $\{x(b): x \in S(\lambda), \lambda \in K\} \subseteq K$, then there exist $u \in C^0(I, \boldsymbol{R}^m)$ and $x \in C^1(I, \boldsymbol{R}^n)$ verifying $x'(t) = f(t, x(t), u(t))$, $u(t) \in U(t, x(t))$ for all $t \in I$, $x(a) = x(b)$.*

PROOF. Choose $k > \max\{L, M(1 - \mu)^{-1}\}$. Throughout this proof, we write Λ to denote the space $\boldsymbol{R}^n$, E_1 for the space $C^0(I, \boldsymbol{R}^n)$, equipped with the norm $\|v\|_1 = \max_{t \in I} e^{-kt}|v(t)|_n$, and E_2 to denote the space $C^0(I, \boldsymbol{R}^m)$, with the norm $\|u\|_2 = \max_{t \in I} e^{-kt}|u(t)|_m$. Of course, these norms are equivalent to the usual ones.

We first define, for every $\lambda \in \Lambda$, $v \in E_1$,

$$\Phi(\lambda, v) = \left\{ u \in E_2 : u(t) \in U\left(t, \lambda + \int_a^t v(\tau)\, d\tau\right) \text{ for all } t \in I \right\}.$$

Owing to the assumptions and [11, Proposition 1.2], we see that the multifunction $t \to U\left(t, \lambda + \int_a^t v(\tau)\, d\tau\right)$ is lower semicontinuous and nonempty, convex, closed-valued. Hence, by [10, Theorem 3.2″], there is a continuous function $u: I \to \boldsymbol{R}^m$ such that $u(t) \in U\left(t, \lambda + \int_a^t v(\tau)\, d\tau\right)$ for all $t \in I$. This implies $\Phi(\lambda, v) \neq \emptyset$. Moreover, the set $\Phi(\lambda, v)$ is convex and closed in E_2, as a simple computation shows. Let us prove that, for any $\lambda \in \Lambda$, the multifunction $v \to \Phi(\lambda, v)$ is a multivalued contraction from E_1 into E_2, with constant Lk^{-1}. Obviously, this is achieved by establishing the inequality

$$d_2^{\star}(\Phi(\lambda, v), \Phi(\lambda, w)) \leq Lk^{-1} \|v - w\|_1 \tag{1}$$

for all $v, w \in E_1$. Pick $\lambda \in \Lambda$, $v, w \in E_1$ and choose $u \in \Phi(\lambda, v)$. Since, due to assumption (c_3), for every $t \in I$ we have

$$\delta_m\left(u(t), U\left(t, \lambda + \int_a^t w(\tau)\, d\tau\right)\right) \leq L \int_a^t |v(\tau) - w(\tau)|_n\, d\tau,$$

Proposition 1.2 guarantees that, for any $\varepsilon > 0$ there is a function $z \in \Phi(\lambda, w)$ fulfilling

$$|u(t) - z(t)|_m \leq L \int_a^t |v(\tau) - w(\tau)|_n\, d\tau + \varepsilon \quad \text{for all } t \in I.$$

The preceding formula yields

$$e^{-kt} |u(t) - z(t)|_m \leq e^{-kt}\left[L\|v - w\|_1 \int_a^t e^{k\tau}\, d\tau + \varepsilon\right] \leq Lk^{-1}\|v - w\|_1 + \varepsilon e^{-ka}, \qquad t \in I.$$

Hence, $d_2(u, \Phi(\lambda, w)) \leq Lk^{-1}\|v - w\|_1$ for every $u \in \Phi(\lambda, v)$. This implies (1).

A quite similar argument may be used to conclude that, for any $v \in E_1$, the multifunction $\lambda \to \Phi(\lambda, v)$ satisfies a Lipschitz condition and so is lower semicontinuous.

Next, let $\Psi: \Lambda \times E_1 \times E_2 \to E_1$ be the mapping defined by

$$\Psi(\lambda, v, u)(t) = f\left(t, \lambda + \int_a^t v(\tau)\, d\tau, u(t)\right), \qquad t \in I,$$

for all $(\lambda, v, u) \in \Lambda \times E_1 \times E_2$. It is a simple matter to see that, for every $(v, u) \in E_1 \times E_2$, the function $\lambda \to \Psi(\lambda, v, u)$ is continuous. Moreover, for any $\lambda \in \Lambda$, the mapping $(v, u) \to \Psi(\lambda, v, u)$ is a contraction from $E_1 \times E_2$ into E_1, with constant $Mk^{-1} + \mu$. To prove this, pick $\lambda \in \Lambda$ and $(v, u), (w, z) \in E_1 \times E_2$. Because of assumption (c_1), one has

$$|\Psi(\lambda, v, u)(t) - \Psi(\lambda, w, z)(t)|_n \leq M \int_a^t |v(\tau) - w(\tau)|_n \, d\tau + \mu |u(t) - z(t)|_m \leq$$

$$\leq M \|v - w\|_1 \int_a^t e^{k\tau} \, d\tau + \mu |u(t) - z(t)|_m \leq e^{kt} (Mk^{-1} \|v - w\|_1 + \mu \|u - z\|_2) \leq$$

$$\leq e^{kt} (Mk^{-1} + \mu) \max \{\|v - w\|_1, \|u - z\|_2\}$$

for all $t \in I$. Consequently,

$$\|\Psi(\lambda, v, u) - \Psi(\lambda, w, z)\|_1 \leq (Mk^{-1} + \mu) \max \{\|v - w\|_1, \|u - z\|_2\}.$$

We have now showed that the multifunction Φ and the function Ψ satisfy all the assumptions of Theorem 2.1.

Let $\Gamma(\lambda, v) = \Psi(\lambda, v, \Phi(\lambda, v))$, $(\lambda, v) \in \Lambda \times E_1$, and let $T: E_1 \to C^1(I, \boldsymbol{R}^n)$ be the operator defined by

$$T(v)(t) = \int_a^t v(\tau) \, d\tau, \qquad t \in I,$$

for all $v \in E_1$. Since, for any $\lambda \in \Lambda$, we have

$$S(\lambda) = v_\lambda + T(Fix(\Gamma(\lambda, \cdot))), \tag{2}$$

where v_λ denotes the function $t \to v_\lambda(t) = \lambda$, $t \in I$, assertion (k_1) follows immediately from the conclusion (i_1) of Theorem 2.1 and the continuity of T.

To verify assertion (k_2), pick $\lambda_1, \ldots, \lambda_p \in \Lambda$ and choose $x_i \in S(\lambda_i)$, $i = 1, \ldots, p$. Owing to (2), for every $i = 1, \ldots, p$, there is $v_i \in Fix(\Gamma(\lambda_i, \cdot))$ such that $x_i = v_{\lambda_i} + T(v_i)$. Thus, the conclusion (i_2) of Theorem 2.1 yields a continuous function $\gamma: \Lambda \to E_1$ with the properties $\gamma(\lambda_i) = v_i$ for each $i = 1, \ldots, p$, and $\gamma(\lambda) \in Fix(\Gamma(\lambda, \cdot))$ for all $\lambda \in \Lambda$. Clearly, the mapping $s: \Lambda \to C^1(I, \boldsymbol{R}^n)$ defined by $s(\lambda) = v_\lambda + T(\gamma(\lambda))$, $\lambda \in \Lambda$, is continuous and one has $s(\lambda_i) = x_i$, $i = 1, \ldots, p$, $s(\lambda) \in S(\lambda)$ for every $\lambda \in \Lambda$.

The proof of assertion (k_3) is easily accomplished bearing in mind the arcwise connectedness of Λ and conclusion (k_2).

Finally, we show that assertion (k_4) is true. Let K be a convex, compact subset of Λ such that

$$\{x(b): x \in S(\lambda), \lambda \in K\} \subseteq K. \tag{3}$$

For any $\lambda \in K$, we set $\sigma(\lambda) = s(\lambda)(b)$, where s is a function given by conclusion (k_2). Obviously, the mapping $\sigma: K \to \Lambda$ is continuous and, due to (3), one has $\sigma(K) \subseteq K$. Thus, by the Schauder Fixed Point Theorem, there exists $\lambda^\star \in K$ such that $\sigma(\lambda^\star) = \lambda^\star$. This produces two functions, $u \in E_2$ and $x \in C^1(I, \boldsymbol{R}^n)$, with the required properties: $x'(t) = f(t, x(t), u(t))$, $u(t) \in U(t, x(t))$ for all $t \in I$; $x(a) = x(b)$. Therefore, the proof is complete. $\triangle$

REMARK 3.1. It is of interest to note that the conclusion (k_1) of the preceding theorem is no longer true without assuming the multifunction U convex-valued. In fact, consider for instance the case when $f(t,x,y) = x + 2^{-1}y$ and $U(t,x) = \{y_1, y_2\}$, $(t,x,y) \in I \times \boldsymbol{R}^n \times \boldsymbol{R}^n$, where y_1 and y_2 are two different points of $\boldsymbol{R}^n$. An easy computation shows that the set $S(\lambda)$ is not arcwise connected for all $\lambda \in \Lambda$, although the function f is continuous, the multifunction U has nonempty, closed values and the hypotheses (c_1)-(c_3) of Theorem 3.3 are fulfilled.

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