Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

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Balls for the Kobayashi distance and extension of the automorphisms of strictly convex domains in C^n with real analytic boundary

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **5** (1994), n.2, p. 193–196.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1994_9_5_2_193_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1994.

Rend. Mat. Acc. Lincei
s. 9, v. 5:193-196 (1994)

Geometria. — *Balls for the Kobayashi distance and extension of the automorphisms of strictly convex domains in* $\mathbf{C}^n$ *with real analytic boundary.* Nota di ANDREA IANNUZZI, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — It is shown that given a bounded strictly convex domain Ω in C^n with real analitic boundary and a point x_0 in Ω, there exists a larger bounded strictly convex domain Ω' with real analitic boundary, close as wished to Ω, such that Ω is a ball for the Kobayashi distance of Ω' with center x_0. The result is applied to prove that if Ω is not biholomorphic to the ball then any automorphism of Ω extends to an automorphism of Ω'.

KEY WORDS: Kobayashi distance; Automorphisms of bounded domains; Complex Monge Ampère equation.

RIASSUNTO. — *Palle per la distanza di Kobayashi ed estensione degli automorfismi di domini strettamente convessi di* C^n *con bordo analitico reale*. Si dimostra che dato un dominio limitato strettamente convesso Ω in C^n con bordo analitico reale e un punto x_0 in Ω, esiste un dominio limitato strettamente convesso Ω' con bordo analitico reale contenente Ω ma vicino ad esso quanto si vuole, tale che Ω è una palla per la distanza di Kobayashi di Ω' con centro in x_0. Come applicazione si dimostra che se Ω non è biolomorfo alla palla, allora ogni suo automorfismo si estende a un automorfismo di Ω'.

1. In this *Note* we apply the well known results of Lempert [5] about the intrinsic metrics of a strictly convex domain in $\mathbf{C}^n$ and properties of the solutions of the complex homogeneous Monge-Ampère equation to prove the following result:

THEOREM 1.1. *Let* Ω *be a bounded strictly convex domain in* $\mathbf{C}^n$ *with real analytic boundary and fix* $x_0 \in \Omega$. *Then given any open neighborhood* U *of* $\overline{\Omega}$ *there exists a bounded strictly convex domain* Ω' *with real analytic boundary with* $\Omega \subset \overline{\Omega} \subset \Omega' \subset U$ *and such that for some* $r > 0$, *if* $k_{\Omega'}$ *denotes the Kobayashi distance on* Ω', *we have* $\Omega = \{x \in \Omega' \mid k_{\Omega'}(x_0, x) < r\}$.

In [8] Vitushkin proved that if D is a strictly pseudoconvex domain in $\mathbf{C}^n$ not biholomorphic to the ball, then there exists a larger strictly pseudoconvex domain D' such that each automorphism of D extends to an automorphism of D'. As a consequence of Theorem 1.1 and of results of Bland-Duchamp-Kalka [3], with an elementary proof, we get a more precise version of Vitushkin's theorem in the case of strictly convex domains:

COROLLARY 1.2. *Let* Ω *be a bounded, strictly convex domain in* $\mathbf{C}^n$ *with real analytic boundary and not biholomorphic to the unit ball of* $\mathbf{C}^n$. *Then given any open neighborhood* U *of* $\overline{\Omega}$ *there exists a bounded strictly convex domain* Ω' *with*

(*) Nella seduta dell'11 dicembre 1993.

real analytic boundary with $\Omega \subset \overline{\Omega} \subset \Omega' \subset U$ *and such that every automorphism of* Ω *extends to an automorphism of* Ω'*; in other words* $\mathrm{Aut}(\Omega) \subset \mathrm{Aut}(\Omega')$.

For sake of simplicity we expressed the «closeness as wished» of the domain Ω' to the domain Ω in topological terms but it should be observed that without much effort the proof yields the same result in terms of any reasonable norm on the space of domains of $\boldsymbol{C}^n$.

2. Let Ω be a bounded strictly convex domain in $\boldsymbol{C}^n$ with smooth boundary, fix $x_0 \in \Omega$ and let k_Ω denote the Kobayashi distance on Ω. We define the *Lempert exhaustion* τ *for* Ω *at* x_0 by $\tau \equiv 1$ on $\partial\Omega$ and $\tau(x) = \tanh^2(k_\Omega(x_0, x))$.

Then we can summarize the properties of τ as follows:

Proposition 2.1. *Let* Ω *be a strictly convex domain in* $\boldsymbol{C}^n$ *with boundary of class* C^ω*, fix* $x_0 \in \Omega$ *and set* $u = \log \tau$ *where* τ *is the Lempert exhaustion for* Ω *at* x_0*. Then* τ *is a continuous exhaustion of* $\overline{\Omega}$ *of class* C^ω *and strictly plurisubharmonic on* $\overline{\Omega} - \{x_0\}$*. Furthermore* $u: \overline{\Omega} - \{x_0\} \to \boldsymbol{R}$ *is the unique function such that*

(*i*) $u = 0$ *on* $\partial\Omega$,

(*ii*) $u \in C^\omega(\overline{\Omega} - \{x_0\})$,

(*iii*) $u = \log|x - x_0|^2 + o(1)$ *as* $x \to x_0$,

(*iv*) $(dd^c u)^n = 0$ *on* $\overline{\Omega} - \{x_0\}$,

(*v*) $dd^c u \geq 0$ *on* $\overline{\Omega} - \{x_0\}$.

The proofs of the statements this proposition may be found in [5-7], and for the uniqueness part, depend on the maximum principle of Bedford and Taylor [4]. In fact corresponding result holds under weaker differentiability assumption. Proposition 2.1 is the key ingredient for the proof of Theorem 1.1

Proof of Theorem 1.1. Let $u = \log \tau$ and, in the neighborhood of $\partial\Omega$ consider the following Cauchy type problem

$$
\begin{cases} (dd^c \upsilon)^n = 0\,, & \\ \upsilon(x) = 0\,, & x \in \partial\Omega\,, \\ \dfrac{d\upsilon}{du}(x) = 1\,, & x \in \partial\Omega\,. \end{cases} \tag{2.1}
$$

The domain Ω is strictly convex, and hence strictly pseudoconvex, with real analytic boundary and u is a global defining function for $\partial\Omega$. Bedford-Burns [2] solved (2.1) as an application of Cauchy-Kovalevski theorem. There is a real analytic solution of (2.1) in a neighbourhood V of $\partial\Omega$ which is an analytic extension of u provided the problem satisfies the non characteristic condition

$$
du \wedge d^c u \wedge (dd^c u)^{n-1} \neq 0\,. \tag{2.2}
$$

Being Ω strictly pseudoconvex and u a defining function, (2.2) is automatically satisfied. Thus we obtain u defined on $V \cup \Omega$ and for sufficiently small $\varepsilon > 0$ we can consider

$\Omega' = \{x \in V \cup \Omega \mid u(x) < \varepsilon\} \subset\subset V \cup \Omega$ and assume that Ω' is strictly convex. Furthermore given any open neighborhood U of $\bar{\Omega}$ it is possible to choose ε so that $\Omega \subset \bar{\Omega} \subset \Omega' \subset U$.

Moreover the Lempert exhaustion τ of $\bar{\Omega}$ at x_0 is strictly plurisubharmonic on $\bar{\Omega} - \{x_0\}$, and hence $dd^c \tau > 0$ on $\bar{\Omega} - \{x_0\}$. But then for ε small enough we may obtain

$$dd^c \tau > 0 \quad \text{on} \quad \bar{\Omega}' - \{x_0\}. \tag{2.3}$$

Set now $u' = u - \varepsilon$. Then u' is a real analytic defining function for $\partial\Omega'$ and by construction we have

(*i*) $u' = 0$ on $\partial\Omega'$,

(*ii*) $u' \in C^\omega(\bar{\Omega} - \{x_0\})$,

(*iii*) $u'(x) = u(x) - \varepsilon = \log|x - x_0|^2 + o(1)$ as $x \to x_0$,

(*iv*) $(dd^c u')^n = 0$ on $\bar{\Omega}' - \{x_0\}$.

Moreover by (2.3) we obtain $dd^c(e^{u'}) > 0$, and this fact along with (*iv*) implies (see for example Remark 2 at page 242 of [9])

(*v*) $dd^c u' \geqslant 0$ on $\bar{\Omega} - \{x_0\}$.

It follows that it is possible to apply Theorem 2.1 to obtain $u' = \log \tau'$ where τ' is the Lempert exhaustion for Ω' at x_0. But then $\Omega = \{x \in \Omega' \mid u' < -\varepsilon\} = \{x \in \Omega' \mid \tau' < e^{-\varepsilon}\}$ and since $\tau'(x) = \tanh^2(k_{\Omega'}(x_0, x))$ this means that Ω is a ball with respect to the distance of Kobayashi defined on Ω'.

3. By proving Corollary 1.2, we now illustrate how it is possible to apply Theorem 1.1 in order to study extension phenomena for the automorphism group of a strictly convex domain. We premise a result due to Bland-Duchamp-Kalka, proved in [3].

Proposition 3.2. *Let Ω and Ω' be two bounded, strictly convex domains in C^n with boundaries of class C^∞ and let Ω_r and Ω'_r be the Kobayashi balls of radius $r > 0$ about $x_0 \in \Omega$ and $y_0 \in \Omega'$. Then every biholomorphism $F: \Omega_r \to \Omega'_r$ sending x_0 to y_0 is restriction of a biholomorphism between Ω and Ω'.*

The further element of the proof of Corollary 1.2 is the fact that if Ω is a strictly convex domain not biholomorphic to the ball, the automorphism group admits a common fixed point x_0 (see for example Abate [1, Theorem 2.5.7]). Because of Theorem 1.1 given any open neighborhood U of $\bar{\Omega}$ there exists a bounded strictly convex domain Ω' with real analytic boundary with $\Omega \subset \bar{\Omega} \subset \Omega' \subset U$ and so that Ω is a ball with respect to the Kobayashi distance of Ω'. Thus it is possible to apply Proposition 3.2 to extend every automorphism of Ω to an automorphism of Ω' and we are done.

Remark. In the case of the ball this result is false. It is well known that there are infinitely many automorphism which extend holomorphically out of the ball, but do not extend as automorphism of a bigger domain. Moreover, even choosing only those automorphism which extend, it is not possible to find a common domain of extension. Indeed consider the family of automorphism $g_t : \Delta \rightarrow \Delta$ of the unit disc Δ defined for $t \in (0, 1)$ by $g_t(\zeta) = -(\zeta - t)/(1 - t\zeta)$.

There exists a unique fixed point $x = (1 - \sqrt{1 - t^2})/t$ of g_t in Δ and so each g_t extends to an automorphism of a domain Ω_t containing Δ. But we also notice that g_t has a singularity $z = 1/t$ and $1/t \rightarrow 1 \in \partial\Delta$ as $t \rightarrow 1$. Thus there is not a common domain of extension for this family of automorphisms. The same argument is valid in higher dimension since every automorphism of Δ can be extended to an automorphism of the unit ball in C^n.

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