Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Giovanni Bellettini, Maurizio Paolini

Convex approximation of an inhomogeneous anisotropic functional

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **5** (1994), n.2, p. 177–187.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1994_9_5_2_177_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1994.

Rend. Mat. Acc. Lincei
s. 9, v. 5:177-187 (1994)

Analisi numerica. — *Convex approximation of an inhomogeneous anisotropic functional.* Nota di GIOVANNI BELLETTINI e MAURIZIO PAOLINI, presentata (*) dal Socio E. Magenes.

ABSTRACT. — The numerical minimization of the functional $\mathcal{F}(u) = \int_\Omega \phi(x, \nu_u)\,|Du| + \int_{\partial\Omega} \mu u\, d\mathcal{H}^{n-1} - \int_\Omega \kappa u\, dx$, $u \in BV(\Omega; \{-1, 1\})$, is addressed. The function ϕ is continuous, has linear growth, and is convex and positively homogeneous of degree one in the second variable. We prove that $\mathcal{F}$ can be equivalently minimized on the convex set $BV(\Omega; [-1, 1])$ and then regularized with a sequence $\{\mathcal{F}_\varepsilon(u)\}_\varepsilon$ of strictly convex functionals defined on $BV(\Omega; [-1, 1])$. Then both $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ can be discretized by continuous linear finite elements. The convexity property of the functionals on $BV(\Omega; [-1, 1])$ is useful in the numerical minimization of $\mathcal{F}$. The $\Gamma - L^1(\Omega)$-convergence of the discrete functionals $\{\mathcal{F}_h\}_h$ and $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ to $\mathcal{F}$, as well as the compactness of any sequence of discrete absolute minimizers, are proven.

KEY WORDS: Calculus of variations; Anisotropic surface energy; Finite elements; Convergence of discrete approximations.

RIASSUNTO. — *Approssimazione convessa di un funzionale non omogeneo ed anisotropo.* Si studia la minimizzazione numerica del funzionale $\mathcal{F}(u) = \int_\Omega \phi(x, \nu_u)\,|Du| + \int_{\partial\Omega} \mu u\, d\mathcal{H}^{n-1} - \int_\Omega \kappa u\, dx$. La funzione ϕ è continua, ha crescita lineare ed è convessa e positivamente omogenea di grado uno nella seconda variabile. Si dimostra che $\mathcal{F}$ può essere equivalentemente minimizzato sull'insieme convesso $BV(\Omega; [-1, 1])$ e successivamente regolarizzato con una successione $\{\mathcal{F}_\varepsilon(u)\}_\varepsilon$ di funzionali strettamente convessi definiti su $BV(\Omega; [-1, 1])$. $\mathcal{F}$ e $\mathcal{F}_\varepsilon$ sono poi discretizzati con elementi finiti lineari continui. La convessità dei funzionali su $BV(\Omega; [-1, 1])$ è utile nella minimizzazione numerica di $\mathcal{F}$. Si dimostra infine la $\Gamma - L^1(\Omega)$-convergenza dei funzionali $\{\mathcal{F}_h\}_h$ e $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ a $\mathcal{F}$ e la compattezza di successioni di punti di minimo discreti assoluti.

0. INTRODUCTION

Several problems in the Calculus of Variations that fall in the general framework proposed by De Giorgi [8], arising in phase transitions [4] and crystal growth [5] involve functionals depending in an inhomogeneous and anisotropic way on an interfacial energy. For instance, let Ω be a bounded smooth domain of $\mathbb{R}^n$, and let $\phi: \Omega \times \mathbb{R}^n \to [0, +\infty[$ be a continuous function with linear growth, convex and positively homogeneous of degree one in the second variable. Given a smooth set $E \subseteq \mathbb{R}^n$, the typical interfacial term is of the form

$$\int_{\Omega \cap \partial E} \phi(x, \nu_E(x))\, d\mathcal{H}^{n-1}(x),$$

where $\nu_E(x)$ denotes the outward unit normal vector of ∂E at the point x, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

The study of minimum problems involving such functionals is also related to the ap-

(*) Nella seduta del 13 novembre 1993.

proximation of the motion of an interface, which propagates with a velocity depending on the position, the normal vector, and the mean curvature [1].

In this paper we generalize the numerical minimization via convex approximation presented in [2] to a model functional with an anisotropic and inhomogeneous surface term. This can be viewed as a preliminary step for the study of the geometric motion of fronts by anisotropic curvature.

More precisely, given two functions $\kappa \in L^\infty(\Omega)$ and $\mu \in L^\infty(\partial\Omega)$, and assuming that $\phi(\cdot, \xi)$ can be extended in a continuous way up to $\partial\Omega$, we consider the minimum problem:

$$\min_{u \in BV(\Omega;\{-1,1\})} \mathcal{F}(u), \qquad \text{where} \qquad \mathcal{F}(u) = \int_\Omega \phi(x, \nu_u)\,|Du| + \int_{\partial\Omega} \mu u \, d\mathcal{H}^{n-1} - \int_\Omega \kappa u \, dx .$$

If the solution to this problem is the characteristic function of a set $A \subseteq \Omega$ (with values 1 in A and -1 in $\Omega \setminus A$) with smooth boundary, one can prove that $\Omega \cap \partial A$ has mean curvature related to κ and ϕ, and that the contact angle at the intersection of ∂A with $\partial\Omega$ is suitably related to μ and ϕ.

Following the ideas in [2], we shall equivalently minimize $\mathcal{F}$ on the larger convex set $BV(\Omega; [-1, 1])$. The (nonstrict) convexity of $\mathcal{F}$ can be exploited for the numerical minimization of $\mathcal{F}$ via linear finite elements discretizations. Since the numerical algorithms perform better for strictly convex functionals, $\mathcal{F}$ is preliminarly regularized by a sequence $\{\mathcal{F}_\varepsilon\}_\varepsilon$ of convex functionals.

The main result of this paper is the Γ-convergence of the discrete functionals $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ to $\mathcal{F}$ when ε and h go to zero independently. Since the compactness of each family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ of discrete absolute minima is also proved, in view of basic properties of Γ-convergence [9], the family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ admits a subsequence converging to a minimum point u of $\mathcal{F}$ and $\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h})$ converges to $\mathcal{F}(u)$.

1. The setting

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ be a bounded open set with Lipschitz continuous boundary and denote by $|\cdot|$ the n-dimensional Lebesgue measure and by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$ [10]. If $f: \Omega \to \mathbf{R}$ is a function and $t \in \mathbf{R}$, we set $\{f > t\} = \{x \in \Omega: f(x) > t\}$, $\{f = t\} = \{x \in \Omega: f(x) = t\}$.

If λ is a (possibly vector-valued) Radon measure, its total variation will be denoted by $|\lambda|$. If λ_0 is a scalar Radon measure on Ω such that λ is absolutely continuous with respect to λ_0, the symbol λ/λ_0 stands for the Radon-Nikodym derivative of λ with respect to λ_0.

The space $BV(\Omega)$ is defined as the space of the functions $u \in L^1(\Omega)$ whose distributional gradient Du is an $\mathbf{R}^n$-valued Radon measure with bounded total variation in Ω. Since no confusion is possible, we denote by $u \in L^1(\partial\Omega)$ the trace of $u \in BV(\Omega)$ on $\partial\Omega$ and set $\nu_u(x) = (Du/|Du|)(x)$ for $|Du|$-almost every $x \in \Omega$. We also set

$$Du = \nabla u \, dx + D^s u ,$$

where ∇u denotes the density of the absolutely continuous part of Du with respect to the Lebesgue measure and $D^s u$ stands for the singular part. One can prove that ∇u coincides almost everywhere with the approximate differential of u.

Let $E \subseteq \mathbf{R}^n$ be a measurable set; we denote by χ_E the characteristic function of E, *i.e.*, $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = -1$ if $x \notin E$, and we set $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ if $x \notin E$. We say that E has finite perimeter in Ω if $\int_\Omega |D1_E| < +\infty$, and we denote by $P(E, \Omega)$ its perimeter. We indicate by $\partial^* E$ the reduced boundary of E. We introduce the two closed subsets of $BV(\Omega)$ as $\tilde{K} = BV(\Omega; \{-1, 1\})$ and $K = BV(\Omega; [-1, 1])$. Given $u \in BV(\Omega)$ we set $S(u) = \{(x, s) \in \Omega \times \mathbf{R}: s < u^+(x)\}$; it turns out that $S(u)$ is a set of finite perimeter in $\Omega \times \mathbf{R}$.

For the definitions and the main properties of the functions of bounded variation and of sets of finite perimeter we refer to [10, 12, 14, 16].

For any $\mathcal{L}: BV(\Omega) \to [\inf \mathcal{L}, +\infty]$ with $-\infty < \inf \mathcal{L}$, we denote by $\overline{\mathcal{L}}: BV(\Omega) \to$ $\to [\inf \mathcal{L}, +\infty]$ the lower semicontinuous envelope (or relaxed functional) of $\mathcal{L}$ with respect to the $L^1(\Omega)$-topology. The functional $\overline{\mathcal{L}}$ is defined as the greatest $L^1(\Omega)$-lower semicontinuous functional less than or equal to $\mathcal{L}$ and can be characterized as

$$\overline{\mathcal{L}}(u) = \inf\left\{\liminf_{h \to +\infty} \mathcal{L}(u_h): \{u_h\}_h \subseteq BV(\Omega),\ u_h \overset{L^1(\Omega)}{\longrightarrow} u\right\}.$$

For the main properties of the relaxed functionals we refer to [3].

From now on $\phi: \overline{\Omega} \times \mathbf{R}^n \to [0, +\infty[$ will be a continuous function satisfying the properties

$$(1.1) \qquad \phi(x, t\xi) = |t|\, \phi(x, \xi) \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbf{R}^n, \quad \forall t \in \mathbf{R},$$

$$(1.2) \qquad \lambda|\xi| \leq \phi(x, \xi) \leq \Lambda|\xi| \quad \forall x \in \overline{\Omega}, \quad \forall \xi \in \mathbf{R}^n,$$

for two suitable positive constants $0 < \lambda \leq \Lambda < +\infty$, and such that $\phi(x, \cdot)$ is convex on $\mathbf{R}^n$ for any $x \in \overline{\Omega}$. Further regularity assumptions on ϕ will be required afterwards (see (1.9)).

Let us recall the following coarea-type formula

$$(1.3) \qquad \int_\Omega \phi(x, \nu_u)\, |Du| = \int_{\mathbf{R}} \int_{\Omega \cap \partial^*\{u > t\}} \phi(x, \nu_t)\, d\mathcal{H}^{n-1}(x)\, dt \qquad \forall u \in BV(\Omega),$$

where ν_t stands for the outer unit normal vector to the set $\Omega \cap \partial^* \{u > t\}$.

1.1 *The continuous functional.* Let $\mu \in L^\infty(\partial\Omega)$ be such that

$$(1.4) \qquad |\mu(x)| \leq \phi(x, \nu_\Omega(x)) \qquad \text{for } \mathcal{H}^{n-1}\text{ - a.e. } x \in \partial\Omega,$$

where $\nu_\Omega(x)$ denotes a unit normal vector to $\partial\Omega$ at the point x. Let $\kappa \in L^\infty(\Omega)$. We define the functional $\mathcal{F}: BV(\Omega) \to [\inf \mathcal{F}, +\infty]$, for any $u \in K$, as

$$\mathcal{F}(u) = \int_\Omega \phi(x, \nu_u)\, |Du| + \int_{\partial\Omega} \mu u\, d\mathcal{H}^{n-1} - \int_\Omega \kappa u\, dx,$$

and set $\mathcal{F} = +\infty$ on $BV(\Omega) \setminus K$. As a consequence of the following semicontinuity result and the boundedness from below, $\mathcal{F}$ admits at least one minimum point.

THEOREM 1.1. *The functional $\mathcal{F}$ is lower semicontinuous on K with respect to the topology of $L^1(\Omega)$.*

PROOF. First we note that any $\mu \in L^\infty(\partial\Omega)$ verifying (1.4) can be approximated in $L^1(\partial\Omega)$ by a sequence of functions $\{\mu^\delta\}_{\delta>0}$ of the form

$$\mu^\delta(x) = \phi(x, \nu_\Omega(x)) \sum_{i=0}^{N^\delta} \mu_i^\delta \, 1_{F_i^\delta}(x),$$

where $-1 = \mu_0^\delta < \ldots < \mu_N^\delta = 1$, and $\{F_0^\delta, \ldots, F_N^\delta\}$ is a measurable partition of $\partial\Omega$. Here F_0^δ and F_N^δ might be empty. Denoting by $\mathcal{F}^\delta$ the functional $\mathcal{F}$ with μ replaced by μ^δ, we have, for any $u \in K$,

$$|\mathcal{F}^\delta(u) - \mathcal{F}(u)| \leq \int_{\partial\Omega} |u| \, |\mu - \mu^\delta| \, d\mathcal{H}^{n-1} \leq \|\mu - \mu^\delta\|_{L^1(\partial\Omega)} \to 0,$$

as $\delta \to 0$. Namely, $\mathcal{F}^\delta \to \mathcal{F}$ uniformly on K as $\delta \to 0$. Since the uniform limit of semicontinuous functions is semicontinuous, the assertion of the theorem is thus reduced to prove that any $\mathcal{F}^\delta$ is $L^1(\Omega)$-lower semicontinuous on K. Since no confusion is possible, we omit the superscript δ.

Set $\alpha_i = (\mu_i - \mu_{i-1})/2 > 0$ and $G_i = \{\mu \geq \mu_i\} \subseteq \partial\Omega$, for all $1 \leq i \leq N$. Note that neither $\mu_0 = -1$ nor $\mu_N = 1$ are necessarily assumed, namely, that $G_1 = \partial\Omega$ and $G_N = \emptyset$ are allowed. Since

$$\sum_{i=1}^{N} \alpha_i = 1 \quad \text{and} \quad \mu(x) = \phi(x, \nu_\Omega) \sum_{i=1}^{N} \alpha_i \chi_{G_i}(x) \quad \text{for} \quad \mathcal{H}^{n-1}\text{- a.e. } x \in \partial\Omega,$$

the functional $\mathcal{F}$ can be represented as a convex combination of functionals $\mathcal{F}^i$ as follows:

$$\mathcal{F}(u) = \sum_{i=1}^{N} \alpha_i \left[\int_\Omega \phi(x, \nu_u) \, |Du| + \int_{\partial\Omega} \phi(x, \nu_\Omega) \chi_{G_i} u \, d\mathcal{H}^{n-1} - \int_\Omega \kappa u \, dx \right] =: \sum_{i=1}^{N} \alpha_i \mathcal{F}^i(u).$$

To prove the lower semicontinuity of $\mathcal{F}$ it will be enough to show that each $\mathcal{F}^i$ is lower semicontinuous. For simplicity we omit the index i, thus denoting $G_i = G$ a measurable subset of $\partial\Omega$, and assume

$$\mu(x) = \phi(x, \nu_\Omega) \chi_G(x). \tag{1.5}$$

Let B be a ball containing $\overline{\Omega}$ and define

$$\Phi(x, \xi) = \begin{cases} \phi(x, \xi) & \text{if } (x, \xi) \in \overline{\Omega} \times \mathbf{R}^n, \\ \Lambda|\xi| & \text{if } (x, \xi) \in (B \setminus \overline{\Omega}) \times \mathbf{R}^n. \end{cases}$$

Then Φ is lower semicontinuous on $B \times \mathbf{R}^n$ (recall (1.2)). We can extend $-\chi_G \in L^1(\partial\Omega)$ to a function $w \in W^{1,1}(B \setminus \overline{\Omega}; [-1, 1])$ with trace $-\chi_G$ on $\partial\Omega$, so that there exists $C > 0$ such that $\|w\|_{W^{1,1}(B\setminus\overline{\Omega})} \leq C \|\chi_G\|_{L^1(\partial\Omega)}$ [11, Theorem 1.II; 12, Theorem 2.16].

For any $u \in K$ we define $U \in BV(B; [-1, 1])$ as follows:

$$U = \begin{cases} u & \text{on } \Omega, \\ w & \text{on } B \setminus \Omega. \end{cases}$$

Obviously B and w do not depend on u, hence $\Lambda \int_{B \setminus \bar{\Omega}} |\nabla w| \, dx$ is a constant, and we shall denote it by c_1; set also $c_2 = \int_{\partial\Omega} \phi(x, \nu_\Omega) \, d\mathcal{H}^{n-1}(x)$. Recalling that $|u| \leq 1$, we find [7]

$$\int_B \Phi(x, \nu_U) \, |DU| = \int_\Omega \phi(x, \nu_u) \, |Du| + \int_{\partial\Omega} \phi(x, \nu_\Omega) \, |u + \chi_G| \, d\mathcal{H}^{n-1} + c_1 =$$

$$= \int_\Omega \phi(x, \nu_u) \, |Du| + \int_{\partial\Omega} \phi(x, \nu_\Omega) u \chi_G \, d\mathcal{H}^{n-1} + c_1 + c_2 .$$

Hence, recalling (1.5) we have

$$\int_\Omega \phi(x, \nu_u) \, |Du| + \int_{\partial\Omega} \mu u \, d\mathcal{H}^{n-1} = \int_B \Phi(x, \nu_U) \, |DU| - (c_1 + c_2) .$$

Recalling the definition and the convexity of Φ, the functional $\int_B \Phi(x, \nu_U) \, |DU|$ is L^1-lower semicontinuous. Since the map $u \to \int_\Omega \kappa u \, dx$ is continuous with respect to the topology of $L^1(\Omega)$, the assertion follows. □

If ϕ is not convex in ξ then $\mathcal{F}$ is not, in general, lower semicontinuous, and the lower semicontinuous envelope of the functional $u \to \int_\Omega \phi(x, \nabla u) \, dx$ on $W^{1,1}(\Omega)$ can be written on $BV(\Omega) \cap L^\infty(\Omega)$ as $\int_\Omega \phi^{**}(x, \nu_u) \, |Du|$, where ϕ^{**} denotes the greatest function that is convex in ξ and less than or equal to $\phi(x, \xi)$ for all $(x, \xi) \in \Omega \times \mathbf{R}^n$. In addition, as in [2], if condition (1.4) is not fulfilled, $\mathcal{F}$ is not lower semicontinuous. Observe that $\mathcal{F}$ admits at least a minimum point $u \in K$ ($u \in \tilde{K}$, respectively), because of condition (1.2) and since $\mathcal{F}$ is lower semicontinuous on K (on $\tilde{K}$, respectively).

The following theorem shows that to minimize $\mathcal{F}$ on $\tilde{K}$ is equivalent to minimize $\mathcal{F}$ on the convex set K, and this reads as a (nonstrictly) convex problem.

Theorem 1.2. *Suppose that $u \in K$ is a minimum point of $\mathcal{F}$ on K. Then*

$$\mathcal{F}(u) = \mathcal{F}(\chi_{\{u > t\}}) \quad \textit{for a.e. } t \in [-1, 1],$$

namely, $\chi_{\{u > t\}} \in \tilde{K}$ is a minimum point of $\mathcal{F}$ on $\tilde{K}$ for almost every $t \in [-1, 1]$.

Proof. For all $v \in K$, from (1.3) and the Cavalieri formula we have

$$\mathcal{F}(v) = \int_{-1}^{1} \int_{\Omega \cap \partial^* \{u > t\}} \phi(x, \nu_t) \, d\mathcal{H}^{n-1} dt + \frac{1}{2} \int_{-1}^{1} \int_{\partial\Omega} \mu \chi_{\{v > t\}} \, d\mathcal{H}^{n-1} dt -$$

$$- \frac{1}{2} \int_{-1}^{1} \int_\Omega \kappa \chi_{\{v > t\}} \, dx \, dt = \frac{1}{2} \int_{-1}^{1} \mathcal{F}(\chi_{\{v > t\}}) \, dt ,$$

that is

$$\int_{-1}^{1} (\mathcal{F}(\chi_{\{v>t\}}) - \mathcal{F}(v))\, dt = 0 \qquad \forall v \in K.$$

The minimality of u on K entails $\mathcal{F}(\chi_{\{u>t\}}) - \mathcal{F}(u) \geq 0$; therefore $\mathcal{F}(u) = \mathcal{F}(\chi_{\{u>t\}})$ for almost every $t \in [-1, 1]$. □

REMARK 1.1. In view of Theorem 1.2, we have that $\min_{v \in \tilde{K}} \mathcal{F}(v) = \min_{v \in K} \mathcal{F}(v)$; moreover $\mathcal{F}$ has a unique minimum point on $\tilde{K}$ if and only if $\mathcal{F}$ has a unique minimum point on K, and they coincide. Note that $\mathcal{F}$ may exhibit relative minima on $\tilde{K}$; in view of the convexity of K, they are no longer relative minima of $\mathcal{F}$ on K.

1.2. *The regularized functionals.* Given $\varepsilon \geq 0$, in analogy with [2], we define a regularization of ϕ as follows

$$\phi_\varepsilon(x, \xi) = \sqrt{\varepsilon^2 + (\phi(x, \xi))^2}, \tag{1.6}$$

for all $(x, \xi) \in \bar{\Omega} \times \boldsymbol{R}^n$. Let us consider the map $G_\varepsilon \colon BV(\Omega) \to [0, +\infty]$ defined by

$$G_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \phi_\varepsilon(x, \nabla u)\, dx & \text{if } u \in W^{1,1}(\Omega), \\ +\infty & \text{elsewhere}. \end{cases}$$

Observe that, by the continuity assumption on ϕ and by (1.1), there exists a continuous function $\omega \colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$, with $\omega(0) = 0$, such that

$$|\phi_\varepsilon(x, \xi) - \phi_\varepsilon(y, \xi)| \leq |\phi(x, \xi) - \phi(y, \xi)| \leq \omega(|x - y|)(1 + |\xi|)$$

for any $x, y \in \Omega$ and any $\xi \in \boldsymbol{R}^n$. Then, applying [7, Theorem 3.2] and observing that $\lim_{t \to 0^+} t\phi_\varepsilon(x, \xi/t) = \phi(x, \xi)$, we find that

$$\overline{G_\varepsilon}(u) = \int_\Omega \phi_\varepsilon(x, \nabla u)\, dx + \int_\Omega \phi\left(x, \frac{D^s u}{|D^s u|}\right) |D^s u| \qquad \forall u \in BV(\Omega).$$

We are now ready to define the regularized functionals $\mathcal{F}_\varepsilon \colon BV(\Omega) \to [\inf \mathcal{F}_\varepsilon, +\infty]$. For any $\varepsilon > 0$ and for any $u \in K$, we set

$$\mathcal{F}_\varepsilon(u) = \int_\Omega \phi_\varepsilon(x, \nabla u)\, dx + \int_\Omega \phi\left(x, \frac{D^s u}{|D^s u|}\right) |D^s u| + \int_{\partial\Omega} \mu u \, d\mathcal{H}^{n-1} - \int_\Omega \kappa u \, dx, \tag{1.7}$$

and we set $\mathcal{F}_\varepsilon = +\infty$ on $BV(\Omega) \setminus K$.

THEOREM 1.3. *For any $\varepsilon > 0$ the functional $\mathcal{F}_\varepsilon$ is lower semicontinuous on K with respect to the topology of $L^1(\Omega)$.*

PROOF. Reasoning as in the proof of Theorem 1.1, and using the same notation, we have

$$\mathcal{F}_\varepsilon(u) + \int_\Omega \kappa u \, dx = \int_B \sqrt{\varepsilon^2 + (\Phi(x, \nabla U))^2}\, dx + \int_B \Phi\left(x, \frac{D^s U}{|D^s U|}\right) |D^s U| - (c_2 + c_3)$$

where

$$c_3 = \int_{B \setminus \bar{\Omega}} \sqrt{\varepsilon^2 + \Lambda^2 |\nabla w|^2}\, dx .$$

As the functional at the right-hand side is L^1-lower semicontinuous (it is a lower semicontinuous envelope by [7]), the theorem follows. □

It is not difficult to show that, if condition (1.4) is not fulfilled, then the functional $\mathcal{F}_\varepsilon$ is not lower semicontinuous.

Observe that the restriction of $\mathcal{F}_\varepsilon$ to K ($\tilde{K}$, respectively) admits at least a minimum point $u \in K$ ($u \in \tilde{K}$, respectively), because of condition (1.2) and since $\mathcal{F}_\varepsilon$ is lower semicontinuous on K (on $\tilde{K}$, respectively). Observe also that, if $\mathcal{F}_\varepsilon$ has a minimum point $u_\varepsilon \in K \cap W^{1,1}_{\mathrm{loc}}(\Omega)$ then, since $\mathcal{F}_\varepsilon$ is strictly convex in $(BV(\Omega) \cap W^{1,1}_{\mathrm{loc}}(\Omega))/\boldsymbol{R}$, the minimum is unique up to a possible additive constant.

Remark 1.2. *We have $\mathcal{F}_\varepsilon \to \mathcal{F}$ uniformly in K as $\varepsilon \to 0$.*

Proof. For any $u \in K$, using (1.1), we have

$$|\mathcal{F}_\varepsilon(u) - \mathcal{F}(u)| = \varepsilon \left| \int_\Omega \sqrt{1 + \left(\phi\left(x, \nabla\left(\frac{u}{\varepsilon}\right)\right)\right)^2}\, dx - \int_\Omega \phi\left(x, \nabla\left(\frac{u}{\varepsilon}\right)\right) dx \right| \leq \varepsilon |\Omega| . \qquad \square$$

1.3. *The discrete functionals.* Let $\{\mathcal{S}_h\}_{h>0}$ denote a *regular* family of partitions of Ω into simplices [6]. Let $h_S \leq h$ denote the diameter of any $S \in \mathcal{S}_h$. For any $h > 0$, let $V_h \subset H^1(\Omega; [-1, 1]) \subset K$ be the piecewise linear finite element space over $\mathcal{S}_h$ with values in $[-1, 1]$ and Π_h be the usual Lagrange interpolation operator over the continuous piecewise linear functions. By C we shall mean an absolute positive constant whose value may vary at each occurrence. For the sake of simplicity, we shall assume that the discrete domain $\Omega_h = \bigcup_{S \in \mathcal{S}_h} S$ coincides with $\bar{\Omega}$. In order to introduce the discrete functionals $\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$, we approximate μ and κ as in [2] by a sequence of continuous piecewise linear functions $\mu_h \to \mu$ and $\kappa_h \to \kappa$ in L^1 as $h \to 0$ such that [6]

$$(1.8) \qquad \begin{aligned} &\|\mu_h\|_{L^\infty(\partial\Omega)} \leq \|\mu\|_{L^\infty(\partial\Omega)}, \qquad \|\nabla\mu_h\|_{L^1(\partial\Omega)} = o(h^{-1}), \\ &\|\kappa_h\|_{L^\infty(\Omega)} \leq \|\kappa\|_{L^\infty(\Omega)}, \qquad \|\nabla\kappa_h\|_{L^1(\Omega)} = o(h^{-1}). \end{aligned}$$

We define the discrete functionals as follows: for any $u \in V_h$ we set

$$\mathcal{F}_{\varepsilon,h}(u) = \sum_{S \in \mathcal{S}_h} \int_S \Pi_h(\phi_\varepsilon(x, \nabla u))\, dx + \int_{\partial\Omega} \Pi_h(\mu_h u)\, d\mathcal{H}^{n-1} - \int_\Omega \Pi_h(\kappa_h u)\, dx ,$$

$\mathcal{F}_{\varepsilon,h} = +\infty$ on $BV(\Omega) \setminus V_h$. Finally we define $\mathcal{F}_h = \mathcal{F}_{0,h}$. The piecewise constant interpolation $\int_\Omega \Pi_h^0(\phi_\varepsilon(x, \nabla u))\, dx$ can also be used in the first term without affecting the convergence result and allowing a simpler implementation of the numerical algorithms.

To prove the main theorem (2.1) we need the assumptions

$$\phi(\cdot, \xi) \in W^{1,\infty}(\Omega), \qquad |\nabla_x \phi(x, \xi)| \leq C|\xi| \qquad \forall (x, \xi) \in \Omega \times \mathbf{R}^n, \tag{1.9}$$

and that $\phi(x, \cdot)$ is Lipschitz continuous uniformly with respect to x.

If $u \in V_h$, by the properties of the Lagrange interpolation operator, noting that (1.6) gives $|\nabla_x \phi_\varepsilon(x, \nabla u)| \leq |\nabla_x \phi(x, \nabla u)|$ and using (1.9) we have

$$\left| \sum_{S \in S_h} \int_S (\Pi_h(\phi_\varepsilon(x, \nabla u)) - \phi_\varepsilon(x, \nabla u))\, dx \right| \leq \sum_{S \in S_h} \|\Pi_h(\phi_\varepsilon(x, \nabla u)) - \tag{1.10}$$

$$- \phi_\varepsilon(x, \nabla u)\|_{L^\infty(S)} |S| \leq Ch \sum_{S \in S_h} \|\nabla_x \phi(x, \nabla u)\|_{L^\infty(S)} |S| \leq Ch \int_\Omega |\nabla u|\, dx.$$

2. Convergence of the discretized functionals

Remark 2.1. *We have* $\lim_{\varepsilon \to 0} \mathcal{F}_{\varepsilon, h} = \mathcal{F}_h$ *uniformly in* V_h *and with respect to* h.

Proof. See Remark 1.2. □

The next main theorem generalizes [2, Theorem 3.1].

Theorem 2.1. *For any* $\varepsilon > 0$ *we have,*

$$\Gamma\text{-}\lim_{h \to 0} \mathcal{F}_h = \mathcal{F} \quad \textit{and} \quad \Gamma\text{-}\lim_{h \to 0} \mathcal{F}_{\varepsilon, h} = \mathcal{F}_\varepsilon \quad \textit{in } L^1(\Omega).$$

Proof. We give a unified proof for both cases $\varepsilon > 0$ and $\varepsilon = 0$, considering $\mathcal{F}_h = \mathcal{F}_{\varepsilon, h}$ and $\mathcal{F} = \mathcal{F}_\varepsilon$ if $\varepsilon = 0$. Hence, let $\varepsilon \geq 0$ be fixed. We split the proof into two steps, namely, we prove that the two following properties hold [9]:

(*i*) for any $u \in BV(\Omega)$ and any sequence $\{u_h\}_h$ in $BV(\Omega)$ converging to u in $L^1(\Omega)$ we have $\mathcal{F}_\varepsilon(u) \leq \liminf_{h \to 0} \mathcal{F}_{\varepsilon, h}(u_h)$;

(*ii*) for any $u \in BV(\Omega)$ there exists a sequence $\{u_h\}_h$ in $BV(\Omega)$ converging to u in $L^1(\Omega)$ such that $\mathcal{F}_\varepsilon(u) = \lim_{h \to 0} \mathcal{F}_{\varepsilon, h}(u_h)$.

Preliminarly we decompose $\mathcal{F}_{\varepsilon, h}(u_h)$, for all $u_h \in V_h$, as follows:

$$\mathcal{F}_{\varepsilon, h}(u_h) = \mathcal{F}_\varepsilon(u_h) + \int_{\partial\Omega} [\Pi_h(\mu_h u_h) - \mu u_h]\, d\mathcal{H}^{n-1} - \int_\Omega [\Pi_h(\kappa_h u_h) - \kappa u_h]\, dx + \tag{2.1}$$

$$+ \sum_{S \in S_h} \int_S (\Pi_h(\phi_\varepsilon(x, \nabla u_h)) - \phi_\varepsilon(x, \nabla u_h))\, dx =: \mathcal{F}_\varepsilon(u_h) + I_h + II_h + III_{\varepsilon, h}.$$

Recalling (1.8) and reasoning as in [2], one gets $\lim_{h \to 0} [|I_h| + |II_h|] = 0$.

Proof of Step (*i*). Let $u \in BV(\Omega)$ and $\{u_h\}_h$ in $BV(\Omega)$ be any sequence so that $u_h \to u$ in $L^1(\Omega)$ as $h \to 0$. We can assume that $u_h \in V_h$ for any h and that $\sup_h \mathcal{F}_{\varepsilon, h}(u_h) < +\infty$. From (1.2) we get $\sup_h \int_\Omega |\nabla u_h|\, dx < +\infty$, so that, in view of (1.10) we have $\lim_{h \to 0} |III_{\varepsilon, h}| = 0$. Then, using (2.1) and the lower semicontinuity of $\mathcal{F}_\varepsilon$

(Theorems 1.1 and 1.3), we conclude that

$$\mathcal{F}_\varepsilon(u) \leq \liminf_{h\to 0} \mathcal{F}_\varepsilon(u_h) = \liminf_{h\to 0} \mathcal{F}_{\varepsilon,h}(u_h),$$

and (*i*) is proved.

Proof of Step (*ii*). We can assume that $u \in K$. Given a ball B containing $\overline{\Omega}$, let $\tilde{u} \in W^{1,1}(B\setminus\overline{\Omega};[-1,1])$ be a function with trace u on $\partial\Omega$ [11] and denote again by $u \in BV(B;[-1,1])$ the function $u(x) = u(x)$ if $x \in \Omega$, $u(x) = \tilde{u}(x)$ if $x \in B\setminus\Omega$. Observe that

$$\int_{\partial\Omega} |Du| = 0. \tag{2.2}$$

Let $\eta_h = o(h^{-1/2})$ and $\{\delta_h\}_h$ be a family of mollifiers defined by $\delta_h(x) = \eta_h^n \delta(\eta_h x)$. Set $\widehat{u}_h(x) = (u * \delta_h)(x)$ for all $x \in B$, where u is extended to 0 outside B. It is well known [12, Proposition 1.15] that, recalling (2.2),

$$\lim_{h\to 0} \|\widehat{u}_h - u\|_{L^1(\Omega)} = 0, \quad \text{and} \quad \lim_{h\to 0} \int_\Omega |\nabla\widehat{u}_h|\,dx = \int_\Omega |Du|. \tag{2.3}$$

Set $u_h = \Pi_h \widehat{u}_h \in V_h$; then [2]

$$\lim_{h\to 0} \|u_h - u\|_{L^1(\Omega)} = 0, \quad \lim_{h\to 0} \int_\Omega |\nabla u_h|\,dx = \int_\Omega |Du|, \tag{2.4}$$

and

$$\lim_{h\to 0} \int_{\partial\Omega} |u_h - u|\,d\mathcal{H}^{n-1} = 0. \tag{2.5}$$

Hence, using Reshetnyak's Theorem [15] (see also [13]), we get

$$\lim_{h\to 0} \int_\Omega \phi(x, \nabla u_h)\,dx = \int_\Omega \phi(x, \nu_u)\,|Du|. \tag{2.6}$$

Using (2.1), (2.4), (2.5), and (2.6), we get (*ii*) when $\varepsilon = 0$.

Let $\varepsilon > 0$. One can prove (see [14, Theorems 1.8 and 1.10]) that the sequence $\{D1_{S(\widehat{u}_h)}\}_h$ converges weakly on $\Omega \times \mathbf{R}$ to $D1_{S(u)}$ and, using (2.2), that

$$\lim_{h\to 0} |D1_{S(\widehat{u}_h)}|(\Omega \times \mathbf{R}) = |D1_{S(u)}|(\Omega \times \mathbf{R}). \tag{2.7}$$

Let $\widetilde{\phi}_\varepsilon : \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^+ \to [0, +\infty]$ be defined by

$$\widetilde{\phi}_\varepsilon(x, s, \xi, t) = \begin{cases} t\phi_\varepsilon\left(x, \dfrac{\xi}{t}\right) & \text{if } t > 0, \\ \phi(x, \xi) & \text{if } t = 0. \end{cases}$$

Then $\widetilde{\phi}_\varepsilon$ is continuous, and the function $(\xi, t) \to \widetilde{\phi}_\varepsilon(x, s, \xi, t)$ is convex and positively homogeneous of degree one on $\mathbf{R}^n \times \mathbf{R}^+$. By [7, Lemma 2.2], for any $u \in K$ we have

$$\int_{\Omega\times\mathbf{R}} \widetilde{\phi}_\varepsilon\left(x, s, \frac{D1_{S(u)}}{|D1_{S(u)}|}\right) |D1_{S(u)}| = \int_\Omega \phi_\varepsilon(x, \nabla u)\,dx + \int_\Omega \phi\left(x, \frac{D^s u}{|D^s u|}\right) |D^s u|.$$

Using again Reshetnyak's Theorem (recall (2.7)) we have

$$\lim_{h\to 0}\int_\Omega \phi_\varepsilon(x,\nabla\widehat{u}_h)\,dx = \lim_{h\to 0}\int_{\Omega\times\boldsymbol{R}} \widetilde{\phi}_\varepsilon\left(x,s,\frac{D1_{S(\widehat{u}_h)}}{|D1_{S(\widehat{u}_h)}|}\right)|D1_{S(\widehat{u}_h)}| = \tag{2.8}$$

$$= \int_{\Omega\times\boldsymbol{R}} \widetilde{\phi}_\varepsilon\left(x,s,\frac{D1_{S(u)}}{|D1_{S(u)}|}\right)|D1_{S(u)}| = \int_\Omega \phi_\varepsilon(x,\nabla u)\,dx + \int_\Omega \phi\left(x,\frac{D^s u}{|D^s u|}\right)|D^s u|\,.$$

Observe that for any h we have

$$\left|\int_\Omega \phi_\varepsilon(x,\nabla\widehat{u}_h)\,dx - \int_\Omega \phi_\varepsilon(x,\nabla u_h)\,dx\right| \leq \int_\Omega |\phi(x,\nabla\widehat{u}_h)\,dx - \phi(x,\nabla u_h)|\,dx \to 0$$

as $h\to 0$, in view of the Lipschitz assumption on $\phi(x,\cdot)$ and the fact that [2]

$$\lim_{h\to 0}\|\widehat{u}_h - u_h\|_{W^{1,1}(\Omega)} = 0\,.$$

Using (2.8) we then find

$$\lim_{h\to 0}\int_\Omega \phi_\varepsilon(x,\nabla u_h)\,dx = \lim_{h\to 0}\int_\Omega \phi_\varepsilon(x,\nabla\widehat{u}_h)\,dx = \int_\Omega \phi_\varepsilon(x,\nabla u)\,dx + \int_\Omega \phi\left(x,\frac{D^s u}{|D^s u|}\right)|D^s u|\,.$$

This, together with (2.5) and (2.4), concludes the proof of (*ii*) when $\varepsilon > 0$. □

A straightforward consequence is the following Γ-convergence result for $\mathcal{F}_{\varepsilon,h}$, as ε and h go to 0 independently.

COROLLARY 2.1. *We have* $\Gamma\text{-}\lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h} = \mathcal{F}$ *in* $L^1(\Omega)$.

Finally, we prove the compactness of any sequence of approximated minima which, in view of basic properties of Γ-convergence gives, up to a subsequence, the convergence to a minimum of the original functional $\mathcal{F}$.

THEOREM 2.2. *Any family of absolute minima of the functionals* $\mathcal{F}_\varepsilon$, $\mathcal{F}_h$, *or* $\mathcal{F}_{\varepsilon,h}$, *is relatively compact in* $L^1(\Omega)$.

PROOF. Let $u_{\varepsilon,h}$ be a minimum point of $\mathcal{F}_{\varepsilon,h}$. Given any $v\in K$, from Corollary 2.1 there exists a sequence $\{v_{\varepsilon,h}\}_{\varepsilon,h}$ converging to v in $L^1(\Omega)$ as $(\varepsilon,h)\to(0,0)$, so that

$$\lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h}) = \mathcal{F}(v)\in\boldsymbol{R}\,.$$

Hence $\sup_{\varepsilon,h}\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}) \leq \sup_{\varepsilon,h}\mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h}) < +\infty$. Then we get

$$\sup_{\varepsilon,h}\int_\Omega |Du_{\varepsilon,h}| < +\infty\,,$$

and the assertion for $\mathcal{F}_{\varepsilon,h}$ follows from the compactness theorem in $BV(\Omega)$. The assertion for $\mathcal{F}_\varepsilon$ and $\mathcal{F}_h$ is similar. □

Work partially supported by NSF Grant DMS-9008999, by MURST (Progetto Nazionale «Equazioni di Evoluzione e Applicazioni Fisico-Matematiche» and «Analisi Numerica e Matematica Computazionale») and CNR (IAN Contracts 92.00833.01, 93.00564.01), Italy.

References

[1] F. Almgren - J. E. Taylor - L. Wang, *Curvature-driven flows: a variational approach.* SIAM J. Control Optim., 31, 1993, 387-437.

[2] G. Bellettini - M. Paolini - C. Verdi, *Convex approximations of functionals with curvature.* Rend. Mat. Acc. Lincei, s. 9, 2, 1991, 297-306.

[3] G. Buttazzo, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Longman Scientific & Technical, Harlow 1989.

[4] G. Caginalp, *The dynamics of a conserved phase field system: Stefan-like, Hele-Shaw, and Cahn-Hilliard models as asymptotic limits.* IMA J. Appl. Math., 44, 1990, 77-94.

[5] J. W. Cahn - C. A. Handwerker - J. E. Taylor, *Geometric models of crystal growth.* Acta Metall., 40, 1992, 1443-1474.

[6] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam 1978.

[7] G. Dal Maso, *Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals.* Manuscripta Math., 30, 1980, 387-416.

[8] E. De Giorgi, *Free discontinuity problems in calculus of variations.* In: R. Dautray (ed.), *Frontieres in Pure and Applied Mathematics.* North-Holland, Amsterdam 1991, 55-62.

[9] E. De Giorgi - T. Franzoni, *Su un tipo di convergenza variazionale.* Atti Acc. Lincei Rend. fis., s. 8, 58, 1975, 842-850.

[10] H. Federer, *Geometric Measure Theory.* Springer-Verlag, Berlin 1968.

[11] E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili.* Rend. Sem. Mat. Univ. Padova, 27, 1957, 284-305.

[12] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, Boston 1984.

[13] S. Luckhaus - L. Modica, *The Gibbs-Thomson relation within the gradient theory of phase transitions.* Arch. Rational Mech. Anal., 107, 1, 1989, 71-83.

[14] M. Miranda, *Superfici cartesiane generalizzate ed insiemi di perimetro localmente finito sui prodotti cartesiani.* Ann. Scuola Norm. Pisa Cl. Sci., (4), 3, 1964, 515-542.

[15] Yu. G. Reshetnyak, *Weak convergence of completely additive functions on a set.* Siberian Math. J., 9, 1968, 1039-1045.

[16] A. I. Vol'pert, *The space BV and quasilinear equations.* Math. USSR Sbornik, 2, 1967, 225-267.

G. Bellettini: Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta S. Donato, 5 - 40127 Bologna

M. Paolini: Dipartimento di Matematica
Università degli Studi di Milano
Via C. Saldini, 50 - 20133 Milano