Convex approximation of an inhomogeneous anisotropic functional

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ABSTRACT. — The numerical minimization of the functional $\mathcal{F}(u) = \int \phi(x, v_u) \|Du\| + \int_{\partial u} \mu u \, d\mathcal{H}^{n-1} - \int_{\Omega} ku \, dx$, $u \in BV(\Omega; [-1, 1])$, is addressed. The function $\phi$ is continuous, has linear growth, and is convex and positively homogeneous of degree one in the second variable. We prove that $\mathcal{F}$ can be equivalently minimized on the convex set $BV(\Omega; [-1, 1])$ and then regularized with a sequence $\{\mathcal{F}_\varepsilon(u)\}_\varepsilon$ of strictly convex functionals defined on $BV(\Omega; [-1, 1])$. Then both $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ can be discretized by continuous linear finite elements. The convexity property of the functionals on $BV(\Omega; [-1, 1])$ is useful in the numerical minimization of $\mathcal{F}$. The $\Gamma - L^1(\Omega)$-convergence of the discrete functionals $\{\mathcal{F}_\varepsilon\}_\varepsilon$ and $\{\mathcal{F}_{\varepsilon,b}\}_\varepsilon, b$ to $\mathcal{F}$, as well as the compactness of any sequence of discrete absolute minimizers, are proven.

KEY WORDS: Calculus of variations; Anisotropic surface energy; Finite elements; Convergence of discrete approximations.

0. INTRODUCTION

Several problems in the Calculus of Variations that fall in the general framework proposed by De Giorgi [8], arising in phase transitions [4] and crystal growth [5] involve functionals depending in an inhomogeneous and anisotropic way on an interfacial energy. For instance, let $\Omega$ be a bounded smooth domain of $\mathbb{R}^n$, and let $\phi: \Omega \times \mathbb{R}^n \to [0, + \infty$ be a continuous function with linear growth, convex and positively homogeneous of degree one in the second variable. Given a smooth set $E \subseteq \mathbb{R}^n$, the typical interfacial term is of the form

$$\int_{\Omega \cap \partial E} \phi(x, v_{E}(x)) \, d\mathcal{H}^{n-1}(x),$$

where $v_{E}(x)$ denotes the outward unit normal vector of $\partial E$ at the point $x$, and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

The study of minimum problems involving such functionals is also related to the ap-
proximation of the motion of an interface, which propagates with a velocity depending on the position, the normal vector, and the mean curvature [1].

In this paper we generalize the numerical minimization via convex approximation presented in [2] to a model functional with an anisotropic and inhomogeneous surface term. This can be viewed as a preliminary step for the study of the geometric motion of fronts by anisotropic curvature.

More precisely, given two functions \( K \in L^\infty(\Omega) \) and \( \mu \in L^\infty(\partial\Omega) \), and assuming that \( \phi(\cdot, \xi) \) can be extended in a continuous way up to \( \partial\Omega \), we consider the minimum problem:

\[
\min_{u \in BV(\Omega; \{-1, 1\})} \mathcal{F}(u), \quad \text{where} \quad \mathcal{F}(u) = \int_{\partial\Omega} \phi(x, \nu_u) \, |Du| + \int_{\partial\Omega} \mu u \, d\mathcal{H}^{n-1} - \int_{\partial\Omega} \kappa u \, dx.
\]

If the solution to this problem is the characteristic function of a set \( A \subset \Omega \) (with values 1 in \( A \) and \(-1 \) in \( \Omega \setminus A \)) with smooth boundary, one can prove that \( \Omega \cap \partial A \) has mean curvature related to \( \kappa \) and \( \phi \), and that the contact angle at the intersection of \( \partial A \) with \( \partial\Omega \) is suitably related to \( \mu \) and \( \phi \).

Following the ideas in [2], we shall equivalently minimize \( \mathcal{F} \) on the larger convex set \( BV(\Omega; [-1, 1]) \). The (nonstrict) convexity of \( \mathcal{F} \) can be exploited for the numerical minimization of \( \mathcal{F} \) via linear finite elements discretizations. Since the numerical algorithms perform better for strictly convex functionals, \( \mathcal{F} \) is preliminarily regularized by a sequence \( \{\mathcal{F}_\epsilon\}_\epsilon \) of convex functionals.

The main result of this paper is the \( I^-\)convergence of the discrete functionals \( \{\mathcal{F}_{\epsilon, b}\}_\epsilon, b \) to \( \mathcal{F} \) when \( \epsilon \) and \( b \) go to zero independently. Since the compactness of each family \( \{u_{\epsilon, b}\}_\epsilon, b \) of discrete absolute minima is also proved, in view of basic properties of \( I^-\)convergence [9], the family \( \{u_{\epsilon, b}\}_\epsilon, b \) admits a subsequence converging to a minimum point \( u \) of \( \mathcal{F} \) and \( \mathcal{F}_{\epsilon, b}(u_{\epsilon, b}) \) converges to \( \mathcal{F}(u) \).

1. THE SETTING

Let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be a bounded open set with Lipschitz continuous boundary and denote by \( |\cdot| \) the \( n \)-dimensional Lebesgue measure and by \( \mathcal{H}^{n-1} \) the \((n - 1)\)-dimensional Hausdorff measure in \( \mathbb{R}^n \) [10]. If \( f: \Omega \to \mathbb{R} \) is a function and \( t \in \mathbb{R} \), we set \( \{f > t\} = \{x \in \Omega: f(x) > t\} \), \( \{f = t\} = \{x \in \Omega: f(x) = t\} \).

If \( \lambda \) is a (possibly vector-valued) Radon measure, its total variation will be denoted by \( |\lambda| \). If \( \lambda_0 \) is a scalar Radon measure on \( \Omega \) such that \( \lambda \) is absolutely continuous with respect to \( \lambda_0 \), the symbol \( \lambda/\lambda_0 \) stands for the Radon-Nikodym derivative of \( \lambda \) with respect to \( \lambda_0 \).

The space \( BV(\Omega) \) is defined as the space of the functions \( u \in L^1(\Omega) \) whose distributional gradient \( Du \) is an \( \mathbb{R}^n \)-valued Radon measure with bounded total variation in \( \Omega \). Since no confusion is possible, we denote by \( u \in L^1(\partial\Omega) \) the trace of \( u \in BV(\Omega) \) on \( \partial\Omega \) and set \( \nu_u(x) = (Du/|Du|)(x) \) for \( |Du| \)-almost every \( x \in \Omega \). We also set

\[
Du = \nabla u \, dx + D' u,
\]

where \( \nabla u \) denotes the density of the absolutely continuous part of \( Du \) with respect to
the Lebesgue measure and $D^1 u$ stands for the singular part. One can prove that $\nabla u$ coincides almost everywhere with the approximate differential of $u$.

Let $E \subseteq \mathbb{R}^n$ be a measurable set; we denote by $\chi_E$ the characteristic function of $E$, i.e., $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = -1$ if $x \notin E$, and we set $1_E(x) = 1$ if $x \in E$, $1_E(x) = 0$ if $x \notin E$. We say that $E$ has finite perimeter in $\Omega$ if $\int_{\partial^* E} |Du| < +\infty$, and we denote by $P(E, \Omega)$ its perimeter. We indicate by $\partial^* E$ the reduced boundary of $E$. We introduce the two closed subsets of $BV(\Omega)$ as $K = BV(\Omega; \{-1, 1\})$ and $K = BV(\Omega; [-1, 1])$.

Given $u \in BV(\Omega)$ we set $S(u) = \{(x, s) \in \Omega \times \mathbb{R}: s < u^+(x)\}$; it turns out that $S(u)$ is a set of finite perimeter in $\Omega \times \mathbb{R}$.

For the definitions and the main properties of the functions of bounded variation and of sets of finite perimeter we refer to [10, 12, 14, 16].

For any $\mathcal{E}: BV(\Omega) \rightarrow [\inf \mathcal{E}, +\infty]$ with $-\infty < \inf \mathcal{E}$, we denote by $\overline{\mathcal{E}}: BV(\Omega) \rightarrow [\inf \mathcal{E}, +\infty]$ the lower semicontinuous envelope (or relaxed functional) of $\mathcal{E}$ with respect to the $L^1(\Omega)$-topology. The functional $\overline{\mathcal{E}}$ is defined as the greatest $L^1(\Omega)$-lower semicontinuous functional less than or equal to $\mathcal{E}$ and can be characterized as

$$
\overline{\mathcal{E}}(u) = \inf \left\{ \liminf_{h \rightarrow +\infty} \mathcal{E}(u_h): \{u_h\}_h \subseteq BV(\Omega), \ u_h \overset{L^1(\Omega)}{\rightarrow} u \right\}.
$$

For the main properties of the relaxed functionals we refer to [3].

From now on $\varphi: \overline{\Omega} \times \mathbb{R}^n \rightarrow [0, +\infty]$ will be a continuous function satisfying the properties

\[
(1.1) \quad \varphi(x, t\xi) = |t| \varphi(x, \xi) \quad \forall x \in \overline{\Omega}, \ \forall \xi \in \mathbb{R}^n, \ \forall t \in \mathbb{R},
\]

\[
(1.2) \quad \lambda |\xi| \leq \varphi(x, \xi) \leq \Lambda |\xi| \quad \forall x \in \overline{\Omega}, \ \forall \xi \in \mathbb{R}^n,
\]

for two suitable positive constants $0 < \lambda \leq \Lambda < +\infty$, and such that $\varphi(x, \cdot)$ is convex on $\mathbb{R}^n$ for any $x \in \overline{\Omega}$. Further regularity assumptions on $\varphi$ will be required afterwards (see (1.9)).

Let us recall the following coarea-type formula

\[
(1.3) \quad \int_{\overline{\Omega}} \varphi(x, \nu_u) |Du| = \int_{\mathbb{R}} \int_{\partial^* \{u > t\}} \varphi(x, \nu_t) d\mathcal{H}^{n-1}(x) dt \quad \forall u \in BV(\Omega),
\]

where $\nu_t$ stands for the outer unit normal vector to the set $\Omega \cap \partial^* \{u > t\}$.

1.1 The continuous functional. Let $\mu \in L^\infty(\partial \Omega)$ be such that

\[
(1.4) \quad |\mu(x)| \leq \varphi(x, \nu_\partial(x)) \quad \text{for } \mathcal{H}^{n-1} - \text{ a.e. } x \in \partial \Omega,
\]

where $\nu_\partial(x)$ denotes a unit normal vector to $\partial \Omega$ at the point $x$. Let $\kappa \in L^\infty(\Omega)$. We define the functional $\mathcal{F}: BV(\Omega) \rightarrow [\inf \mathcal{F}, +\infty]$, for any $u \in K$, as

$$
\mathcal{F}(u) = \int_{\partial^*} \varphi(x, \nu_u) |Du| + \int_{\partial \Omega} \mu u d\mathcal{H}^{n-1} - \int_{\partial \Omega} k u dx,
$$

and set $\mathcal{F} = +\infty$ on $BV(\Omega) \setminus K$. As a consequence of the following semicontinuity result and the boundedness from below, $\mathcal{F}$ admits at least one minimum point.
Theorem 1.1. The functional $\mathcal{F}$ is lower semicontinuous on $K$ with respect to the topology of $L^1(\Omega)$.

Proof. First we note that any $\mu \in L^\infty(\partial\Omega)$ verifying (1.4) can be approximated in $L^1(\partial\Omega)$ by a sequence of functions $\{\mu^\varepsilon\}_{\varepsilon > 0}$ of the form

$$
\mu^\varepsilon(x) = \varphi(x, \nu_\partial(x)) \sum_{i=0}^{N^\varepsilon} \mu^\varepsilon_i 1_{F^\varepsilon_i}(x),
$$

where $-1 = \mu^\varepsilon_0 < \ldots < \mu^\varepsilon_N = 1$, and $\{F^\varepsilon_0, \ldots, F^\varepsilon_N\}$ is a measurable partition of $\partial\Omega$. Here $F^\varepsilon_0$ and $F^\varepsilon_N$ might be empty. Denoting by $\mathcal{F}^\varepsilon$ the functional $\mathcal{F}$ with $\mu$ replaced by $\mu^\varepsilon$, we have, for any $u \in K$,

$$
|\mathcal{F}^\varepsilon(u) - \mathcal{F}(u)| \leq \int_{\partial\Omega} |u| |\mu - \mu^\varepsilon| d\mathcal{H}^{n-1} \leq \|\mu - \mu^\varepsilon\|_{L^1(\partial\Omega)} \to 0,
$$
as $\varepsilon \to 0$. Namely, $\mathcal{F}^\varepsilon \to \mathcal{F}$ uniformly on $K$ as $\varepsilon \to 0$. Since the uniform limit of semicontinuous functions is semicontinuous, the assertion of the theorem is thus reduced to prove that any $\mathcal{F}^\varepsilon$ is $L^1(\Omega)$-lower semicontinuous on $K$. Since no confusion is possible, we omit the superscript $\varepsilon$.

Set $\alpha_i = (\mu_i - \mu_{i-1})/2 > 0$ and $G_i = \{\mu \geq \mu_i\} \subseteq \partial\Omega$, for all $1 \leq i \leq N$. Note that neither $\mu_0 = -1$ nor $\mu_N = 1$ are necessarily assumed, namely, that $G_1 = \partial\Omega$ and $G_N = \emptyset$ are allowed. Since

$$
\sum_{i=1}^{N} \alpha_i = 1 \quad \text{and} \quad \mu(x) = \varphi(x, \nu_\partial) \sum_{i=1}^{N} \alpha_i \chi_{G_i}(x) \quad \text{for} \quad \mathcal{H}^{n-1} - \text{a.e.} \ x \in \partial\Omega,
$$

the functional $\mathcal{F}$ can be represented as a convex combination of functionals $\mathcal{F}^i$ as follows:

$$
\mathcal{F}(u) = \sum_{i=1}^{N} \alpha_i \left[ \int_{\Omega} \varphi(x, \nu_u) |Du| + \int_{\partial\Omega} \varphi(x, \nu_\partial) \chi_{G_i} \mu d\mathcal{H}^{n-1} - \int_{\Omega} \kappa u dx \right] = \sum_{i=1}^{N} \alpha_i \mathcal{F}^i(u).
$$

To prove the lower semicontinuity of $\mathcal{F}$ it will be enough to show that each $\mathcal{F}^i$ is lower semicontinuous. For simplicity we omit the index $i$, thus denoting $G_i = G$ a measurable subset of $\partial\Omega$, and assume

$$
(1.5) \quad \mu(x) = \varphi(x, \nu_\partial) \chi_G(x).
$$

Let $B$ be a ball containing $\overline{\Omega}$ and define

$$
\Phi(x, \xi) = \begin{cases} 
\varphi(x, \xi) & \text{if } (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n, \\
\Lambda|\xi| & \text{if } (x, \xi) \in (B \setminus \overline{\Omega}) \times \mathbb{R}^n.
\end{cases}
$$

Then $\Phi$ is lower semicontinuous on $B \times \mathbb{R}^n$ (recall (1.2)). We can extend $-\chi_G \in L^1(\partial\Omega)$ to a function $w \in W^{1,1}(B \setminus \overline{\Omega}; [-1, 1])$ with trace $-\chi_G$ on $\partial\Omega$, so that there exists $C > 0$ such that $\|w\|_{W^{1,1}(B \setminus \overline{\Omega})} \leq C \|\chi_G\|_{L^1(\partial\Omega)}$ [11, Theorem 1.12; 12, Theorem 2.16].
For any $u \in K$ we define $U \in BV(B; [-1, 1])$ as follows:

$$
U = \begin{cases} 
    u & \text{on } \Omega, \\
    w & \text{on } B \setminus \Omega.
\end{cases}
$$

Obviously $B$ and $w$ do not depend on $u$, hence $\Lambda \int B \nabla w \, dx$ is a constant, and we shall denote it by $c_1$; set also $c_2 = \int_{\partial B} \phi(x, u) \, d\mathcal{H}^{n-1}(x)$. Recalling that $|u| \leq 1$, we find [7]

$$
\int_B \Phi(x, \nu_U) \, |DU| = \int_\Omega \phi(x, \nu_u) \, |Du| + \int_{\partial \Omega} \phi(x, \nu_u) \, |u + \chi_G| \, d\mathcal{H}^{n-1} + c_1 = 
$$

$$
\int_\Omega \phi(x, \nu_u) \, |Du| + \int_{\partial \Omega} \phi(x, \nu_u) \, u \chi_G \, d\mathcal{H}^{n-1} + c_1 + c_2.
$$

Hence, recalling (1.5) we have

$$
\int_\Omega \phi(x, \nu_u) \, |Du| + \int_{\partial \Omega} \mu u \, d\mathcal{H}^{n-1} = \int_B \Phi(x, \nu_U) \, |DU| - (c_1 + c_2).
$$

Recalling the definition and the convexity of $\Phi$, the functional $\int_B \Phi(x, \nu_U) \, |DU|$ is $L^1$-lower semicontinuous. Since the map $u \rightarrow \int_\Omega \kappa u \, dx$ is continuous with respect to the topology of $L^1(\Omega)$, the assertion follows. □

If $\phi$ is not convex in $\xi$ then $\mathcal{F}$ is not, in general, lower semicontinuous, and the lower semicontinuous envelope of the functional $u \rightarrow \int_\Omega \phi(x, \nabla u) \, dx$ on $W^{1,1}(\Omega)$ can be written on $BV(\Omega) \cap L^\infty(\Omega)$ as $\int_\Omega \phi^{**}(x, \nu_u) \, |Du|$, where $\phi^{**}$ denotes the greatest function that is convex in $\xi$ and less than or equal to $\phi(x, \xi)$ for all $(x, \xi) \in \Omega \times \mathbb{R}^n$. In addition, as in [2], if condition (1.4) is not fulfilled, $\mathcal{F}$ is not lower semicontinuous. Observe that $\mathcal{F}$ admits at least a minimum point $u \in K$ (or $\bar{K}$, respectively), because of condition (1.2) and since $\mathcal{F}$ is lower semicontinuous on $K$ (on $\bar{K}$, respectively).

The following theorem shows that to minimize $\mathcal{F}$ on $\bar{K}$ is equivalent to minimize $\mathcal{F}$ on the convex set $K$, and this reads as a (nonstrictly) convex problem.

**Theorem 1.2.** Suppose that $u \in K$ is a minimum point of $\mathcal{F}$ on $K$. Then

$$
\mathcal{F}(u) = \mathcal{F}(\chi_{\{u > t\}}) \quad \text{for a.e. } t \in [-1, 1],
$$

namely, $\chi_{\{u > t\}} \in \bar{K}$ is a minimum point of $\mathcal{F}$ on $\bar{K}$ for almost every $t \in [-1, 1]$.

**Proof.** For all $v \in K$, from (1.3) and the Cavalieri formula we have

$$
\mathcal{F}(v) = \int_{-1}^1 \int_{\Omega \cap \{u > t\}} \phi(x, \nu_v) \, d\mathcal{H}^{n-1} \, dt + \frac{1}{2} \int_{-1}^1 \int_{\partial \Omega} \mu \chi_{\{v > t\}} \, d\mathcal{H}^{n-1} \, dt - 
$$

$$
- \frac{1}{2} \int_{-1}^1 \int_\Omega \kappa \chi_{\{v > t\}} \, dx \, dt = \frac{1}{2} \int_{-1}^1 \mathcal{F}(\chi_{\{v > t\}}) \, dt,
$$

where $\mathcal{F}(t) = \mathcal{F}(\chi_{\{t > 1\}})$.
that is
\[ \int_{-1}^{1} (\mathcal{F}(\chi_{\{u > 1\}}) - \mathcal{F}(v)) \, dt = 0 \quad \forall v \in K. \]

The minimality of \( u \) on \( K \) entails \( \mathcal{F}(\chi_{\{u > 1\}}) - \mathcal{F}(u) \geq 0 \); therefore \( \mathcal{F}(u) = \mathcal{F}(\chi_{\{u > 1\}}) \) for almost every \( t \in [-1, 1] \). \( \square \)

**Remark 1.1.** In view of Theorem 1.2, we have that \( \min \mathcal{F}(v) = \min \mathcal{F}(v) \); moreover \( \mathcal{F} \) has a unique minimum point on \( \tilde{K} \) if and only if \( \mathcal{F} \) has a unique minimum point on \( K \), and they coincide. Note that \( \mathcal{F} \) may exhibit relative minima on \( \tilde{K} \); in view of the convexity of \( K \), they are no longer relative minima of \( \mathcal{F} \) on \( K \).

### 1.2. The regularized functionals

Given \( \epsilon \geq 0 \), in analogy with [2], we define a regularization of \( \phi \) as follows

\[
\phi_\epsilon(x, \xi) = \sqrt{\epsilon^2 + (\phi(x, \xi))^2},
\]

for all \((x, \xi) \in \bar{\Omega} \times \mathbb{R}^n\). Let us consider the map \( G_\epsilon : BV(\Omega) \rightarrow [0, +\infty) \) defined by

\[
G_\epsilon(u) = \left\{ \begin{array}{ll}
\int_{\Omega} \phi_\epsilon(x, \nabla u) \, dx & \text{if } u \in W^{1,1}(\Omega), \\
+\infty & \text{elsewhere}.
\end{array} \right.
\]

Observe that, by the continuity assumption on \( \phi \) and by (1.1), there exists a continuous function \( \omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), with \( \omega(0) = 0 \), such that

\[
|\phi_\epsilon(x, \xi) - \phi_\epsilon(y, \xi)| \leq |\phi(x, \xi) - \phi(y, \xi)| \leq \omega(|x - y|)(1 + |\xi|)
\]

for any \( x, y \in \Omega \) and any \( \xi \in \mathbb{R}^n \). Then, applying [7, Theorem 3.2] and observing that \( \lim_{t \to 0^+} t\phi_\epsilon(x, \xi/t) = \phi(x, \xi) \), we find that

\[
\overline{G}_\epsilon(u) = \int_{\bar{\Omega}} \phi_\epsilon(x, \nabla u) \, dx + \int_{\Omega} \phi(x, \frac{D'u}{|D'u|}) \, |D'u| \quad \forall u \in BV(\Omega).
\]

We are now ready to define the regularized functionals \( \mathcal{F}_\epsilon : BV(\Omega) \rightarrow [\inf \mathcal{F}_\epsilon, +\infty] \). For any \( \epsilon > 0 \) and for any \( u \in K \), we set

\[
\mathcal{F}_\epsilon(u) = \int_{\Omega} \phi_\epsilon(x, \nabla u) \, dx + \int_{\Omega} \phi(x, \frac{D'u}{|D'u|}) \, |D'u| + \int_{\partial\Omega} \mu u \, \nu d\mathcal{H}^{n-1} - \int_{\Omega} \kappa u \, dx,
\]

and we set \( \mathcal{F}_\epsilon = +\infty \) on \( BV(\Omega) \setminus K \).

**Theorem 1.3.** For any \( \epsilon > 0 \) the functional \( \mathcal{F}_\epsilon \) is lower semicontinuous on \( K \) with respect to the topology of \( L^1(\Omega) \).

**Proof.** Reasoning as in the proof of Theorem 1.1, and using the same notation, we have

\[
\mathcal{F}_\epsilon(u) + \int_{\Omega} \kappa u \, dx = \int_{\Omega} \sqrt{\epsilon^2 + (\Phi(x, \nabla U))^2} \, dx + \int_{\Omega} \Phi(x, \frac{D'u}{|D'u|}) \, |D'u| - (c_2 + c_3)
\]
where
\[ c_3 = \int_{B \setminus \bar{O}} \sqrt{\epsilon^2 + \Lambda^2 |\nabla w|^2} \, dx. \]

As the functional at the right-hand side is \( L^1 \)-lower semicontinuous (it is a lower semicontinuous envelope by [7]), the theorem follows. \( \Box \)

It is not difficult to show that, if condition (1.4) is not fulfilled, then the functional \( \mathcal{F}_\epsilon \) is not lower semicontinuous.

Observe that the restriction of \( \mathcal{F}_\epsilon \) to \( K (\bar{K}, \text{respectively}) \) admits at least a minimum point \( u \in K (u \in \bar{K}, \text{respectively}) \), because of condition (1.2) and since \( \mathcal{F}_\epsilon \) is lower semicontinuous on \( K \) (on \( \bar{K} \), respectively). Observe also that, if \( \mathcal{F}_\epsilon \) has a minimum point \( u \in K \cap W_{loc}^{1,1} (\Omega) \) then, since \( \mathcal{F}_\epsilon \) is strictly convex in \( (BV(\Omega) \cap W_{loc}^{1,1} (\Omega)) / \mathbb{R} \), the minimum is unique up to a possible additive constant.

**Remark 1.2.** We have \( \mathcal{F}_\epsilon \to \mathcal{F} \) uniformly in \( K \) as \( \epsilon \to 0 \).

**Proof.** For any \( u \in K \), using (1.1), we have
\[ |\mathcal{F}_\epsilon (u) - \mathcal{F}(u)| = \epsilon \left| \int_{\Omega} \sqrt{1 + \left( \phi (x, \nabla \left( \frac{u}{\epsilon} \right) ) \right)^2} \, dx - \int_{\Omega} \phi (x, \nabla \left( \frac{u}{\epsilon} \right) ) \, dx \right| \leq \epsilon |\Omega|. \]

### 1.3. The discrete functionals
Let \( \{ S_h \}_{h > 0} \) denote a regular family of partitions of \( \Omega \) into simplices [6]. Let \( h_b \leq h \) denote the diameter of any \( S \in S_h \). For any \( h > 0 \), let \( V_h \subset H^1 (\Omega; [-1, 1]) \subset K \) be the piecewise linear finite element space over \( S_h \) with values in \([-1, 1]\) and \( \Pi_h \) be the usual Lagrange interpolation operator over the continuous piecewise linear functions. By \( C \) we shall mean an absolute positive constant whose value may vary at each occurrence. For the sake of simplicity, we shall assume that the discrete domain \( \Omega_h = \bigcup S \) coincides with \( \bar{O} \). In order to introduce the discrete functionals \( \mathcal{F}_h \) and \( \mathcal{F}_{\epsilon, b} \), we approximate \( \mu \) and \( \kappa \) as in [2] by a sequence of continuous piecewise linear functions \( \mu_h \to \mu \) and \( \kappa_h \to \kappa \) in \( L^1 \) as \( h \to 0 \) such that [6]
\begin{align}
&\|\mu_h\|_{L^\infty (\partial \Omega)} \leq \|\mu\|_{L^\infty (\partial \Omega)}, \quad \|\nabla \mu_h\|_{L^1 (\partial \Omega)} = o(h^{-1}), \\
&\|\kappa_h\|_{L^\infty (\Omega)} \leq \|\kappa\|_{L^\infty (\Omega)}, \quad \|\nabla \kappa_h\|_{L^1 (\Omega)} = o(h^{-1}).
\end{align}

We define the discrete functionals as follows: for any \( u \in V_h \) we set
\[ \mathcal{F}_{\epsilon, b} (u) = \sum_{S \in S_h} \int_S \Pi_b (\phi (x, \nabla u)) \, dx + \int_{\partial \Omega} \Pi_b (\mu_h u) \, d\sigma \|a^{-1} - \int_{\partial \Omega} \Pi_b (\kappa_h u) \, dx, \]
\[ \mathcal{F}_{\epsilon, b} = + \infty \text{ on } BV(\Omega) \setminus V_h. \]
Finally we define \( \mathcal{F}_b = \mathcal{F}_{0, b} \). The piecewise constant interpolation \( \int_{\partial \Omega} \Pi_b (\phi (x, \nabla u)) \, dx \) can also be used in the first term without affecting the convergence result and allowing a simpler implementation of the numerical algorithms.
To prove the main theorem (2.1) we need the assumptions 
(1.9) \( \phi(\cdot, \xi) \in W^{1, \infty}(\Omega), \quad |\nabla \phi(x, \xi)| \leq C|\xi|, \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n \),
and that \( \phi(x, \cdot) \) is Lipschitz continuous uniformly with respect to \( x \).

If \( u \in V_b \), by the properties of the Lagrange interpolation operator, noting that 
(1.6) gives \( |\nabla_x \phi(x, \nabla u)| \leq |\nabla_x \phi(x, \nabla u)| \) and using (1.9) we have

\[
(1.10) \quad \left| \sum_{S \in \mathcal{S}_h} \left( \int_S (I_\beta(\phi(x, \nabla u)) - \phi(x, \nabla u)) \, dx \right) \right| \leq \sum_{S \in \mathcal{S}_h} \|I_\beta(\phi(x, \nabla u)) - \phi(x, \nabla u)\|_{L^\infty} \leq C \int_\Omega |\nabla u| \, dx.
\]

2. Convergence of the discretized functionals

**Remark 2.1.** We have \( \lim_{\beta \to 0} \mathcal{F}_{\varepsilon, b} = \mathcal{F}_b \) uniformly in \( V_b \) and with respect to \( b \).

**Proof.** See Remark 1.2. \( \Box \)

The next main theorem generalizes [2, Theorem 3.1].

**Theorem 2.1.** For any \( \varepsilon > 0 \) we have,

\[
\Gamma^* \lim_{b \to 0} \mathcal{F}_b = \mathcal{F} \quad \text{and} \quad \Gamma^* \lim_{b \to 0} \mathcal{F}_{\varepsilon, b} = \mathcal{F}_\varepsilon \quad \text{in} \quad L^1(\Omega).
\]

**Proof.** We give a unified proof for both cases \( \varepsilon > 0 \) and \( \varepsilon = 0 \), considering \( \mathcal{F}_b = \mathcal{F}_{\varepsilon, b} \) and \( \mathcal{F} = \mathcal{F}_\varepsilon \) if \( \varepsilon = 0 \). Hence, let \( \varepsilon \geq 0 \) be fixed. We split the proof into two steps, namely, we prove that the two following properties hold [9]:

(i) for any \( u \in BV(\Omega) \) and any sequence \( \{u_b\}_b \) in \( BV(\Omega) \) converging to \( u \) in \( L^1(\Omega) \) we have \( \mathcal{F}_\varepsilon(u) \leq \liminf_{b \to 0} \mathcal{F}_{\varepsilon, b}(u_b) \);

(ii) for any \( u \in BV(\Omega) \) there exists a sequence \( \{u_b\}_b \) in \( BV(\Omega) \) converging to \( u \) in \( L^1(\Omega) \) such that \( \mathcal{F}_\varepsilon(u) = \lim_{b \to 0} \mathcal{F}_{\varepsilon, b}(u_b) \).

Preliminarily we decompose \( \mathcal{F}_{\varepsilon, b}(u_b) \), for all \( u_b \in V_b \), as follows:

\[
(2.1) \quad \mathcal{F}_{\varepsilon, b}(u_b) = \mathcal{F}_\varepsilon(u_b) + \int_\Omega [I_\beta(\mu u_b) - \mu u_b] \, d\mathcal{H}^{n-1} - \int_\Omega [I_\beta(\kappa u_b) - \kappa u_b] \, dx + \sum_{S \in \mathcal{S}_h} \int_S (I_\beta(\phi(x, \nabla u_b)) - \phi(x, \nabla u_b)) \, dx =: \mathcal{F}_\varepsilon(u_b) + I_b + II_b + III_{\varepsilon, b}.
\]

Recalling (1.8) and reasoning as in [2], one gets \( \lim_{b \to 0} \|I_b\| + \|II_b\| = 0 \).

**Proof of Step (i).** Let \( u \in BV(\Omega) \) and \( \{u_b\}_b \) in \( BV(\Omega) \) be any sequence so that \( u_b \to u \) in \( L^1(\Omega) \) as \( b \to 0 \). We can assume that \( u_b \in V_b \) for any \( b \) and that \( \sup_b \mathcal{F}_{\varepsilon, b}(u_b) < +\infty \). From (1.2) we get \( \sup_b \int_\Omega |\nabla u_b| \, dx < +\infty \), so that, in view of (1.10) we have \( \lim_{b \to 0} \|III_{\varepsilon, b}\| = 0 \). Then, using (2.1) and the lower semicontinuity of \( \mathcal{F}_\varepsilon \)
(Theorems 1.1 and 1.3), we conclude that
\[ \mathcal{F}_\varepsilon(u) \leq \liminf_{b \to 0} \mathcal{F}_b(u_b) = \liminf_{b \to 0} \mathcal{F}_{\varepsilon, b}(u_b) , \]
and (i) is proved.

**Proof of Step (ii).** We can assume that \(u \in K\). Given a ball \(B\) containing \(\overline{\Omega}\), let \(\tilde{u} \in W^{1,1}(B \setminus \overline{\Omega}; [-1, 1])\) be a function with trace \(u\) on \(\partial \Omega\) [11] and denote again by \(u \in BV(B; [-1, 1])\) the function \(u(x) = u(x)\) if \(x \in \Omega\), \(u(x) = \tilde{u}(x)\) if \(x \in B \setminus \Omega\). Observe that
\[ (2.2) \quad \int_{\partial \Omega} |D\tilde{u}| = 0 . \]

Let \(\eta_b = o(b^{-1/2})\) and \(\{\delta_b\}_b\) be a family of mollifiers defined by \(\delta_b(x) = \eta_b \delta(\eta_b x)\). Set \(\tilde{u}_b(x) = (u \ast \delta_b)(x)\) for all \(x \in B\), where \(u\) is extended to 0 outside \(B\). It is well known [12, Proposition 1.15] that, recalling (2.2),
\[ (2.3) \quad \lim_{b \to 0} \|\tilde{u}_b - u\|_{L^1(\Omega)} = 0 , \quad \text{and} \quad \lim_{b \to 0} \int_{\Omega} |\nabla \tilde{u}_b| \, dx = \int_{\Omega} |D\tilde{u}| . \]

Set \(u_b = \Pi B \tilde{u}_b \in V_b\); then [2]
\[ (2.4) \quad \lim_{b \to 0} \|u_b - u\|_{L^1(\Omega)} = 0 , \quad \lim_{b \to 0} \int_{\Omega} |\nabla u_b| \, dx = \int_{\Omega} |D\tilde{u}| , \]
and
\[ (2.5) \quad \lim_{b \to 0} \int_{\partial \Omega} |u_b - u| \, d\mathcal{H}^{n-1} = 0 . \]

Hence, using Reshetnyak's Theorem [15] (see also [13]), we get
\[ (2.6) \quad \lim_{b \to 0} \int_{\Omega} \phi(x, \nabla u_b) \, dx = \int_{\Omega} \phi(x, \nabla u) \, |Du| . \]

Using (2.1), (2.4), (2.5), and (2.6), we get (ii) when \(\varepsilon = 0\).

Let \(\varepsilon > 0\). One can prove (see [14, Theorems 1.8 and 1.10]) that the sequence \(\{D_{1(S B)}\}_b\) converges weakly on \(\Omega \times R\) to \(D_{1(S u)}\) and, using (2.2), that
\[ (2.7) \quad \lim_{b \to 0} \int_{\Omega} |D_{1(S B)}| (\Omega \times R) = |D_{1(S u)}| (\Omega \times R) . \]

Let \(\tilde{\phi}_\varepsilon : \Omega \times R \times R^n \times R^+ \to [0, + \infty] \) be defined by
\[ \tilde{\phi}_\varepsilon(x, s, \xi, t) = \begin{cases} t \phi_\varepsilon(x, \frac{s}{t}, \frac{\xi}{t}) & \text{if } t > 0 , \\ \phi(x, \xi) & \text{if } t = 0 . \end{cases} \]

Then \(\tilde{\phi}_\varepsilon\) is continuous, and the function \((\xi, t) \to \tilde{\phi}_\varepsilon(x, s, \xi, t)\) is convex and positively homogeneous of degree one on \(R^n \times R^+\). By [7, Lemma 2.2], for any \(u \in K\) we have
\[ \int_{\Omega \times R} \tilde{\phi}_\varepsilon(x, s, \frac{D_{1(S u)}}{|D_{1(S u)}|}) |D_{1(S u)}| = \int_{\Omega} \phi_\varepsilon(x, \nabla u) \, dx + \int_{\Omega} \phi(x, \frac{D'u}{|D'u|}) \, |D'u| . \]
Using again Reshetnyak’s Theorem (recall (2.7)) we have

\[
\lim_{b \to 0} \int_{\Omega} \phi_\varepsilon(x, \nabla u_b) \, dx = \lim_{b \to 0} \int_{\Omega \times \mathbb{R}} \tilde{\phi}_\varepsilon \left( x, s, \frac{D\mathcal{S}(u_b)}{|D\mathcal{S}(u_b)|} \right) |D\mathcal{S}(u_b)| =
\]

\[
= \int_{\Omega \times \mathbb{R}} \tilde{\phi}_\varepsilon \left( x, s, \frac{D\mathcal{S}(u)}{|D\mathcal{S}(u)|} \right) |D\mathcal{S}(u)| = \int_{\Omega} \phi_\varepsilon(x, \nabla u) \, dx + \int_{\Omega} \phi \left( x, \frac{D^i u}{|D^i u|} \right) |D^i u|.
\]

Observe that for any \( b \) we have

\[
\left| \int_{\Omega} \phi_\varepsilon(x, \nabla u_b) \, dx - \int_{\Omega} \phi_\varepsilon(x, \nabla u_b) \, dx \right| \leq \int_{\Omega} |\phi(x, \nabla u_b) \, dx - \phi(x, \nabla u_b) | \, dx \to 0
\]
as \( b \to 0 \), in view of the Lipschitz assumption on \( \phi(x, \cdot) \) and the fact that \[2\]

\[
\lim_{b \to 0} \| \tilde{u}_b - u_b \|_{W^{1,1}(\Omega)} = 0.
\]

Using (2.8) we then find

\[
\lim_{b \to 0} \int_{\Omega} \phi_\varepsilon(x, \nabla u_b) \, dx = \lim_{b \to 0} \int_{\Omega} \phi_\varepsilon(x, \nabla u_b) \, dx = \int_{\Omega} \phi_\varepsilon(x, \nabla u) \, dx + \int_{\Omega} \phi \left( x, \frac{D^i u}{|D^i u|} \right) |D^i u|.
\]

This, together with (2.5) and (2.4), concludes the proof of (\( ii \)) when \( \varepsilon > 0 \).

A straightforward consequence is the following \( I \)-convergence result for \( \mathcal{F}_{\varepsilon, h} \), as \( \varepsilon \) and \( h \) go to 0 independently.

**Corollary 2.1.** We have \( I \)-\( \lim_{(\varepsilon, h) \to (0, 0)} \mathcal{F}_{\varepsilon, h} = \mathcal{F} \) in \( L^1(\Omega) \).

Finally, we prove the compactness of any sequence of approximated minima which, in view of basic properties of \( I \)-convergence gives, up to a subsequence, the convergence to a minimum of the original functional \( \mathcal{F} \).

**Theorem 2.2.** Any family of absolute minima of the functionals \( \mathcal{F}_\varepsilon, \mathcal{F}_h, \) or \( \mathcal{F}_{\varepsilon, h} \), is relatively compact in \( L^1(\Omega) \).

**Proof.** Let \( u_{\varepsilon, h} \) be a minimum point of \( \mathcal{F}_{\varepsilon, h} \). Given any \( v \in K \), from Corollary 2.1 there exists a sequence \( \{v_{\varepsilon, h}\}_{\varepsilon, h} \) converging to \( v \) in \( L^1(\Omega) \) as \( (\varepsilon, h) \to (0, 0) \), so that

\[
\lim_{(\varepsilon, h) \to (0, 0)} \mathcal{F}_{\varepsilon, h}(v_{\varepsilon, h}) = \mathcal{F}(v) \in \mathbb{R}.
\]

Hence \( \sup_{\varepsilon, h} \mathcal{F}_{\varepsilon, h}(u_{\varepsilon, h}) \leq \sup_{\varepsilon, h} \mathcal{F}_{\varepsilon, h}(v_{\varepsilon, h}) < +\infty \). Then we get

\[
\sup_{\varepsilon, h} \int_{\Omega} |D u_{\varepsilon, h}| < +\infty,
\]

and the assertion for \( \mathcal{F}_{\varepsilon, h} \) follows from the compactness theorem in \( BV(\Omega) \). The assertion for \( \mathcal{F}_\varepsilon \) and \( \mathcal{F}_h \) is similar.

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