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Solution sets of multivalued Sturm-Liouville problems in Banach spaces

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Equazioni differenziali ordinarie. — *Solution sets of multivalued Sturm-Liouville problems in Banach spaces.* Nota (*) di ALESSANDRO MARGHERI e PIETRO ZECCA, presentata dal Corrisp. R. Conti.

ABSTRACT. — We give some results about the topological structure of solution sets of multivalued Sturm-Liouville problems in Banach spaces.

KEY WORDS: Solution set; Boundary value problem; Differential equations.

RIASSUNTO. — *Sull'insieme delle soluzioni per problemi di Sturm-Liouville multivoci in spazi di Banach.* Riducendo il problema ai limiti in un problema di punto fisso per contrazioni multivoche a valori convessi o decomponibili, vengono dati due risultati sulla struttura topologica dell'insieme delle soluzioni per problemi di Sturm-Liouville multivoci in spazi di Banach.

1. INTRODUCTION

In this *Note* we want to give some results on the topological structure of the solution set for the multivalued Sturm-Liouville problem

$$x''(t) \in F(t, x(t), x'(t)), \quad \text{a.e. on } J = [a, b], \quad x \in X; \tag{1}$$

$$\begin{cases} c_1 x(a) - d_1 x'(a) = x_a, \\ c_2 x(b) - d_2 x'(b) = x_b, \end{cases} \tag{2}$$

where X is a Banach space, $F: J \times X^2 \to 2^X \setminus \phi$, and, following [4], $d_i \in \{0, 1\}$ $i = 1, 2$, $c_i = 1$ if $d_i = 0$ $i = 1, 2$.

We extend to this context the results obtained in [3] about the topological structure of the solution sets of (1) for the case $X = \boldsymbol{R}^q$, $d_i = 0$, $i = 1, 2$, $x_a = x_b = 0$. The results are obtained reducing the boundary value problem to a fixed point one, and then using two theorems concerning the topological structure of the fixed points sets of multivalued contraction mappings, respectively with decomposable [1] and convex [8] values.

We think that this extension to the infinite dimensional case is justified because, as far as we know, these are the first results of this kind presented for Banach spaces.

2. SOME DEFINITIONS AND NOTATIONS

We will denote by $L^p(J, X)$, $1 \leq p < \infty$ the Banach space of the (equivalence classes) of Bochner p-integrable functions from J to X, by $C^k(J, X)$ the Banach space of the functions from J to X continuously differentiable up to the order k, and by $W^{2,1}(J, X)$

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the Banach space of the functions from J to X with absolutely continuous first derivative. The norms on this spaces are the standard ones. $\|\cdot\|: X \to [0, +\infty)$ will denote the X norm, while the other ones will be identified by means of subscripts.

We will assume that the homogeneous problem, *i.e.* $F = \{0\}$, $x_a = x_b = 0$ has only the trivial solution. Then it is well known [4] that for $u \in L^1(J, X)$ the only solution of $x''(t) = u(t)$ a.e. on J with bound conditions (2) is given by

$$x(t) = h(t) + \int_a^b g(t,s)\,u(s)\,ds \quad \text{on } J,$$

where $h \in C^2(J, X)$ is the solution of $v'' = 0$ with conditions (2) and $g \in C(J^2, \boldsymbol{R})$ is the so-called Green function of the problem. We let $\|g\|_{\infty, J^2} = \sup_{J^2} |g(t,s)|$, and analogously for g_t the derivative of g with respect to t.

Given A, B bounded sets in X we denote by $D_X(A, B)$ their Hausdorff distance defined by

$$D_X(A, B) = \max\{\sup_{b \in B} d_X(b, A), \sup_{a \in A} d_X(a, B)\}$$

(see [4, Definition 2.1]).

A function $F: J \times X^2 \to 2^X \setminus \phi$ is said almost lower semicontinuous (almost lsc) if for any $\varepsilon > 0$ and $G \subset X^2$ nonempty compact set, there exists a closed set $J_\varepsilon \subset J$ with $\mu(J \setminus J_\varepsilon) < \varepsilon$ (μ is the Lebesgue measure on J) such that $F_{|J_\varepsilon \times G}$ is lsc and $\overline{\operatorname{span} F(J_\varepsilon \times G)}$ is separable.

We will write $u(\cdot) \in F(\cdot, q(\cdot), q'(\cdot))$ instead of «u is a strongly measurable selection of $F(\cdot, q(\cdot), q'(\cdot))$».

A subset $K \subset L^1(J, X)$ is called decomposable if for every $u, v \in K$ and every measurable $A \subset J$: $u\chi_A + v\chi_{J \setminus A} \in K$.

3. Results

In the sequel we will be very «dry», remarking only those passages that need particular care because of infinite dimension. For further details we invite the reader to see [3].

Theorem 3.1. *Let $F: J \times X^2 \to 2^X \setminus \phi$ be almost lsc with compact values. Assume that $L^1(J, X)$ is separable and that there exist $m \in L^1(J, X)$, $\alpha, \beta > 0$ such that*

(3) $$D_X(F(t, 0, 0), \{0\}) \leq m(t) \quad \text{a.e. on } J,$$

(4) $$D_X(F(t, x_1, y_1), F(t, x_2, y_2)) \leq \alpha\|x_1 - x_2\| + \beta\|y_1 - y_2\|$$

for any (t, x_1, y_1), $(t, x_2, y_2) \in J \times X^2$, where

(5) $$K = \alpha\|g\|_{\infty, J^2} + \beta\|g_t\|_{\infty, J^2} < 1.$$

Then the set S_F of solutions of (1), (2) *is a retract of* $W^{2,1}(J, X)$.

PROOF. For $u \in L^1(J, X)$, let $x(u)\colon J \to X$ be the solution of $x'' = u(t)$ with conditions (2), *i.e.*

$$x(u)(t) = h(t) + \int_a^b g(t,s)\,u(s)\,ds \qquad t \in J\,, \tag{6}$$

and let $\mathcal{U}(u) = \{G(t) \mid G(\cdot) \in F(\,\cdot\,, x(u)(\cdot), x'(u)(\cdot))\}$. $\mathcal{U}(u)$ is a non empty closed decomposable subset of $L^1(J, X)$. From (3), (4) it follows that $\mathcal{U}(u)$ is bounded in $L^1(J, X)$ so that $\mathcal{U}\colon L^1(J, X) \to 2^{L^1(J,X)} \setminus \phi$ is a multivalued map with bounded closed and decomposable values.

We will show that $\mathcal{U}$ is a contraction mapping, *i.e.*, if $u_1, u_2 \in L^1(J, X)$

$$D_{L^1(J,X)}(\mathcal{U}(u_1), \mathcal{U}(u_2)) \leqslant K\|u_1 - u_2\|_{L^1(J,X)}\,, \tag{7}$$

where K is given by (5). By (6) we get

$$\|x(u_1) - x(u_2)\|_{C(J,X)} \leqslant \|g\|_{\infty, J^2}\|u_1 - u_2\|_{L^1(J,X)}\,, \tag{8}$$

$$\|x'(u_1) - x'(u_2)\|_{C(J,X)} \leqslant \|g_t\|_{\infty, J^2}\|u_1 - u_2\|_{L^1(J,X)}\,. \tag{9}$$

Let $G_1 \in \mathcal{U}(u_1)$. By [4, Proposition 3.5] there exists $G_2 \in \mathcal{U}(u_2)$ such that $\|G_1(t) - G_2(t)\| = d_X(G_1(t), F(t, x(u_2)(t), x'(u_2)(t)))$ a.e. on J. It follows that

$$\|G_1 - G_2\|_{L^1(J,X)} = \int_a^b d_X(G_1(t), F(t, x(u_2)(t), x'(u_2)(t)))\,dt \leqslant$$

$$\leqslant \int_a^b D_X(F(t, x(u_1)(t), x'(u_1)(t)), F(t, x(u_2)(t), x'(u_2)(t)))\,dt\,.$$

By (3), (4), (5), (8), (9) we have $\|G_1 - G_2\|_{L^1(J,X)} \leqslant K\|u_1 - u_2\|_{L^1(J,X)}$, and then

$$\sup_{G_1 \in \mathcal{U}(u_1)} d_{L^1(J,X)}(G_1, \mathcal{U}(u_2)) \leqslant K\|u_1 - u_2\|_{L^1(J,X)}\,.$$

Interchanging u_1 and u_2 we obtain (7). If we let $\text{Fix}\,\mathcal{U} = \{u \in L^1(J, X) \mid u \in \mathcal{U}(u)\}$, by [7] $\text{Fix}\,\mathcal{U} \neq \emptyset$, and by [1] $\text{Fix}\,\mathcal{U}$ is a retract of $L^1(J, X)$. Let $r\colon L^1(J, X) \to \text{Fix}\,\mathcal{U}$ be a retraction; then the map R defined on $W^{2,1}(J, X)$ by

$$(Rx)(t) = h(t) + \int_a^b g(t,s)\,r(x'')(s)\,ds$$

is the desired retraction of $W^{2,1}(J, X)$ onto S_F. □

THEOREM 3.2. *Let $F\colon J \times X^2 \to 2^X \setminus \phi$ be almost lsc with compact convex values. Assume that X is reflexive and separable and that* (3), (4) *still hold but $m \in L^2(J, X)$ and α, β are such that*

$$K' = (b - a)(\alpha + \beta)\max\{\|g\|_{\infty, J^2}, \|g_t\|_{\infty, J^2}\} < 1\,. \tag{10}$$

Then S_F is a retract of $C^1(J, X)$.

PROOF. For $q \in C^1(J, X)$ let us define the following closed convex subset of $L^1(J, X)$: $\mathcal{U}(q) = \{u(t) \mid u(\cdot) \in F(\cdot, q(\cdot), q'(\cdot))\}$, and let us consider the differential inclusion

$$x''(t) \in F(t, q(t), q'(t)) \quad \text{a.e. on } J \tag{11}$$

with bound condition (2). Let Σ: $q \to \Sigma(q) \subset C^1(J, X)$ be the solution map of (11), (2), *i.e.* the map that to each $q \in C^1(J, X)$ associates the set

$$\Sigma(q) = \left\{ y \in C^1(J, X) \mid y(t) = h(t) + \int_a^b g(t, s) u(s)\, ds,\ u \in \mathcal{U}(q) \right\}$$

of solutions of (11), (2). $\Sigma(q)$ is non empty and convex. We show it is also closed. Let $\{x_n\}$ be a sequence in $\Sigma(q)$, $x_n \to x$, in $C^1(J, X)$, and let $\{u_n\} \subset \mathcal{U}(q)$ be such that

$$x_n(t) = h(t) + \int_a^b g(t, s) u_n(s)\, ds \qquad t \in J .$$

Since by (3), (4)

$$\|u_n(s)\| \leq m(s) + \alpha\|q(s)\| + \beta\|q'(s)\| \quad \text{a.e. on J},\tag{12}$$

$\{u_n\}$ is a bounded sequence in $L^2(J, X)$. As X is reflexive it is well known (see [2]) that $L^2(J, X)$ is reflexive too, so that we may assume $\{u_n\}$ weakly convergent in $L^2(J, X)$ to a function u, *i.e.* for any $\varphi \in L(J, X)^*$

$$\langle \varphi, u_n \rangle \to \langle \varphi, u \rangle \quad \text{in } \boldsymbol{R} \text{ as } n \to \infty .\tag{13}$$

Another consequence of the reflexivity of X is that the dual space $L^2(J, X)^*$ is $L^2(J, X^*)$, and since X is separable we have by [5] that each $\varphi \in L^2(J, X)^*$ is representable in the form

$$\langle \varphi, x \rangle = \int_a^b \langle x^*(t), x(t) \rangle\, dt ,$$

where $x^* \in L^2(J, X^*)$, and the duality product is in $L^2(J, X)$ and in X respectively on the left and on the right side of the previous equality.

Consider now an arbitrary $\varphi \in L^2(J, X)^*$. We have

$$\langle \varphi, x_n \rangle = \int_a^b \left\langle x^*(t), h(t) + \int_a^b g(t, s) u_n(s)\, ds \right\rangle dt = \int_a^b z_n(t)\, dt .\tag{14}$$

For «t» fixed in J $x^*(t) \in X^*$, and we can take $x^*(t)$ under the integral sign (see [6]) justifying the following

$$\left\langle x^*(t), h(t) + \int_a^b g(t, s) u_n(s)\, ds \right\rangle = \langle x^*(t), h(t) \rangle + \int_a^b \langle x^*(t), g(t, s) u_n(s) \rangle\, ds .$$

By (13) we get

$$z_n(t) \to z(t) = \langle x^*(t), h(t)\rangle + \int_a^b \langle x^*(t), g(t,s)u(s)\rangle\, ds = \left\langle x^*(t), h(t) + \int_a^b g(t,s)u(s)\, ds\right\rangle \quad \text{on } J.$$

As $\|z_n(t)\| \leq \|x^*(t)\|_{X^*}\|g\|_{\infty,J^2}\|u_n\|_{L^1(J,X)}$, by (12) there exists $C>0$ such that $\|z_n(t)\| \leq C\|x^*(t)\|_{X^*}$, a.e. on J, so that we can apply the Lebesgue's dominated convergence theorem in (14) obtaining

$$\langle \varphi, x\rangle = \int_a^b z(t)\, dt = \int_a^b \left\langle x^*(t), h(t) + \int_a^b g(t,s)u(s)\, ds\right\rangle dt.$$

The arbitrariety of φ implies that $x(t) = h(t) + \int_a^b g(t,s)u(s)\, ds$, $t \in J$. By Mazur's theorem on convex sets, $u(\cdot) \in F(\cdot, q(\cdot), q'(\cdot))$, so that $x \in \Sigma(q)$ that consequently is closed.

Now we show that Σ is a multivalued contraction mapping, *i.e.*

$$D_{C^1(J,X)}(\Sigma(q_1), \Sigma(q_2)) \leq K'\|q_1 - q_2\|_{C^1(J,X)}, \tag{15}$$

where K' is like in (10). Let $y_1 \in \Sigma(q_1)$ and let $u_1 \in \mathcal{U}(q_1)$ be such that $y_1(t) = h(t) + \int_a^b g(t,s)u_1(s)\, ds$. Again by [4, Proposition 3.5] there exists $u_2 \in \mathcal{U}(q_2)$ such that

$$\|u_1(t) - u_2(t)\| = d_X(u_1(t), F(t, q_2(t), q_2'(t))) \quad \text{a.e. on } J. \tag{16}$$

Let $y_2(t) = h(t) + \int_a^b g(t,s)u_2(s)\, ds \in \Sigma(q_2)$. Then, as in the proof of Theorem 1.1, using (16) we have

$$\|y_1 - y_2\|_{C(J,X)} \leq \|g\|_{\infty,J^2}(b-a)(\alpha\|q_1-q_2\|_{C(J,X)} + \beta\|q_1' - q_2'\|_{C(J,X)}) \leq K'\|q_1 - q_2\|_{C^1(J,X)}.$$

In a similar way we get $\|y_1' - y_2'\|_{C(J,X)} \leq K'\|q_1 - q_2\|_{C^1(J,X)}$, and going on as in Theorem 1.1 we obtain (15). Now, from [8] we conclude that Fix $\Sigma = S_F$ is a retract of $C^1(J,X)$.

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