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The Curvature of a Set with Finite Area

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Calcolo delle variazioni. — *The Curvature of a Set with Finite Area.* Nota di ELISABETTA BAROZZI, presentata (*) dal Corrisp. M. Miranda.

ABSTRACT. — In a paper, by myself, E. Gonzalez and I. Tamanini (see [2]), it was proven that all sets of finite perimeter do have a non trivial variational property, connected with the mean curvature of their boundaries. In the present article, that variational property is made more precise.

KEY WORDS: Calculus of variations; Geometric measure theory; Mean curvature; Boundaries of finite measure.

RIASSUNTO. — *Curvatura delle frontiere degli insiemi di perimetro finito.* In un lavoro di E. Gonzalez, I. Tamanini e me stessa (v. [2]), fu provato che tutti gli insiemi di perimetro finito hanno una notevole proprietà variazionale connessa con la curvatura media delle loro frontiere. Nel presente articolo si fanno due osservazioni su tale proprietà variazionale degli insiemi di perimetro finito.

1. INTRODUCTION

For $H \in L^1(R^n)$ define

$$\mathcal{F}_H(X) = P(X) + \int_X H(x)\,dx\,, \tag{1.1}$$

where X is a (Lebesgue) measurable subset of R^n and $P(X)$ is the perimeter of X. Here and in the sequel we use the basic properties of sets of finite perimeter, for which we refer to [3, 8]. The functional (1.1) was introduced by U. Massari in [6, 7] (see also the survey paper [4]).

Let E be a minimizer of $\mathcal{F}_H$, suppose H is continuous at $x \in \partial E$ and ∂E is smooth near x. Then is easy to see that the mean curvature of ∂E at x is given by $-H(x)/ /(n-1)$. This fact suggests the following

DEFINITION 1.1. *A set E is said to have (variational) mean curvature $H \in L^1(R^n)$ if E is a minimizer of $\mathcal{F}_H$, i.e.*

$$\mathcal{F}_H(E) \leqslant \mathcal{F}_H(X) \tag{1.2}$$

for every measurable set $X \subset R^n$.

If E is a set with finite perimeter, we introduce the space

$$\mathcal{H}^1(E) = \{H \in L^1(R^n): (1.2) \text{ holds}\}\,. \tag{1.3}$$

In a previous work (see [2]) it was proved that $\mathcal{H}^1(E) \neq \emptyset$, *i.e.* every set $E \subset R^n$ with finite perimeter has a variational curvature $H \in L^1(R^n)$. Actually, there exist infinity many variational curvatures associated with the set E. In fact, if H belongs to $\mathcal{H}^1(E)$ and $\phi \in L^1(R^n)$, $\phi(x) \leqslant 0$ a.e. $x \in E$, $\phi(x) \geqslant 0$ a.e. $x \in R^n - E$, then $H + \phi$ still belongs to $\mathcal{H}^1(E)$. The paper is organized as follows.

(*) Nella seduta del 12 febbraio 1994.

In section 2 we shall prove the existence of a function $H_E \in \mathcal{H}^1(E)$ minimizing the L^1-norm in $\mathcal{H}^1(E)$; precisely we have

$$\text{(1.4)} \qquad \begin{cases} \|H_E\|_{L^1(E)} = \inf\{\|H\|_{L^1(E)} : H \in \mathcal{H}^1(E)\}, \\ \|H_E\|_{L^1(R^n - E)} = \inf\{\|H\|_{L^1(R^n - E)} : H \in \mathcal{H}^1(E)\}. \end{cases}$$

In order to obtain this result it is helpful to note first that

$$\text{(1.5)} \qquad \begin{cases} \inf\{\|H\|_{L^1(E)} : H \in \mathcal{H}^1(E)\} = P(E), \\ \inf\{\|H\|_{L^1(R^n - E)} : H \in \mathcal{H}^1(E)\} = P(E). \end{cases}$$

The function H_E is defined in a very natural way, since its level sets are the minima E_λ of certain variational functionals $\mathcal{B}_\lambda$.

In section 3 we define, for $p > 1$,

$$\text{(1.6)} \qquad \mathcal{H}^p(E) = \{H \in \mathcal{H}^1(E) : H \in L^p(E)\}.$$

If $\mathcal{H}^p(E) \neq \emptyset$ and the n-dimensional measure of E is finite, it is easy to show the existence of a (unique) element of $\mathcal{H}^p(E)$ minimizing the $L^p(E)$-norm (this fails if the measure of E is not finite). We shall prove that such a minimizing curvature is the function H_E previously introduced.

2. The function H_E

Let $E \subset R^n$ be a set with finite perimeter. From the isoperimetric inequality we find that $\min\{|E|, |R^n - E|\} < +\infty$, where $|X|$ denotes the Lebesgue measure of the set $X \subset R^n$. Let $h \in L^1(R^n)$ be a function such that $h(x) > 0$ a.e. $x \in R^n$ and let $d\alpha = h \cdot dx$ be the measure with density h, *i.e.*

$$\alpha(X) = \int_X h(x)\,dx, \qquad X \subset R^n.$$

If $|E| < +\infty$, we can take $h(x) = 1$ $\forall x \in E$.

For $\lambda > 0$ consider the functional

$$\text{(2.1)} \qquad \mathcal{B}_\lambda(X) = P(X) + \lambda\alpha(E - X)$$

and let E_λ be a solution of the problem

$$\text{(2.2)} \qquad \begin{cases} \mathcal{B}_\lambda(X) \longrightarrow \min \\ \text{with the constraint } X \subset E. \end{cases}$$

For $\lambda < \mu$ we have $E_\lambda \subset E_\mu$. On the other hand, from the inequality

$$\mathcal{B}_\lambda(E_\lambda) \leqslant \mathcal{B}_\lambda(E)$$

we derive

$$\text{(2.3)} \qquad P(E_\lambda) + \lambda\alpha(E - E_\lambda) \leqslant P(E).$$

Inequality (2.3) implies that the sets E_λ converge to E in the $L^1_{\text{loc}}(R^n)$-sense as λ goes to $+\infty$. Therefore

$$|E - \cup\{E_\lambda : \lambda > 0\}| = 0$$

and we can define, for $x \in E$

$$H_E(x) = -\inf\{\lambda h(x): x \in E_\lambda, \lambda \in Q\}. \tag{2.4}$$

It is clear that $H_E(x) \leqslant 0$ a.e. $x \in E$.

For brevity put $\mathcal{F}_E = \mathcal{F}_{H_E}$; it will be proved that

$$\mathcal{F}_E(E) \leqslant \mathcal{F}_E(X) \quad \forall X \subset E. \tag{2.5}$$

Now, we define H_E outside E by putting

$$H_E(x) = -H_{R^n - E}(x), \qquad x \in R^n - E. \tag{2.6}$$

We have $H_E(x) \geqslant 0$ a.e. $x \in R^n - E$. Since

$$\mathcal{F}_{R^n - E}(R^n - E) \leqslant \mathcal{F}_{R^n - E}(Y) \quad \forall Y \subset R^n - E, \tag{2.7}$$

we infer, for $X = R^n - Y$, that

$$\mathcal{F}_E(E) \leqslant \mathcal{F}_E(X) \quad \forall X \supset E. \tag{2.8}$$

From (2.5), (2.8) and the well known inequality

$$P(A \cup B) + P(A \cap B) \leqslant P(A) + P(B), \tag{2.9}$$

we conclude that

$$\mathcal{F}_E(E) \leqslant \mathcal{F}_E(X) \quad \forall X \subset R^n, \tag{2.10}$$

i.e. $H \in \mathcal{H}^1(E)$.

So, we must prove inequality (2.5) (inequality (2.7) follows from (2.5), with E replaced by $R^n - E$). This will be done in the second part of the proof of Theorem 2.1.

Remark 2.1. Let $H \in \mathcal{H}^1(E)$; from $\mathcal{F}_H(E) \leqslant \mathcal{F}_H(\emptyset)$ we get

$$P(E) \leqslant -\int_E H(x)\,dx \leqslant \|H\|_{L^1(E)}. \tag{2.11}$$

From (2.11) we derive that, if

$$\|H\|_{L^1(E)} = P(E), \tag{2.12}$$

then $H(x) \leqslant 0$ a.e. $x \in E$ and $\mathcal{F}_H(E) = \mathcal{F}_H(\emptyset) = 0$. So, if (2.12) holds, the functional $\mathcal{F}_H$ has also the empty set as a minimizer.

In the same way, from $\mathcal{F}_H(E) \leqslant \mathcal{F}_H(R^n)$, we get

$$\|H\|_{L^1(R^n - E)} \geqslant P(E). \tag{2.13}$$

Theorem 2.1. *For every set $E \subset R^n$ with $P(E) < +\infty$ we have $H_E \in \mathcal{H}^1(E)$ and*

$$\|H_E\|_{L^1(R^n)} = 2P(E). \tag{2.14}$$

Proof. We begin by proving equality (2.14). Observe that (2.14) together with Remark 2.1 implies that

$$\|H_E\|_{L^1(R^n)} = \inf\{\|H\|_{L^1(R^n)}: H \in \mathcal{H}^1(E)\}. \tag{2.15}$$

The computation of $\|H_E\|_{L^1(R^n)}$ proceeds as follows. For $k \in N$ define

$$\theta_k(j) = j/2^k, \quad j \in N \tag{2.16}$$

and let E_j be a minimizer of (2.1) with $\lambda = \theta_k(j)$. Let $E_0 = \emptyset$ and define

$$S_j = E_j - E_{j-1}, \quad j \in N. \tag{2.17}$$

From the very definition of H_E it is clear that

$$-(j/2^k)h(x) \leqslant H_E(x) \leqslant -[(j-1)/2^k]h(x) \quad \forall x \in S_j \tag{2.18}$$

and therefore

$$\sum_{j=1}^{\infty} [(j-1)/2^k]\,\alpha(S_j) \leqslant \|H_E\|_{L^1(E)} \leqslant \sum_{j=1}^{\infty} (j/2^k)\,\alpha(S_j). \tag{2.19}$$

For $0 < \lambda < \mu$ we have $\mathcal{B}_\lambda(E_\lambda) \leqslant \mathcal{B}_\lambda(E_\mu) \leqslant \mathcal{B}_\mu(E_\mu) \leqslant \mathcal{B}_\mu(E_\lambda)$, so that

$$\lambda\alpha(E_\mu - E_\lambda) \leqslant P(E_\mu) - P(E_\lambda) \leqslant \mu\alpha(E_\mu - E_\lambda). \tag{2.20}$$

In particular, if $\lambda = (j-1)/2^k$, $\mu = j/2^k$, we get

$$[(j-1)/2^k]\,\alpha(S_j) \leqslant P(E_j) - P(E_{j-1}) \leqslant (j/2^k)\,\alpha(S_j). \tag{2.21}$$

From (2.3), recalling the lower semicontinuity of the perimeter, we obtain

$$\lim_{\lambda \to +\infty} P(E_\lambda) = P(E). \tag{2.22}$$

Now, from the first inequality in (2.21) and (2.22) we find

$$\sum_{j=1}^{\infty} (j/2^k)\,\alpha(S_j) = \sum_{j=1}^{\infty} (j/2^k - (j-1)/2^k)\,\alpha(S_j) + \tag{2.23}$$

$$+ \sum_{j=1}^{\infty} ((j-1)/2^k)\,\alpha(S_j) \leqslant 2^{-k}\alpha(E) + \lim_{N} P(E_N) = 2^{-k}\alpha(E) + P(E).$$

REMARK 2.1. Let $H \in$

$$P(E) \leqslant -\int_E H(x)\,dx \leqslant \|H\|_{L^1(E)}. \tag{2.11}$$

From (2.11) we derive that, if

$$\|H\|_{L^1(E)} = P(E), \tag{2.12}$$

then $H(x) \leqslant 0$ a.e. $x \in E$ and $\mathcal{F}_H(E) = \mathcal{F}_H(\emptyset) = 0$. So, if (2.12) holds, the functional $\mathcal{F}_H$ has also the empty set as a minimizer.

In the same way, from $\mathcal{F}_H(E) \leqslant \mathcal{F}_H(R^n)$, we get

$$\|H\|_{L^1(R^n - E)} \geqslant P(E). \tag{2.13}$$

THEOREM 2.1. *For every set $E \subset R^n$ with $P(E) < +\infty$ we have $H_E \in \mathcal{H}^1(E)$ and*

$$\|H_E\|_{L^1(R^n)} = 2P(E). \tag{2.14}$$

PROOF. We begin by proving equality (2.14). Observe that (2.14) together with Remark 2.1 implies that

$$\|H_E\|_{L^1(R^n)} = \inf\{\|H\|_{L^1(R^n)} : H \in \mathcal{H}^1(E)\}. \tag{2.15}$$

Since $|E - \cup\{S_j: 1 \leqslant j \leqslant +\infty\}| = 0$, for almost every $x \in E$ there exists $j \in N$ such that $x \in S_j$ and therefore $H_k(x) = -(j/2^k)h(x)$. So, we have

$$H_E(x) = -\inf\{\lambda h(x): x \in E_\lambda\} \geqslant -(j/2^k)h(x) = H_k(x),$$

that is

$$H_E(x) \geqslant H_k(x) \quad \text{a.e. } x \in E. \tag{2.29}$$

If $x \in S_1 = E_1$, then $H_k(x) = -2^{-k}h(x)$; since $H_E(x) \leqslant 0$ a.e. $x \in E$, from (2.29) we obtain

$$-2^{-k}h(x) = H_k(x) \leqslant H_E(x) \leqslant 0. \tag{2.30}$$

If $x \in S_j$, $j \geqslant 2$, then $x \notin E_{j-2}$ and one has thus

$$-((j-1)/2^k)h(x) \leqslant \inf\{\lambda h(x): x \in E_\lambda\} = -H_E(x),$$

i.e.

$$H_E(x) \leqslant -((j-2)/2^k)h(x) = H_k(x) + h(x)/2^{k-1}. \tag{2.31}$$

From (2.31) and (2.29) we get

$$H_k(x) \leqslant H_E(x) \leqslant H_k(x) + h(x)/2^{k-1}. \tag{2.32}$$

From inequalities (2.30), (2.32) we conclude that

$$H_E(x) = \lim_{k \to +\infty} H_k(x) \quad \text{a.e. } x \in E. \tag{2.33}$$

In a similar way it can be proved that (2.33) holds a.e. $x \in R^n - E$.

From the Lebesgue Theorem we conclude now that $H_k \xrightarrow[k]{} H_E$ in $L^1(R^n)$ as k goes to $+\infty$. Since $\mathcal{H}^1(E)$ is a closed subset of $L^1(R^n)$, it only remains to prove that $H_k \in \mathcal{H}^1(E)$ $\forall k \in N$. To this aim, recalling (2.21), it can be easily proved by induction that

$$P(E_j) + \int_{E_j} H_k(x)\,dx \leqslant P(X) + \int_{X \cap E_j} H_k(x)\,dx \quad \forall j \in N, \ \forall X \subset R^n. \tag{2.34}$$

Letting $j \to +\infty$, one finds that E is a minimizer of $\mathcal{F}_{H_k}$, so that $H_k \in \mathcal{H}^1(E)$. This concludes the proof of Theorem 2.1.

Remark 2.2. Let $\varepsilon_k > 0$ be a sequence of positive number converging to 0 and, for each $k \in N$, let $\theta_k(j)$ be an increasing sequence such that

i) $\lim_{j \to +\infty} \theta_k(j) = +\infty \ \forall k \in N$,

ii) $\theta_k(j+1) - \theta_k(j) < \varepsilon_k \ \forall j \in N, \forall k \in N$.

Defining H_k as in the proof of Theorem 2.1, one can prove that

$$H_E(x) = \lim_{k \to +\infty} H_k(x) \quad \text{a.e. } x \in R^n.$$

If in addition, for any $k \in N$ the sequence θ_{k+1} is a refinement of θ_k, the $H_k \xrightarrow[k]{} H_E$ in $L^1(R^n)$ as k goes to $+\infty$.

REMARK 2.3. It is evident that, if $\lambda > 0$, then $H_{E_\lambda}(x) = H_E(x)$ a.e. $x \in E_\lambda$. Then, from Theorem 2.1, we have

$$\int_{E_\lambda} |H_{E_\lambda}(x)|\, dx = P(E_\lambda) \quad \forall \lambda > 0 .$$

We conclude that $\mathcal{F}_E(E_\lambda) = \mathcal{F}_{E_\lambda}(E_\lambda) = 0$ $\forall\lambda > 0$, so that for each $\lambda > 0$ the set E_λ is a minimizer for $\mathcal{F}_E$.

REMARK 2.4. Define

$$E_\lambda^+ = \bigcap_{\mu > \lambda} E_\mu , \quad E_\lambda^- = \bigcup_{\eta < \lambda} E_\eta , \quad S_\lambda = E_\lambda^+ - E_\lambda^- .$$

It is easy to show that E_λ^+, E_λ^- are minimizers (the maximal and minimal ones) for $\mathcal{B}_\lambda$ and that $H_E(x) = -\lambda h(x) \Leftrightarrow x \in S_\lambda$.

REMARK 2.5. There may be many functions in $\mathcal{H}^1(E)$ minimizing the L^1-norm; note the following examples.

EXAMPLE 2.1. Let $E = \{(x,y) \in R^2 : x^2 + y^2 \leq \rho^2\}$, $\rho > 0$. Then $H_E(x,y) = -2/\rho$ $\forall (x,y) \in E$, $x^2 + y^2 > 0$. Define

$$H(x,y) = \begin{cases} -(x^2+y^2)^{-1/2} & \text{for } (x,y) \in E,\ x^2+y^2 > 0 , \\ H_E(x,y) & \text{for } (x,y) \notin E . \end{cases}$$

Then $H \in \mathcal{H}^1(E)$ and $\|H\|_{L^1(E)} = P(E)$. Note that $H_E \in L^\infty(E)$ while $H \notin L^p(E)$ $\forall p \geq 2$.

EXAMPLE 2.2. Let $E = \{(x,y) \in R^2 : |x| \leq L,\ |y| \leq L\}$, $L > 0$, let $h(x,y) = 1$ $\forall (x,y) \in E$.

For $0 < r < L$ define Q_r as the union of the open balls of radius r inside E.

Let $\lambda_0 = (1 + \sqrt{\pi}/2)\cdot(1/L)$. With elementary computations one finds that

$$E_\lambda^+ = \begin{cases} \emptyset & \text{if } \lambda < \lambda_0 , \\ \text{closure of } Q_{1/\lambda} & \text{if } \lambda \geq \lambda_0 , \end{cases}$$

$$E_\lambda^- = \begin{cases} \emptyset & \text{if } \lambda \leq \lambda_0 , \\ Q_{1/\lambda} & \text{if } \lambda > \lambda_0 . \end{cases}$$

We have $E = E_{\lambda_0}^+ \cup \bigcup_{\lambda > \lambda_0} \partial E_\lambda \cup \{P_1, P_2, P_3, P_4\}$, where P_i's are the vertices of the square E, and

$$H_E(x,y) = \begin{cases} -\lambda_0 & \text{if } (x,y) \in E_{\lambda_0}^+ , \\ -\lambda & \text{if } (x,y) \in \partial E_\lambda ,\ \lambda > \lambda_0 . \end{cases}$$

It is clear that $\|H_E\|_{L^2(E)} = +\infty$; in fact, this is a consequence of the following theorem in [5]: «If E is a subset of R^2 and $H_E \in L^2(R^2)$, then the density of E at the boundary points is $1/2$». By a direct computation one can easily find that $H_E \in L^p(E)$ $\forall p < 2$ but $H_E \notin L^2(E)$.

Let $Q_0 = E^+_{\lambda_0}$ and, for $0 < \sigma < \lambda_0$, define

$$H_\sigma(x, y) = \begin{cases} -\lambda_0 + \sigma & \text{if } (x, y) \in Q_0, \\ H_E(x, y) - \sigma|Q_0|/|E - Q_0| & \text{if } (x, y) \in E - Q_0. \end{cases}$$

Clearly $H_\sigma \in \mathcal{H}^1(E)$ and $\|H_\sigma\|_{L^1(E)} = \|H_E\|_{L^1(E)}$, while $\|H_\sigma\|_{L^p(E)} > \|H_E\|_{L^p(E)}$ $\forall p$: $1 < p < 2$.

3. The L^p-norm of H_E

Suppose $\mathcal{H}^p(E) \neq \emptyset$ for some p, $1 < p < +\infty$; it is easy to see that $\mathcal{H}^p(E)$ is a convex subset of $L^p(E)$. If, in addition, $|E| < +\infty$, then the Hölder's inequality implies that $\mathcal{H}^p(E)$ is also closed in $L^p(E)$. From these remarks and the Clarkson's inequalities (see [1]) we immediately get the following

Theorem 3.1. Suppose

$$|E| < +\infty \tag{3.1}$$

and suppose $\mathcal{H}^p(E) \neq \emptyset$ for some p, $1 < p < +\infty$. Then $\mathcal{H}^p(E)$ contains a unique function that minimizes the $L^p(E)$-norm.

Remark 3.1. The condition (3.1) can't be avoided. To see this, consider the set $E = \{(x, y) \in R^2 : x^2 + y^2 > 1\}$ and let $c_i = 2/(i^2 - 1)$, $i \in N$, $i > 1$. The function

$$H_i(x, y) = \begin{cases} -c_i & \text{if } 1 < x^2 + y^2 < i^2, \\ 0 & \text{if } x^2 + y^2 \geqslant i^2, \\ 2 & \text{if } x^2 + y^2 \leqslant 1 \end{cases}$$

belongs to $\mathcal{H}^2(E)$ for each $i \in N$, $i > 1$ and $\|H_i\|_{L^2(E)} \to 0$ as i goes to $+\infty$.

The following theorem shows that the curvature with minimal $L^p(E)$-norm, whose existence was asserted in Theorem 1, is exactly the function H_E; precisely, we have

Theorem 3.2. Same hypothesis as in Theorem 3.1. We have

$$\|H_E\|_{L^p(E)} \leqslant \|H\|_{L^p(E)} \quad \forall H \in \mathcal{H}^p(E) \tag{3.2}$$

and equality in (3.2) implies

$$H(x) = H_E(x) \quad \text{a.e. } x \in E. \tag{3.3}$$

Proof. The uniqueness result was proved in Theorem 3.1. So it remains only to prove the minimum property (3.2). As $|E| < +\infty$, we can choose $h = 1$ (*i.e.* $d\alpha = dx$) in the construction of H_E inside E. If $H \in \mathcal{H}^p(E)$, so it is the function

$$H^*(x) = \begin{cases} H(x) & \text{for } x \in R^n - E, \\ \min\{H(x), 0\} & \text{for } x \in E \end{cases}$$

and clearly $\|H^*\|_{L^p(E)} \leqslant \|H\|_{L^p(E)}$, equality holding if and only if $H(x) \leqslant 0$ a.e. $x \in E$. So, we can suppose $H(x) \leqslant 0$ inside E. From $\mathcal{F}_H(E) \leqslant \mathcal{F}_H(E_\lambda)$ and the Hölder's inequality we get

$$P(E) - P(E_\lambda) \leqslant \left[\int_{E - E_\lambda} |H|^p \, dx \right]^{1/p} |E - E_\lambda|^{1 - 1/p} . \tag{3.4}$$

Then, from

$$\lambda |E - E_\lambda| \leqslant P(E) - P(E_\lambda) \quad (\text{see } (2.3)), \tag{3.5}$$

we obtain

$$|E - E_\lambda| \leqslant \frac{1}{\lambda^p} \int_{E - E_\lambda} |H|^p \, dx . \tag{3.6}$$

For $1 < q < p$ we have

$$\int_E |H_E|^q \, dx = \int_0^{+\infty} |\{x \in E: |H_E(x)|^q > t\}| \, dt \leqslant$$

$$\leqslant |E| + \int_1^{+\infty} |\{x \in E: |H_E(x)|^q > t\}| \, dt = |E| + q \int_1^{+\infty} |E - E_\lambda| \lambda^{q-1} \, d\lambda .$$

From this inequality and (3.6) we find the estimate

$$\int_E |H_E|^q \, dx \leqslant |E| + \frac{q}{p - q} \int_E |H|^p \, dx . \tag{3.7}$$

In particular $H_E \in L^q(E)$ for $q < p$.

Now let $\theta_k(j)$, E_j, S_j be as in (2.16), (2.17) and define

$$\alpha_j(k) = \frac{1}{|S_j|} \int_{S_j} |H_E| \, dx , \qquad \beta_j(k) = \frac{1}{|S_j|} \int_{S_j} |H| \, dx .$$

For brevity in the sequel we shall write α_j, β_j instead $\alpha_j(k)$, $\beta_j(k)$. We have

$$\alpha_j^q |S_j| \leqslant \int_{S_j} |H_E|^q \, dx , \qquad \beta_j^q |S_j| \leqslant \int_{S_j} |H|^q \, dx$$

so that

$$\begin{cases} \displaystyle\sum_{i=1}^{j} \alpha_i^q |S_i| \leqslant \int_{E_j} |H_E|^q \, dx , \\[2ex] \displaystyle\sum_{i=1}^{j} \beta_i^q |S_i| \leqslant \int_{E_j} |H|^q \, dx . \end{cases} \tag{3.8}$$

On the other hand it is not difficult to show that

$$\lim_{k \to +\infty} \sum_{j=1}^{\infty} \alpha_j^q |S_j| = \int_E |H_E|^q \, dx . \tag{3.9}$$

In fact, from $(j-1)/2^k \leqslant |H_E| \leqslant j/2^k$ $\forall x \in S_j$, $(j-1)/2^k \leqslant \alpha_j \leqslant j/2^k$ we get

$$0 \leqslant \int_E |H_E|^q\,dx - \sum_{j=1}^{\infty} \alpha_j^q |S_j| \leqslant \frac{1}{2^{kq}}|S_1| + \frac{1}{2^{kq}} \sum_{j=2}^{\infty} [j^q - (j-1)^q]|S_j| \leqslant$$

$$\leqslant \frac{1}{2^{kq}}|S_1| + \frac{1}{2^{kq}} \sum_{j=2}^{\infty} j^{q-1}|S_j| \leqslant \frac{1}{2^{kq}}|S_1| + \frac{2^{q-1}}{2^{kq}} \sum_{j=2}^{\infty} (j-1)^{q-1}|S_j| =$$

$$= \frac{1}{2^{kq}}|S_1| + \frac{2^{q-1}}{2^k} \sum_{j=2}^{\infty} \left[\frac{j-1}{2^k}\right]^{q-1} |S_j| \leqslant \frac{1}{2^{kq}}|S_1| + \frac{2^{q-1}}{2^k} \int_{E-E_1} |H_E|^{q-1}\,dx$$

and (3.9) follows. The next step is to prove

$$\sum_{j=1}^{\infty} \beta_j^q |S_j| \geqslant \sum_{j=1}^{\infty} \alpha_j^q |S_j|\,. \tag{3.10}$$

Let

$$I = \int_E |H|\,dx - \int_E |H_E|\,dx \geqslant 0 \tag{3.11}$$

and $\gamma_j = \beta_j - \alpha_j$, $j \in N$. We have

$$\sum_{j=1}^{\infty} \gamma_j |S_j| = I\,, \tag{3.12}$$

$$\sum_{j=1}^{m} \gamma_j |S_j| \leqslant I \quad \forall m \in N\,, \tag{3.13}$$

$$\sum_{j=1}^{\infty} \beta_j^q |S_j| - \sum_{j=1}^{\infty} \alpha_j^q |S_j| \geqslant q \sum_{j=1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j|\,. \tag{3.14}$$

Equality (3.12) follows from the very definition of α_j, β_j. Now, note that

$$\sum_{j=1}^{m} \gamma_j |S_j| = \int_{E_m} |H|\,dx - \int_{E_m} |H_E|\,dx = -\mathscr{F}_H(E_m) + \mathscr{F}_E(E_m) =$$

$$= -\mathscr{F}_H(E_m) \leqslant -\mathscr{F}_H(E) = -\mathscr{F}_E(E) - \int_E (H - H_E)\,dx = \int_E (|H| - |H_E|)\,dx = I\,,$$

which proves (3.13).

(3.14) follows from the elementary inequality $(1+x)^q \geqslant 1 + qx$ for $x \geqslant -1$ applied with $x = \gamma_j/\alpha_j$. From (3.12), (3.13) and the monotonicity of the sequence α_j one easily infers (by induction on m) that

$$\sum_{j=1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| \geqslant \alpha_1^{q-1} I - \alpha_m^{q-1} \sum_{j=m+1}^{\infty} \gamma_j |S_j| + \sum_{j=m+1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| \tag{3.15}$$

and therefore

$$\sum_{j=1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| \geqslant \alpha_1^{q-1} I - 2 \sum_{j=m+1}^{\infty} \alpha_j^{q-1} |\gamma_j|\,|S_j|\,, \tag{3.16}$$

because $\alpha_j \geqslant \alpha_m$ for $j \geqslant m$. Hölder's inequality gives

$$\sum_{j=1}^{\infty} \alpha_j^{q-1} |\gamma_j| \, |S_j| \leqslant \left[\sum_{j=1}^{\infty} \alpha_j^q |S_j| \right]^{(q-1)/q} \left[\sum_{j=1}^{\infty} |\gamma_j|^q |S_j| \right]^{1/q} < +\infty$$

and then, letting $m \to +\infty$ in (3.16), we obtain

$$\sum_{j=1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| \geqslant \alpha_1^{q-1} I. \tag{3.17}$$

Combining (3.14) and (3.17) we get the invoked inequality (3.10). Finally, from (3.8), (3.9) and (3.10) we get

$$\int_E |H_E|^q \, dx \leqslant \int_E |H|^q \, dx \quad \forall q < p$$

and therefore (3.2).

REMARK 3.2. Let E_j, S_j as in (2.17). Recalling Remark 2.3 we have

$$P(E_j) - P(E_{j-1}) = -\int_{S_j} H_E \, dx .$$

Therefore, if $H_E \in L^p(E)$, we get

$$P(E_j) - P(E_{j-1}) \leqslant \left[\int_{S_j} |H_E|^p \, dx \right]^{1/p} |S_j|^{1-1/p}$$

from which

$$\sum_{j=1}^{\infty} \frac{P(E_j) - P(E_{j-1})}{|S_j|^{1-1/p}} \leqslant c \quad \forall k \in N, \tag{3.18}$$

where $c = \|H_E\|_{L^p(E)}$. Note that the reciprocal is also true, *i.e.* (3.18) implies that $H_E \in L^p(E)$. In fact, from the minimum property of E_{j-1}, E_j we obtain (see (2.20)) the estimates

$$\lambda_{j-1} |S_j| \leqslant P(E_j) - P(E_{j-1}) \leqslant \lambda_j |S_j|$$

so that

$$\lambda_j = \lambda_{j-1} + 1/2^k \leqslant (P(E_j) - P(E_{j-1}))/|S_j| + 1/2^k . \tag{3.19}$$

Define H_k as in (2.28): from (3.19) we derive

$$\|H_k\|^p_{L^p(E)} = \sum_{j=1}^{\infty} \lambda_j^p |S_j| \leqslant \sum_{j=1}^{\infty} \left[\frac{P(E_j) - P(E_{j-1})}{|S_j|} + \frac{1}{2^k} \right]^p |S_j| \leqslant$$

$$\leqslant 2^{p-1} \sum_{j=1}^{\infty} \left[\frac{P(E_j) - P(E_{j-1})}{|S_j|^{1-1/p}} \right]^p + \frac{2^{p-1}}{2^{kp}} |E| \leqslant 2^{p-1} \cdot c^p + \frac{|E|}{2}$$

and therefore $H_E \in L^p(E)$.

References

[1] R. A. Adams, *Sobolev Spaces.* Academic Press, New York-London-Toronto-Sydney-San Francisco 1984.

[2] E. Barozzi - E. Gonzalez - I. Tamanini, *The Mean Curvature of a Set of Finite Perimeter.* Proc. A.M.S., 99, 1987, 313-316.

[3] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, Boston-Basel-Stuttgart 1984.

[4] E. H. A. Gonzalez - U. Massari, *Variational mean curvature.* Rend. Sem. Mat. Univer. Politecnic. Torino, to appear.

[5] E. H. A. Gonzales - U. Massari - I. Tamanini, *Boundaries of prescribed mean curvature.* Rend. Mat. Acc. Lincei, s. 9, v. 4, 1993, 197-206.

[6] U. Massari, *Esistenza e regolarità delle ipersuperfici di curvatura media assegnata in* $\boldsymbol{R}^n$. Arch. Rat. Mech. An., 55, 1974, 357-382.

[7] U. Massari, *Frontiere orientate di curvatura media assegnata in* L^p. Rend. Sem. Mat. Univ. Padova, 53, 1975, 37-52.

[8] U. Massari - M. Miranda, *Minimal Surfaces of Codimension One.* North-Holland, Amsterdam 1984.

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