Elisabetta Barozzi

The Curvature of a Set with Finite Area


Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1994_9_5_2_149_0>

L’utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l’utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma bdim (Biblioteca Digitale Italiana di Matematica)
SIMAI & UMI
http://www.bdim.eu/

ABSTRACT. — In a paper, by myself, E. Gonzalez and I. Tamanini (see [2]), it was proven that all sets of finite perimeter do have a non trivial variational property, connected with the mean curvature of their boundaries. In the present article, that variational property is made more precise.

KEY WORDS: Calculus of variations; Geometric measure theory; Mean curvature; Boundaries of finite measure.

RIASSUNTO. — Curvatura delle frontiere degli insiemi di perimetro finito. In un lavoro di E. Gonzalez, I. Tamanini e me stessa (v. [2]), fu provato che tutti gli insiemi di perimetro finito hanno una notevole proprietà variazionale connessa con la curvatura media delle loro frontiere. Nel presente articolo si fanno due osservazioni su tale proprietà variazionale degli insiemi di perimetro finito.

1. INTRODUCTION

For $H \in L^1(R^n)$ define

$$\mathcal{F}_H(X) = P(X) + \int_X H(x) \, dx,$$

where $X$ is a (Lebesgue) measurable subset of $R^n$ and $P(X)$ is the perimeter of $X$. Here and in the sequel we use the basic properties of sets of finite perimeter, for which we refer to [3,8]. The functional (1.1) was introduced by U. Massari in [6,7] (see also the survey paper [4]).

Let $E$ be a minimizer of $\mathcal{F}_H$, suppose $H$ is continuous at $x \in \partial E$ and $\partial E$ is smooth near $x$. Then it is easy to see that the mean curvature of $\partial E$ at $x$ is given by $-H(x)/(n - 1)$. This fact suggests the following

DEFINITION 1.1. A set $E$ is said to have (variational) mean curvature $H \in L^1(R^n)$ if $E$ is a minimizer of $\mathcal{F}_H$, i.e.

$$\mathcal{F}_H(E) \leq \mathcal{F}_H(X)$$

for every measurable set $X \subset R^n$.

If $E$ is a set with finite perimeter, we introduce the space

$$\mathcal{C}^1(E) = \{H \in L^1(R^n); (1.2) \text{ holds}\}.$$

In a previous work (see [2]) it was proved that $\mathcal{C}^1(E) \neq \emptyset$, i.e. every set $E \subset R^n$ with finite perimeter has a variational curvature $H \in L^1(R^n)$. Actually, there exist infinity many variational curvatures associated with the set $E$. In fact, if $H$ belongs to $\mathcal{C}^1(E)$ and $\phi \in L^1(R^n)$, $\phi(x) \leq 0$ a.e. $x \in E$, $\phi(x) \geq 0$ a.e. $x \in R^n - E$, then $H + \phi$ still belongs to $\mathcal{C}^1(E)$. The paper is organized as follows.

(*) Nella seduta del 12 febbraio 1994.
In section 2 we shall prove the existence of a function \( H_E \in \mathcal{C}^1(E) \) minimizing the \( L^1 \)-norm in \( \mathcal{C}^1(E) \); precisely we have

\[
\begin{align*}
\|H_E\|_{L^1(E)} &= \inf \{\|H\|_{L^1(E)} : H \in \mathcal{C}^1(E) \}, \\
\|H_E\|_{L^1(R^n - E)} &= \inf \{\|H\|_{L^1(R^n - E)} : H \in \mathcal{C}^1(E) \}.
\end{align*}
\]

In order to obtain this result it is helpful to note first that

\[
\begin{align*}
\inf \{\|H\|_{L^1(E)} : H \in \mathcal{C}^1(E) \} &= P(E), \\
\inf \{\|H\|_{L^1(R^n - E)} : H \in \mathcal{C}^1(E) \} &= P(E).
\end{align*}
\]

The function \( H_E \) is defined in a very natural way, since its level sets are the minima of certain variational functionals \( \beta_\lambda \).

In section 3 we define, for \( p > 1 \),

\[
\mathcal{C}^p(E) = \{ H \in \mathcal{C}^1(E) : H \in L^p(E) \}.
\]

If \( \mathcal{C}^p(E) \neq \emptyset \) and the \( n \)-dimensional measure of \( E \) is finite, it is easy to show the existence of a (unique) element of \( \mathcal{C}^p(E) \) minimizing the \( L^p(E) \)-norm (this fails if the measure of \( E \) is not finite). We shall prove that such a minimizing curvature is the function \( H_E \) previously introduced.

### 2. The function \( H_E \)

Let \( E \subset \mathbb{R}^n \) be a set with finite perimeter. From the isoperimetric inequality we find that \( \min \{ |E|, |\mathbb{R}^n - E| \} < +\infty \), where \( |X| \) denotes the Lebesgue measure of the set \( X \subset \mathbb{R}^n \). Let \( b \in L^1(\mathbb{R}^n) \) be a function such that \( b(x) > 0 \) a.e. \( x \in \mathbb{R}^n \) and let \( dx = b \cdot dx \) be the measure with density \( b \), i.e.

\[
\alpha(X) = \int_X b(x) \, dx, \quad X \subset \mathbb{R}^n.
\]

If \( |E| < +\infty \), we can take \( b(x) = 1 \) \( \forall x \in E \).

For \( \lambda > 0 \) consider the functional

\[
\beta_\lambda(X) = P(X) + \lambda \alpha(E - X)
\]

and let \( E_\lambda \) be a solution of the problem

\[
\begin{align*}
\beta_\lambda(X) &\longrightarrow \min \\
\text{with the constraint } X \subset E.
\end{align*}
\]

For \( \lambda < \mu \) we have \( E_\lambda \subset E_\mu \). On the other hand, from the inequality

\[
\beta_\lambda(E_\lambda) \leq \beta_\lambda(E)
\]

we derive

\[
P(E_\lambda) + \lambda \alpha(E - E_\lambda) \leq P(E).
\]

Inequality (2.3) implies that the sets \( E_\lambda \) converge to \( E \) in the \( L^1_{\text{loc}}(\mathbb{R}^n) \)-sense as \( \lambda \) goes to \( +\infty \). Therefore

\[
|E - \bigcup \{ E_\lambda : \lambda > 0 \}| = 0
\]
and we can define, for \( x \in E \)
\[(2.4) \quad H_{E}(x) = -\inf \{ \lambda b(x); x \in E, \lambda \in Q \}. \]
It is clear that \( H_{E}(x) \leq 0 \) a.e. \( x \in E \).

For brevity put \( \mathcal{J}_{E} = \mathcal{J}_{H_{E}} \); it will be proved that
\[(2.5) \quad \mathcal{J}_{E}(E) \leq \mathcal{J}_{E}(X) \quad \forall X \subset E. \]
Now, we define \( H_{E} \) outside \( E \) by putting
\[(2.6) \quad H_{E}(x) = -H_{R^{n}-E}(x), \quad x \in R^{n}-E. \]
We have \( H_{E}(x) \geq 0 \) a.e. \( x \in R^{n}-E \). Since
\[(2.7) \quad \mathcal{J}_{R^{n}-E}(R^{n}-E) \leq \mathcal{J}_{R^{n}-E}(Y) \quad \forall Y \subset R^{n}-E, \]
we infer, for \( X = R^{n}-Y \), that
\[(2.8) \quad \mathcal{J}_{E}(E) \leq \mathcal{J}_{E}(X) \quad \forall X \subset R^{n}, \]
From (2.5), (2.8) and the well known inequality
\[(2.9) \quad P(A \cup B) + P(A \cap B) \leq P(A) + P(B), \]
we conclude that
\[(2.10) \quad \mathcal{J}_{E}(E) \leq \mathcal{J}_{E}(X) \quad \forall X \subset R^{n}, \]
i.e. \( H \in \mathcal{H}^{1}(E) \).

So, we must prove inequality (2.5) (inequality (2.7) follows from (2.5), with \( E \)
replaced by \( R^{n}-E \)). This will be done in the second part of the proof of The­
orem 2.1.

**Remark 2.1.** Let \( H \in \mathcal{H}^{1}(E) \); from \( \mathcal{J}_{H}(E) \leq \mathcal{J}_{H}(\emptyset) \) we get
\[(2.11) \quad P(E) \leq -\int_{E} H(x) \, dx \leq \|H\|_{L^{1}(E)}. \]
From (2.11) we derive that, if
\[(2.12) \quad \|H\|_{L^{1}(E)} = P(E), \]
then \( H(x) \leq 0 \) a.e. \( x \in E \) and \( \mathcal{J}_{H}(E) = \mathcal{J}_{H}(\emptyset) = 0. \) So, if (2.12) holds, the functional \( \mathcal{J}_{H} \) has also the empty set as a minimizer.

In the same way, from \( \mathcal{J}_{H}(E) \leq \mathcal{J}_{H}(R^{n}) \), we get
\[(2.13) \quad \|H\|_{L^{1}(R^{n}-E)} = P(E). \]

**Theorem 2.1.** For every set \( E \subset R^{n} \) with \( P(E) < +\infty \) we have \( H_{E} \in \mathcal{H}^{1}(E) \) and
\[(2.14) \quad \|H_{E}\|_{L^{1}(R^{n})} = 2P(E). \]

**Proof.** We begin by proving equality (2.14). Observe that (2.14) together with Re­
mark 2.1 implies that
\[(2.15) \quad \|H_{E}\|_{L^{1}(R^{n})} = \inf \{ \|H\|_{L^{1}(R^{n})}; H \in \mathcal{H}^{1}(E) \}. \]
The computation of \( \|H_E\|_{L^1(\mathbb{R}^n)} \) proceeds as follows. For \( k \in \mathbb{N} \) define
\[
\theta_k(j) = j/2^k, \quad j \in \mathbb{N}
\]
and let \( E_j \) be a minimizer of (2.1) with \( \lambda = \theta_k(j) \). Let \( E_0 = \emptyset \) and define
\[
S_j = E_j - E_{j-1}, \quad j \in \mathbb{N}.
\]
From the very definition of \( H_E \) it is clear that
\[
-(j/2^k) b(x) \leq H_E(x) \leq -[(j - 1)/2^k] b(x) \quad \forall x \in S_j
\]
and therefore
\[
\sum_{j=1}^{\infty} ((j - 1)/2^k) \alpha(S_j) \leq \|H_E\|_{L^1(E)} \leq \sum_{j=1}^{\infty} (j/2^k) \alpha(S_j).
\]
For \( 0 < \lambda < \mu \) we have \( \mathcal{B}_\lambda(E_\lambda) \leq \mathcal{B}_\lambda(E_\mu) \leq \mathcal{B}_\mu(E_\mu) \leq \mathcal{B}_\mu(E_\lambda) \), so that
\[
\lambda \alpha(E_\mu - E_\lambda) \leq P(E_\mu) - P(E_\lambda) \leq \mu \alpha(E_\mu - E_\lambda).
\]
In particular, if \( \lambda = (j - 1)/2^k, \mu = j/2^k \), we get
\[
[(j - 1)/2^k] \alpha(S_j) \leq P(E_j) - P(E_{j-1}) \leq (j/2^k) \alpha(S_j).
\]
From (2.3), recalling the lower semicontinuity of the perimeter, we obtain
\[
\lim_{\lambda \to +\infty} P(E_\lambda) = P(E).
\]
Now, from the first inequality in (2.21) and (2.22) we find
\[
\sum_{j=1}^{\infty} (j/2^k) \alpha(S_j) = \sum_{j=1}^{\infty} (j/2^k - (j - 1)/2^k) \alpha(S_j) +
\sum_{j=1}^{\infty} \left( ((j - 1)/2^k) \alpha(S_j) \leq 2^{-k} \alpha(E) \right) + \lim_{N \to \infty} P(E_N) = 2^{-k} \alpha(E) + P(E).
\]
(2.11) \( P(E) \leq -\int_E H(x) \, dx \leq \|H\|_{L^1(E)} \).

From (2.11) we derive that, if
\[
\|H\|_{L^1(E)} = P(E),
\]
then \( H(x) \leq 0 \) a.e. \( x \in E \) and \( \mathcal{F}_H(E) = \mathcal{F}_H(\emptyset) = 0 \). So, if (2.12) holds, the functional \( \mathcal{F}_H \) has also the empty set as a minimizer.

In the same way, from \( \mathcal{F}_H(E) \leq \mathcal{F}_H(\mathbb{R}^n) \), we get
\[
\|H\|_{L^1(\mathbb{R}^n - E)} \geq P(E).
\]

**Theorem 2.1.** For every set \( E \subset \mathbb{R}^n \) with \( P(E) < +\infty \) we have \( H_E \in \mathcal{C}^1(E) \) and
\[
\|H_E\|_{L^1(\mathbb{R}^n)} = 2P(E).
\]

**Proof.** We begin by proving equality (2.14). Observe that (2.14) together with Remark 2.1 implies that
\[
\|H_E\|_{L^1(\mathbb{R}^n)} = \inf \{\|H\|_{L^1(\mathbb{R}^n)} : H \in \mathcal{C}^1(E)\}.
\]
Since \(|E - \bigcup \{S_j: 1 \leq j \leq +\infty\}| = 0\), for almost every \(x \in E\) there exists \(j \in \mathbb{N}\) such that \(x \in S_j\) and therefore \(H_k(x) = -(j/2^k)b(x)\). So, we have
\[
H_E(x) = -\inf \{\lambda b(x): x \in E\} = -(j/2^k)b(x) = H_k(x),
\]
that is
\[
H_E(x) \geq H_k(x) \quad \text{a.e. } x \in E. \tag{2.29}
\]
If \(x \in S_1 = E_1\), then \(H_k(x) = -2^{-k}b(x)\); since \(H_E(x) \leq 0\) a.e. \(x \in E\), from (2.29) we obtain
\[
-2^{-k}b(x) = H_k(x) \leq H_E(x) \leq 0. \tag{2.30}
\]
If \(x \in S_j, j \geq 2\), then \(x \notin E_{j-2}\) and one has thus
\[
-((j - 1)/2^k)b(x) \leq \inf \{\lambda b(x): x \in E\} = -H_E(x),
\]
i.e.
\[
H_E(x) \leq -((j - 2)/2^k)b(x) = H_k(x) + b(x)/2^{k-1}. \tag{2.31}
\]
From (2.31) and (2.29) we get
\[
H_k(x) \leq H_E(x) \leq H_k(x) + b(x)/2^{k-1}. \tag{2.32}
\]
From inequalities (2.30), (2.32) we conclude that
\[
H_E(x) = \lim_{k \to +\infty} H_k(x) \quad \text{a.e. } x \in E. \tag{2.33}
\]
In a similar way it can be proved that (2.33) holds a.e. \(x \in \mathbb{R}^n - E\).

From the Lebesgue Theorem we conclude now that \(H_k \to H_E\) in \(L^1(\mathbb{R}^n)\) as \(k\) goes to +\(\infty\). Since \(\mathcal{C}^1(E)\) is a closed subset of \(L^1(\mathbb{R}^n)\), it only remains to prove that \(H_k \in \mathcal{C}^1(E) \forall k \in \mathbb{N}\). To this aim, recalling (2.21), it can be easily proved by induction that
\[
P(E_j) + \int_{E_j} H_k(x) \, dx \leq P(X) + \int_X H_k(x) \, dx \quad \forall j \in \mathbb{N}, \ \forall X \subset \mathbb{R}^n. \tag{2.34}
\]
Letting \(j \to +\infty\), one finds that \(E\) is a minimizer of \(\mathcal{F}_{H_k}\), so that \(H_k \in \mathcal{C}^1(E)\). This concludes the proof of Theorem 2.1.

REMARK 2.2. Let \(\varepsilon_k > 0\) be a sequence of positive number converging to 0 and, for each \(k \in \mathbb{N}\), let \(\theta_k(j)\) be an increasing sequence such that
\[
i) \lim_{j \to +\infty} \theta_k(j) = +\infty \quad \forall k \in \mathbb{N},
\]
\[
ii) \theta_k(j + 1) - \theta_k(j) < \varepsilon_k \quad \forall j \in \mathbb{N}, \ \forall k \in \mathbb{N}.
\]
Defining \(H_k\) as in the proof of Theorem 2.1, one can prove that
\[
H_E(x) = \lim_{k \to +\infty} H_k(x) \quad \text{a.e. } x \in \mathbb{R}^n. \tag{2.34}
\]
If in addition, for any \(k \in \mathbb{N}\) the sequence \(\theta_{k+1}\) is a refinement of \(\theta_k\), the \(H_k \to H_E\) in \(L^1(\mathbb{R}^n)\) as \(k\) goes to +\(\infty\).
REMARK 2.3. It is evident that, if $A > 0$, then $H_{E_x}(x) = H_{E_x}(x)$ a.e. $x \in E_{\lambda}$. Then, from Theorem 2.1, we have

$$
\int_{E_{\lambda}} |H_{E_x}(x)| \, dx = P(E_{\lambda}) \quad \forall \lambda > 0.
$$

We conclude that $\mathcal{F}_E(E_{\lambda}) = \mathcal{F}_{E_x}(E_{\lambda}) = 0 \ \forall \lambda > 0$, so that for each $\lambda > 0$ the set $E_{\lambda}$ is a minimizer for $\mathcal{F}_E$.

REMARK 2.4. Define

$$
E_{\lambda}^+ = \bigcap_{\mu > \lambda} E_{\mu}, \quad E_{\lambda}^- = \bigcup_{\eta < \lambda} E_{\eta}, \quad S_{\lambda} = E_{\lambda}^+ - E_{\lambda}^-.
$$

It is easy to show that $E_{\lambda}^+$, $E_{\lambda}^-$ are minimizers (the maximal and minimal ones) for $\mathcal{B}_x$ and that $H_E(x) = -\lambda b(x) \Leftrightarrow x \in S_{\lambda}$.

REMARK 2.5. There may be many functions in $\mathcal{X}^1(E)$ minimizing the $L^1$-norm; note the following examples.

EXAMPLE 2.1. Let $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq \rho^2\}$, $\rho > 0$. Then $H_E(x, y) = -2/\rho$ $\forall (x, y) \in E$, $x^2 + y^2 > 0$. Define

$$
H(x, y) = \begin{cases} 
-\rho (x^2 + y^2)^{-1/2} & \text{for } (x, y) \in E, \ x^2 + y^2 > 0, \\
H_E(x, y) & \text{for } (x, y) \notin E.
\end{cases}
$$

Then $H \in \mathcal{X}^1(E)$ and $\|H\|_{L^1(E)} = P(E)$. Note that $H_E \in L^\infty(E)$ while $H \notin L^p(E)$ $\forall p \geq 2$.

EXAMPLE 2.2. Let $E = \{(x, y) \in \mathbb{R}^2 : |x| \leq L, |y| \leq L\}$, $L > 0$, let $b(x, y) = 1$ $\forall (x, y) \in E$.

For $0 < r < L$ define $Q_r$ as the union of the open balls of radius $r$ inside $E$.

Let $\lambda_0 = (1 + \sqrt{\pi}/2) \cdot (1/L)$. With elementary computations one finds that

$$
E_{\lambda_0}^+ = \begin{cases} 
\emptyset & \text{if } \lambda < \lambda_0, \\
\text{closure of } Q_{1/\lambda} & \text{if } \lambda \geq \lambda_0.
\end{cases}
$$

$$
E_{\lambda_0}^- = \begin{cases} 
\emptyset & \text{if } \lambda \leq \lambda_0, \\
Q_{1/\lambda} & \text{if } \lambda > \lambda_0.
\end{cases}
$$

We have $E = E_{\lambda_0}^+ \cup \bigcup_{\lambda > \lambda_0} \partial E_{\lambda} \cup \{P_1, P_2, P_3, P_4\}$, where $P_i$’s are the vertices of the square $E$, and

$$
H_E(x, y) = \begin{cases} 
-\lambda_0 & \text{if } (x, y) \in E_{\lambda_0}^+, \\
-\lambda & \text{if } (x, y) \in \partial E_{\lambda}, \ \lambda > \lambda_0.
\end{cases}
$$

It is clear that $\|H_E\|_{L^2(E)} = +\infty$; in fact, this is a consequence of the following theorem in [5]: «If $E$ is a subset of $\mathbb{R}^2$ and $H_E \in L^2(\mathbb{R}^2)$, then the density of $E$ at the boundary points is 1/2». By a direct computation one can easily find that $H_E \in L^p(E)$ $\forall p < 2$ but $H_E \notin L^2(E)$.  


Let $Q_0 = E_{\lambda_0}^+$ and, for $0 < \sigma < \lambda_0$, define
\[
H_\sigma(x, y) = \begin{cases} 
-\lambda_0 + \sigma & \text{if } (x, y) \in Q_0, \\
H_E(x, y) - \sigma |Q_0| / |E - Q_0| & \text{if } (x, y) \in E - Q_0.
\end{cases}
\]
Clearly $H_\sigma \in \mathcal{C}^1(E)$ and $\|H_\sigma\|_{L^1(E)} = \|H_E\|_{L^1(E)}$, while $\|H_\sigma\|_{L^p(E)} > \|H_E\|_{L^p(E)}$ for $1 < p < 2$.

\section{The $L^p$-Norm of $H_E$}

Suppose $\mathcal{C}^p(E) \neq \emptyset$ for some $p$, $1 < p < +\infty$; it is easy to see that $\mathcal{C}^p(E)$ is a convex subset of $L^p(E)$. If, in addition, $|E| < +\infty$, then the Hölder’s inequality implies that $\mathcal{C}^p(E)$ is also closed in $L^p(E)$. From these remarks and the Clarkson’s inequalities (see [1]) we immediately get the following

**Theorem 3.1.** Suppose
\begin{equation}
|E| < +\infty
\end{equation}
and suppose $\mathcal{C}^p(E) \neq \emptyset$ for some $p$, $1 < p < +\infty$. Then $\mathcal{C}^p(E)$ contains a unique function that minimizes the $L^p(E)$-norm.

**Remark 3.1.** The condition (3.1) can’t be avoided. To see this, consider the set $E = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ and let $c_i = 2/(i^2 - 1)$, $i \in \mathbb{N}$, $i > 1$. The function
\[
H_i(x, y) = \begin{cases} 
-c_i & \text{if } 1 < x^2 + y^2 < i^2, \\
0 & \text{if } x^2 + y^2 \geq i^2, \\
2 & \text{if } x^2 + y^2 \leq 1
\end{cases}
\]
belongs to $\mathcal{C}^2(E)$ for each $i \in \mathbb{N}$, $i > 1$ and $\|H_i\|_{L^2(E)} \to 0$ as $i$ goes to $+\infty$.

The following theorem shows that the curvature with minimal $L^p(E)$-norm, whose existence was asserted in Theorem 1, is exactly the function $H_E$; precisely, we have

**Theorem 3.2.** Same hypothesis as in Theorem 3.1. We have
\begin{equation}
\|H_E\|_{L^p(E)} \leq \|H\|_{L^p(E)} \quad \forall H \in \mathcal{C}^p(E)
\end{equation}
and equality in (3.2) implies
\begin{equation}
H(x) = H_E(x) \quad \text{a.e. } x \in E.
\end{equation}

**Proof.** The uniqueness result was proved in Theorem 3.1. So it remains only to prove the minimum property (3.2). As $|E| < +\infty$, we can choose $b = 1$ (i.e. $d\alpha = dx$) in the construction of $H_E$ inside $E$. If $H \in \mathcal{C}^p(E)$, so it is the function
\[
H^*(x) = \begin{cases} 
H(x) & \text{for } x \in \mathbb{R}^n - E, \\
\min \{H(x), 0\} & \text{for } x \in E
\end{cases}
\]
and clearly \( \|H^*\|_{L^p(E)} \leq \|H\|_{L^p(E)} \), equality holding if and only if \( H(x) = 0 \) a.e. \( x \in E \). So, we can suppose \( H(x) \equiv 0 \) inside \( E \). From \( \mathcal{F}_H(E) \leq \mathcal{F}_H(E_\lambda) \) and the Hölder’s inequality we get

\[
P(E) - P(E_\lambda) = \left[ \int_{E - E_\lambda} |H|^p \, dx \right]^{1/p} |E - E_\lambda|^{1 - 1/p}.
\]

Then, from

\[
\lambda |E - E_\lambda| \leq P(E) - P(E_\lambda) \quad \text{(see (2.3))},
\]

we obtain

\[
|E - E_\lambda| \leq \frac{1}{\lambda P} \int_{E - E_\lambda} |H|^p \, dx.
\]

For \( 1 < q < p \) we have

\[
\int_{E} |H_E|^q \, dx = \int_{1}^{+\infty} \{x \in E: |H_E(x)|^q > t\} \, dt \leq |E| + \int_{1}^{+\infty} \{x \in E: |H_E(x)|^q > t\} \, dt = |E| + q \int_{1}^{+\infty} |E - E_\lambda| \lambda^{q - 1} \, d\lambda.
\]

From this inequality and (3.6) we find the estimate

\[
\int_{E} |H_E|^q \, dx \leq |E| + \frac{q}{p - q} \int_{E} |H|^p \, dx.
\]

In particular \( H_E \in L^q(E) \) for \( q < p \).

Now let \( \theta_k(j), E_j, S_j \) be as in (2.16), (2.17) and define

\[
\alpha_j(k) = \frac{1}{|S_j|} \int_{S_j} |H_E| \, dx, \quad \beta_j(k) = \frac{1}{|S_j|} \int_{S_j} |H| \, dx.
\]

For brevity in the sequel we shall write \( \alpha_j, \beta_j \) instead \( \alpha_j(k), \beta_j(k) \). We have

\[
\alpha_j^q |S_j| \leq \int_{S_j} |H_E|^q \, dx, \quad \beta_j^q |S_j| \leq \int_{S_j} |H|^q \, dx
\]

so that

\[
\begin{align*}
\sum_{i=1}^{j} \alpha_i^q |S_i| & \leq \int_{E_j} |H_E|^q \, dx, \\
\sum_{i=1}^{j} \beta_i^q |S_i| & \leq \int_{E_j} |H|^q \, dx.
\end{align*}
\]

On the other hand it is not difficult to show that

\[
\lim_{k \to +\infty} \sum_{j=1}^{\infty} \alpha_j^q |S_j| = \int_{E} |H_E|^q \, dx.
\]
In fact, from \((j - 1)/2^k \leq |H_E| \leq j/2^k\) \(\forall x \in S_j\), \((j - 1)/2^k \leq \alpha_j \leq j/2^k\) we get

\[
0 \leq \int_E |H_E|^q \, dx - \sum_{j=1}^{\infty} \alpha_j^q |S_j| \leq \frac{1}{2^{kq}} |S_1| + \frac{1}{2^{kq}} \sum_{j=2}^{\infty} [j^q - (j - 1)^q] |S_j| \leq
\]

\[
\leq \frac{1}{2^{kq}} |S_1| + \frac{1}{2^{kq}} \sum_{j=2}^{\infty} j^{q-1} |S_j| \leq \frac{1}{2^{kq}} |S_1| + \frac{2^{q-1}}{2^{kq}} \sum_{j=2}^{\infty} (j - 1)^{q-1} |S_j| =
\]

\[
= \frac{1}{2^{kq}} |S_1| + \frac{2^{q-1}}{2^k} \left[ \sum_{j=2}^{\infty} \left( \frac{j - 1}{2^k} \right)^{q-1} |S_j| \right] \leq \frac{1}{2^{kq}} |S_1| + \frac{2^{q-1}}{2^k} \int_{E-E_i} |H_E|^{q-1} \, dx
\]

and (3.9) follows. The next step is to prove

(3.10) \[\sum_{j=1}^{\infty} \beta_j^q |S_j| \geq \sum_{j=1}^{\infty} \alpha_j^q |S_j| .\]

Let

(3.11) \[I = \int_E |H| \, dx - \int_E |H_E| \, dx \geq 0\]

and \(\gamma_j = \beta_j - \alpha_j, j \in \mathbb{N}\). We have

(3.12) \[\sum_{j=1}^{\infty} \gamma_j |S_j| = I ,\]

(3.13) \[\sum_{j=1}^{\infty} \gamma_j |S_j| \leq I \quad \forall m \in \mathbb{N},\]

(3.14) \[\sum_{j=1}^{\infty} \beta_j^q |S_j| - \sum_{j=1}^{\infty} \alpha_j^q |S_j| \geq q \sum_{j=1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| .\]

Equality (3.12) follows from the very definition of \(\alpha_j, \beta_j\). Now, note that

\[
\sum_{j=1}^{m} \gamma_j |S_j| = \int_{E_m} |H| \, dx - \int_{E_m} |H_E| \, dx = -\mathcal{F}_H(E_m) + \mathcal{F}_E(E_m) =
\]

\[
= -\mathcal{F}_H(E_m) \leq -\mathcal{F}_H(E) = -\mathcal{F}_E(E) - \int_E (H - H_E) \, dx = \int_E (|H| - |H_E|) \, dx = I ,
\]

which proves (3.13).

(3.14) follows from the elementary inequality \((1 + x)^q \geq 1 + qx\) for \(x \geq -1\) applied with \(x = \gamma_j/\alpha_j\). From (3.12), (3.13) and the monotonicity of the sequence \(\alpha_j\) one easily infers (by induction on \(m\)) that

(3.15) \[\sum_{j=1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| \geq \alpha_1^{q-1} I - \alpha_m^{q-1} \sum_{j=m+1}^{\infty} \gamma_j |S_j| + \sum_{j=m+1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| \]

and therefore

(3.16) \[\sum_{j=1}^{\infty} \alpha_j^{q-1} \gamma_j |S_j| \geq \alpha_1^{q-1} I - 2 \sum_{j=m+1}^{\infty} \alpha_j^{q-1} |\gamma_j| |S_j| ,\]
because \( \alpha_j \geq \alpha_m \) for \( j \geq m \). Hölder’s inequality gives
\[
\sum_{j=1}^{\infty} \alpha_j^{q-1} |\gamma_j| |S_j| \leq \left( \sum_{j=1}^{\infty} \alpha_j^q |S_j| \right)^{(q-1)/q} \left( \sum_{j=1}^{\infty} |\gamma_j|^q |S_j| \right)^{1/q} < +\infty
\]
and then, letting \( m \to +\infty \) in (3.16), we obtain
\[
(3.17) \quad \sum_{j=1}^{\infty} \alpha_j^{q-1} |\gamma_j| |S_j| \geq \alpha_1^{q-1} I.
\]
Combining (3.14) and (3.17) we get the invoked inequality (3.10). Finally, from (3.8), (3.9) and (3.10) we get
\[
\int_E |H_E|^q dx \leq \int_E |H|^q dx \quad \forall q < p
\]
and therefore (3.2).

**REMARK 3.2.** Let \( E_j, S_j \) as in (2.17). Recalling Remark 2.3 we have
\[
P(E_j) - P(E_{j-1}) = - \int_{S_j} H_E dx.
\]
Therefore, if \( H_E \in L^p(E) \), we get
\[
P(E_j) - P(E_{j-1}) \leq \left( \int_{S_j} |H_E|^p dx \right)^{1/p} |S_j|^{1-1/p}
\]
from which
\[
(3.18) \quad \sum_{j=1}^{\infty} \frac{P(E_j) - P(E_{j-1})}{|S_j|^{1-1/p}} \leq c \quad \forall k \in \mathbb{N},
\]
where \( c = \|H_E\|_{L^p(E)} \). Note that the reciprocal is also true, i.e. (3.18) implies that \( H_E \in L^p(E) \). In fact, from the minimum property of \( E_{j-1}, E_j \) we obtain (see (2.20)) the estimates
\[
\lambda_{j-1} |S_j| \leq P(E_j) - P(E_{j-1}) \leq \lambda_j |S_j|
\]
so that
\[
(3.19) \quad \lambda_j = \lambda_{j-1} + 1/2^k \leq (P(E_j) - P(E_{j-1}))/|S_j| + 1/2^k.
\]
Define \( H_k \) as in (2.28): from (3.19) we derive
\[
\|H_k\|_{L^p(E)} = \sum_{j=1}^{\infty} \lambda_j^p |S_j| \leq \sum_{j=1}^{\infty} \left[ \frac{P(E_j) - P(E_{j-1})}{|S_j|} + \frac{1}{2^k} \right]^p |S_j| \leq
\]
\[
\leq 2^{p-1} \sum_{j=1}^{\infty} \left[ \frac{P(E_j) - P(E_{j-1})}{|S_j|^{1-1/p}} \right]^p + \frac{2p-1}{2^{kp}} |E| \leq 2^{p-1} \cdot c^p + \frac{|E|}{2}
\]
and therefore \( H_E \in L^p(E) \).
REFERENCES


Dipartimento di Matematica
Università degli Studi di Trento
38050 Povo TN