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Finite groups with eight non-linear irreducible characters

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Teoria dei gruppi. — *Finite groups with eight non-linear irreducible characters.* Nota di Yakov Berkovich, presentata (*) dal Socio G. Zappa.

Abstract. — This *Note* contains the complete list of finite groups, having exactly eight non-linear irreducible characters. In section 4 we consider in full details some typical cases.

Key words: Finite groups; Representation of groups; Characters.

Riassunto. — *Gruppi finiti con esattamente otto caratteri irriducibili non lineari.* La *Nota* contiene la lista completa dei gruppi finiti con esattamente otto caratteri irriducibili non lineari. Sono riportate le dimostrazioni di alcuni casi tipici.

1. Introduction

G. Seitz [8] classified all finite groups G with $n(G) = 1$, where $n(G)$ is the number of all non-linear irreducible complex characters of G. In [5] C. Hanşen and J. M. Nielsen classified all finite groups G with $n(G) = 2$. Ya. G. Berkovich [1] classified all finite groups G with $3 \leq n(G) \leq 7$. In this *Note* we give the list of all G with $n(G) = 8$. The proof of our main theorem is very long and complicated. Some comments to the proof of the main theorem one can find in section 4.

2. Notation

Let G be a finite group of order g, let G' be its commutator subgroup of order g' and let $k(G)$ be the class number of G. Then $n(G) = k(G) - g/g'$. If G is a p-group (p is a prime) then $z_i = |\{x \in G \mid |G: C_G(x)| = p^i\}$, where $C_G(x)$ is the centralizer of an element x in G. Let:

p:	prime number;
$C(m)$:	cyclic group of order m;
$E(p^m)$:	elementary abelian group of order p^m;
$ES(m, p)$:	extra-special group of order p^{1+2m};
$D(2m)$:	dihedral group of order $2m$;
$Q(2^m)$:	generalized quaternion group of order 2^m;
$SD(2^m)$:	semi-dihedral group of order 2^m;
A_m:	alternating group of degree m;
S_m:	symmetric group of degree m;
$S(64)$:	group of order 64 with $Z(G) = G' = \Phi(G) = E(4)$, where $\Phi(G)$ is the Frattini subgroup of G and $Z(G)$ is the centre of G;

(*) Nella seduta del 13 novembre 1993.

$A[B$:	splittable extension of B by A;
(A, B):	Frobenius group with the Frobenius kernel B and a complement A;
$A \times B$:	direct product of A and B;
H^m:	direct product of m copies of a group H;
$GL(n, p^m)$:	n-dimensional general linear group over $GF(p^m)$;
$AGL(n, p^m)$:	$GL(n, p^m)[E(p^{nm})$ where the left factor acts irreducibly on the right one;
$\mathrm{Irr}(G)$:	set of all ordinary irreducible characters of G.

We write $h(G) = (h_1^{a(1)}, \ldots, h_k^{a(k)})$ if G has exactly $a(i)$ classes of length h_i, $i = 1, 2, \ldots, k$, $h_1 = 1$ and $a(1) + a(2) + \ldots + a(k) = k(G)$. Obviously $a(1) = |Z(G)|$. We write $d(G) = (d_1^{b(1)}, \ldots, d_s^{b(s)})$ if $\mathrm{Irr}(G)$ contains exactly $b(i)$ characters of degree d_i, $i = 1, 2, \ldots, s$, $b(1) + \ldots + b(s) = |\mathrm{Irr}(G)| = k(G)$. If $d_1 = 1$ then $b(1) = g/g'$. Obviously $cd\ G = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\} = \{d_1, \ldots, d_s\}$. We have $a(1)h_1 + \ldots + a(k)h_k = g = b(1)d_1^2 + \ldots + b(s)d_s^2$, h_i and d_i divide g.

3. The main theorem

In this section we formulate our main theorem:

Main theorem. If $n(G) = 8$ then G is one of the following groups:

A. *G is a non-trivial direct product.*

A1. $T \times C(2)$, $n(T) = 4$.

A2. $T \times C(4)$, $n(T) = 2$.

A3. $T \times C(8)$, $n(T) = 1$.

A4. $A_4 \times B$, $B \in \{Q(8), D(8), AGL(1, 5)\}$.

A5. $D(6) \times D(10)$.

B. *G is a p-group.*

B1. $g = 2^{2m}$, $g' = 2$, $z_0 = 16$.

B2. $g = 2^{2m}$, $g' = 8$, $7z_0 + 3z_1 + z_2 = 64$.

B3. $g = 2^{2m}$, $g' = 32$, $31z_0 + 15z_1 + 7z_2 + 3z_3 + z_4 = 256$.

B4. $g = 3^{2m}$, $g' = 9$, $\{z_0, z_1\} = \{9, 0\}$ or $\{3, 24\}$.

C. *Infinite series.*

C1. $(C((p^\alpha - 1)/8), E(p^\alpha))$.

C2. $Z(G) = C(2)$, $g/g' = (p^\alpha - 1)/2$, $G/Z(G) = (C((p^\alpha - 1)/4), E(p^\alpha))$.

C3. $|Z(G)| = 4$, $g/g' = 2(p^\alpha - 1)$, $G/Z(G) = (C((p^\alpha - 1)/2), E(p^\alpha))$.

C4. $|Z(G)| = 8$, $g/g' = 8(p^\alpha - 1)$, $G/Z(G) = (C(p^\alpha - 1), E(p^\alpha))$.

C5. $(C((3^\alpha - 1/2), G'))$, G' is a special group of order $3^{2\alpha}$, $|Z(G')| = 3^\alpha$, $\alpha \geqslant 3$ is odd.

C6. $C(3^\alpha - 1)[G'$, $g' = 3^{3\alpha}$, $|G''| = 3^{2\alpha}$, $Z(G') = E(3^\alpha)$, $n(G/Z(G')) = 7$.

C7. $(C(6(q-1))[C(q), E(p^\alpha))$, $q = 1 + 2^m 3^n$ is a prime, $p^\alpha = 1 + 12(q-1)q$, $a > 1$.

D. $g/g' = 1$.

D1. $SL(2, 5)$.

D2. $PSL(2, 8)$.

D3. $PSL(2, 13)$.

D4. A_7.

D5. $G/E(16) = A_5$, $Z(G) = 1$.

E. $g/g' = 2$.

E1. $C(2)[ES(1, 3)$ is of exponent 6, $h(G) = (1^3, 6^4, 9^3)$.

E2. $D(6)[A$, $A \in \{E(16), C(4)^2\}$, $h(G) = (1, 3^3, 6, 12^4, 32)$.

E3. $D(10)[E(16)$, $h(G) = (1, 5^3, 20^4, 32^2)$.

F. $g/g' = 3$.

F1. $C(3)[ES(2, 2)$.

F2. Aut $PSL(2, 8)$.

G. $g/g' = 4$.

G1. $C(4)[G'$, $G' \in \{C(9), E(9)\}$, $Z(G) = C(2)$.

G2. $A[C(3)$, $A \in \{D(16), Q(16), SD(16)\}$, $C_G(C(3))$ is not abelian.

G3. $Q(8)[E(27)$, $Z(G) = 1$.

G4. $(Q(16), E(81))$.

G5. $(C(4)[C(5), E(81))$.

G6. $C(4)[E(27)$, $h(G) = (1, 2, 4^6, 9, 18, 27^2)$.

G7. $A[G'$, $G' = (C(3), S(64))$, $A \in \{C(4), E(4)\}$, $h(G) = (1, 3, 12, 16, 48^2, 96^4, 128^2)$.

H. $g/g' = 5$.

H1. $C(5)[ES(2, 2)$.

I. $g/g' = 6$.

I1. $(C(6), E(49))$.

I2. $C(6)[G', g' = 256, h(G) = (1, 3, 6^2, 48, 64, 96^4, 256^4)$. If R is a normal four-subgroup of G then $R < G'$, $G'/R = S(64)$, $C(3)G' = (C(3), G')$.

I3. $C(6)[E(8), h(G) = (1^2, 2^2, 3^2, 4^6, 6^2)$.

I4. $SL(2, 3) * C(4)$, the central product of order 48.

I5. $C(6)[E(9)[E(64), h(G) = (1, 9, 54, 96, 128, 144, 192^2, 288, 384^2, 432, 576^2)$.

J. $g/g' = 8$.

J1. $C(8)[ES(1, 3)$ is of exponent 24, $h(G) = (1^2, 2^2, 9^2, 12^4, 18^2, 27^4)$.

J2. $(C(4) \times D(6))[E(25), h(G) = (1, 12^2, 15^2, 25^3, 50^4, 60^2, 75^2)$.

J3. $C(8)[(C(3), E(25)), h(G) = (1^2, 12^4, 25^2, 50^4, 75^4)$.

J4. $C(8)[G', g' = 243, h(G) = (1, 8, 18, 72^3, 81^3, 162^3, 243^4)$.

J5. $P[C(5), |P| = 32, n(P) = 3, h(G) = (1^2, 2, 4^3, 8^3, 10^2, 20^5)$.

K. $g/g' = 10$.

K1. $AGL(1, 11)[C(3), h(G) = (1, 2, 10^3, 11^4, 22^4, 33^5)$.

L. $g/g' = 12$.

L1. $C(12)[(E(4) \times C(5)), h(G) = (1, 3, 4^3, 5^3, 12, 15^3, 16^2, 20^6)$.

L2. $AGL(1, 13)[E(4), h(G) = (1, 3, 12^4, 13^3, 39^3, 52^8)$.

L3. $AGL(1, 13)[C(5), h(G) = (1, 4, 12^5, 13^2, 52^2, 65^9)$.

L4. $A[G', A \in \{C(12), E(4) \times C(3)\}, g' = 1024, h(G) = (1, 3, 12, 48, 192, 256^3, 768^4, 1024^8)$.

M. $g/g' = 14$.

M1. $C(14)[G', g' = 512, G'/Z(G') \in \{E(64), C(4)^3\}, h(G) = (1, 7, 28^2, 64, 112^4, 448, 512^{12})$.

N. $g/g' = 16$.

N1. $(C(2) \times AGL(1, 9))[C(5), h(G) = (1, 4, 5, 8, 9^3, 16^2, 36^3, 40, 45^{11})$.

N2. $(C(2) \times C(8))[E(81), h(G) = (1, 8^2, 9^2, 16^4, 72^2, 81^{13})$.

N3. $C(16)[C(5)[E(81), h(G) = (1^2, 40^4, 81^2, 324^4, 405^{12})$.

N4. $P[E(9), |P| = 32, G/C_P(E(9)) = (Q(8), E(9)), n(P) = 4, h(G) = (1^4, 8^4, 9^4, 18^{12})$.

N5. $(C(4) \times AGL(1, 5))[E(81), h(G) = (1, 40^2, 45^2, 81^3, 324^4, 360^2, 405^{10})$.

N6. $ES(2, 2)[E(81)$, $h(G) = (1, 16^3, 18^3, 32, 81, 144^3, 162^{12})$.

N7. $P[E(81)$, $h(G) = (1, 16, 18, 32^2, 81^7, 144, 162^{11})$, $|P| = 32$.

O. $g/g' = 18$.

O1. $(C(2) \times E(9))[E(49)$, $h(G) = (1, 6^2, 7^4, 18^2, 42^4, 49^{13})$.

P. $g/g' = 20$.

P1. $(Q(8) \times C(5), E(121))$.

P2. $C(20)[G'$, $G' \cong ES(1, 11)$ is of exponent 11, $G/Z(G') = (C(20), E(121))$, $|G: C_G(Z(G'))| = 10$.

Q. $g/g' = 60$.

Q1. $A[G'$, $A \in \{C(60), E(4) \times C(15)\}$, $G' \in \{E(256), C(4)^4\}$.

Q2. $C(60)[G'$, $G' \cong ES(1, 11)$ is of exponent 11, $G/Z(G') = (C(60), E(121))$, $|G: C_G(Z(G'))| = 10$.

For classification of all groups G with $n(G) \leqslant 3$ see Chapter 19 of book [4]. All groups G with $n(G) = 4$ were classified in [5] (for classification of all G with $n(G) = 5$ see the following issue of the same collection).

Let $\lambda(G)$ be the number of prime factors of $|G|$. I, and independently, E. Bertram conjectured that for all finite groups G the following inequality holds: $\lambda(G) \leqslant k(G)$. I know only two groups G for which $\lambda(G) = k(G)$: $G = M_{22}$ and $PSL(3, 4)$. May be for G solvable one has $\lambda(G) < k(G)$.

4. Comments to the proof of the main theorem

In this section I would like to give some comments to the proof of the main theorem.

We suppose that the complete classifications of finite groups G satisfying to $n(G) \leqslant 7$ or $k(G) \leqslant 12$ are known (see [1, 2]). So in the sequel we assume that $n(G) = 8$ and $k(G) > 12$. Then $g/g' > 4$.

(1) Assume that G' is not semi-simple. Then there exists in G' an abelian minimal normal subgroup R of G. Set $|R| = p^n$ where p is a prime, $n > 0$. Choose R so that p^n is as minimal as possible. Since the intersection of kernels of all non-linear irreducible characters of any non-abelian group is trivial one has $n(G/R) \leqslant 7$. Hence the structure of G/R is known [2]. By the assumption one has $|G/R: G'/R| \geqslant 5$ and G/R is solvable by [2]. Therefore G is solvable in this case.

Now I consider some possibilities supposing that G is solvable.

(2) Assume that $G = A \times B$ where A and B are non-identity subgroups of G. Set $|A| = a$, $|A'| = a'$, $|B| = b$, $|B'| = b'$. Then $8 = n(G) = n(A)n(B) + an(B)/a' + bn(A)/b'$. Suppose that A and B are not abelian, *i.e.*, $n(A) \geqslant 1$, $n(B) \geqslant 1$. Then $n(A) \leqslant 2$,

$n(B) \leqslant 2$. If $n(A) = 2$ then $n(B) = 1$, $b/b' = 2$, $a/a' = 2$. So $B \cong S_3$ by [8] and $A \cong D(10)$ by [5] (see also Ch. 19 in [4]). Now let $n(A) = n(B) = 1$. Then $a/a' + b/b' = 7$ (let $a/a' \geqslant b/b'$). It follows from [8] that $a/a' \neq 5$. Then $a/a' = 4$, $b/b' = 3$, $A \in \in \{Q(8), D(8), AGL(1, 5)\}$ and $B \cong A_4$. Now suppose that A is not abelian and B is abelian. Then $bn(A) = 8$ and $\{n(A), b\} = \{1, 8\}, \{2, 4\}, \{4, 2\}$, so that [2] yields the structure of A.

It is easy to show that $G = A_5 \times C(2)$ is the only non-solvable group with $n(G) = 8$ which is a non-trivial direct product.

(3) Assume that $n(G/R) = 0$ *i.e.,* G/R is abelian. Then $R = G'$. Let G be nilpotent. Then $n = 1$ and $G = P \times A$ where P is a Sylow p-subgroup of G and A is abelian. By (1*a*) we may assume that $A = 1$. Set $|G| = p^m$. Suppose that $|Z(G)| = z_0 = p$. Then $k(G) = |G: G'| + n(G) = p^{m-1} + 8 = z_0 + z_1/p = p + (p^m - p)/p = p^{m-1} + + p - 1$, and $p - 1 = 8$ – a contradiction. Hence $z_0 = p^s > p$. Obviously $cd\, G = \{1, p^k\}$ where $2k = m - s$. Then, as easy to see, $n(G) = p^{s-1}(p - 1) = 8$ and $p = 2$, $s = 4$.

Suppose that G is not nilpotent. Then R is not contained in $\Phi(G)$ (Wielandt) hence there exists a maximal subgroup A of G such that $G = A\,[\,R$. Then $C_A(R) = Z(G)$ and $G/Z(G)$ is a Frobenius group with the kernel $RZ(G)/Z(G)$ and a cyclic complement $A/Z(G)$. It is easy to see that $8 = n(G) = |Z(G)|\, n(G/Z(G))$. In this case we obtain groups C1-C4.

(4) Let $n(G/R) = 1$. By Seitz's result [8] G/R is an extraspecial 2-group or $G = (C(p^a - 1), E(p^a))$. We consider only the first case since the second too difficult to expose in this short *Note*.

Thus let $G/R = ES(m, 2)$.

Suppose that $p = 2$. Then $n = 1$ and $|G'| = 2^2$. Let $d_2, \ldots, d_8$ be the degrees of characters from Irr (G) – Irr (G/R). Then $|g| - |G/R| = 2^{2m+1} = d_2^2 + \ldots + d_8^2 \equiv 1$ (mod 3) – a contradiction (note that d_i is a power of 2 by the Ito theorem on degrees of irreducible characters; see Theorem 6.15 in [7]).

Let now $p > 2$. In view of the minimal choice of $|R|$ one has $C_G(R) = R$. If $Q \in \in \mathrm{Syl}_2(G')$ then $C_G(Q) = P \in \mathrm{Syl}_2(G)$. Therefore $G' = (Q, R)$ is a Frobenius group. Denote by $k_G(M)$ the number of G-classes containing elements from M. If G is a Frobenius group then $P \cong Q(8)$, $k_G(R) = 7$ so $|R| - 1 = p^n - 1 = 7|P| = 56$ – a contradiction. Thus G is not a Frobenius group. In this case $k_G(R) \leqslant 6$. If $x \in R - \{1\}$ then $|G: C_G(x)| \leqslant 2^{m+1}$ (since G' is a Frobenius group). Now we can step by step consider all six possibilities for $k_G(R)$.

Let $k_G(R) = 1$. Then $p^n - 1 = 2^{m+c}$ where $c > 0$. Since G' is a Frobenius group then Sylow 2-subgroup P_0 of $C_G(x)$ is elementary abelian and $C_G(R) = R$. If $y_1, \ldots, y_t$ are pairwise distinct elements of $P_0^{\#}$ then $y_1x, \ldots, y_tx$ are pairwise non-conjugate elements of G. So $t \leqslant 6$ and $|P_0| \leqslant 4$. Now $C_G(R) = R$ implies $n > 1$. Hence $p^n = 3^2$, $2^{m+c} = 2^3$. Since Sylow 2-subgroup of Aut (R) is isomorphic to $SD(16)$, one obtains $m = 1$. Then $2^3(3^2 - 1) = d_2^2 + \ldots + d_8^2$ where $d_i = 2^{c(i)}$, $s(i) \leqslant 3$, which is impossible.

Similarly we can consider the remaining possibilities $k_G(R) = 2, \ldots, 6$. We only show that $p = 3$. Set $d_i = 2^{c(i)}$, $i = 2, \ldots, 8$. Then $2^{2m+1}(p^n - 1) = 2^{2c(2)} + \ldots + 2^{2c(8)} \equiv 1 \pmod 3$. This congruence implies $p = 3$.

(5) In the sequel we assume that G' is semisimple. We know that in this case $|G: G'| \geqslant 5$. Let $R \leqslant G'$ be a minimal normal subgroup of G. Since $k_G(R) \leqslant 8$ then R is simple. Now $k_G(R) \geqslant 3$ so $n(G/R) \leqslant 5$. Then G/R is solvable by [2]. Denote by $\psi(G)$ the number of cosets $xG' \neq G'$ such that $k_G(xG') > 1$. In our case $\psi(G) + k_G(G') \leqslant 8$. We use constantly in our reasoning the equality $|\mathrm{Irr}(G)| = k(G)$.

Suppose that G has a normal p-complement H for a prime divisor p of $|G|$. Take $P \in \mathrm{Syl}_P(G)$ and x, an element of order p in $Z(P)$. Since $\langle x, H\rangle$ is not a Frobenius group (Thompson) then there is in $H - \{1\}$ an element y, which commute with x. Obviously $k_G(\langle xy\rangle) \geqslant p - 1$. Since $k_G(R) + k_G(\langle xy\rangle) \leqslant 8$ then $p \leqslant 5$. Now $k_G(R) \geqslant 3$. If $k_G(R) = 3$ then $R \cong A_5$ and $RP = R \times P$ for $p > 2$, a contradiction, since $R \leqslant H$. Thus $k_G(R) \geqslant 4$. In any case $p > 2$ by Odd Order Theorem. Suppose that $p = 5$. Then $k_G(R) = 4$ and R is isomorphic to $PSL(2, 7)$ or $PSL(2, 8)$ by known classification theorems. Then x centralizes R, and it is easy to see that $n(G) > 8$ – a contradiction. Thus $p = 3$. Since $3 \nmid |R|$ then $R \cong Sz(q)$ where q is a power of 2 (Thompson, Glauberman). It is easy to see that $n(G) > 8$ in this case. Thus G has no a normal p-complement for any $p \mid |G|$.

Now if $p^a \,|g/g'$ and $p^a > n(G) - 2$ then G is solvable or p-nilpotent (Isaacs-Passman [6]). So by the above, if $p^a \,|g/g'$ then $p^a \leqslant 5$. Thus $g/g' | 4 \cdot 3 \cdot 5 = 60$.

If $5 \,|\, g/g'$ then $\psi(G) \geqslant 4$ (Sylow or Thompson). Since $k_G(G') \geqslant 3$ then $4 \nmid g/g'$, $3 \nmid g/g'$. Thus $k_G(G') \leqslant 4$, $G' \in \{A_5, PSL(2, 7), PSL(2, 8)\}$. In view of the structure of $\mathrm{Aut}(G')$ one obtains a contradiction. Thus $g/g' \,|\, 12$.

Suppose that $g/g' = 12$. We note that a Sylow 2-subgroup P of G is not of maximal class since $P \cap G'$ is not cyclic. So if x is a 2-element from $G - G'$ then $|C_P(x)| \geqslant 8$ (Suzuki). If y is an 3-element from $G - G'$ then $|C_{G'}(y)| > 1$. Thus $\psi(G) \geqslant 5$ so $k_G(G') \leqslant 3$. Hence $G' \cong A_5$. It is easy to see that in this case $n(G) > 8$.

Since $g/g' > 4$ then $g/g' = 6$. As above we may assume that $\psi(G) < 5$. If x is an element from $G - G'$ such that $\langle x, G'\rangle = G$ then $C_G(x) = \langle x\rangle$ hence $G = \langle x\rangle [\, G'$ and x induces via conjugation a fixed point free automorphism of G'. By well known result G' is solvable – a contradiction.

These are simplest but in the some sense typical cases.

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