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On iterations of Green type integrals for matrix factorizations of the Laplace operator

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Matematica. — *On iterations of Green type integrals for matrix factorizations of the Laplace operator.* Nota di ALEXANDRE A. SHLAPUNOV, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — Convergence of special Green integrals for matrix factorization of the Laplace operator in $\boldsymbol{R}^n$ is proved. Explicit formulae for solutions of $\bar\partial$-equation in strictly pseudo-convex domains in $\boldsymbol{C}^n$ are obtained.

KEY WORDS: Green integral; Differential operator with injective symbol; Dolbeault complex.

RIASSUNTO. — *Iterazioni di integrali di Green per fattorizzazioni matriciali dell'operatore di Laplace.* Si dimostra la convergenza di integrali di Green per fattorizzazione dell'operatore di Laplace. Si stabiliscono formule esplicite per soluzioni di equazioni di Cauchy-Riemann in domini strettamente pseudoconvessi di $\boldsymbol{C}^n$.

In 1978 two papers of A.V. Romanov devoted to the iterations of the Martinelli-Bochner integral were published (see [14, 15]). In particular, in [15] the following result was obtained.

THEOREM 0.1 (A.V. Romanov [15]). *Let D be a bounded domain in $\boldsymbol{C}^n$ with a connected boundary ∂D of class C^1, and let M be the Martinelli-Bochner integral (on ∂D) defined on the Sobolev space $W^{1,2}(D)$. Then, in the strong operator topology in $W^{1,2}(D)$, $\lim_{\nu \to \infty} M^\nu = \Pi$ where Π is a projection from $W^{1,2}(D)$ onto the closed subspace of holomorphic $W^{1,2}(D)$-functions.*

Using this theorem Romanov (see [15]) obtained a multi-dimensional analogue of the Cauchy-Green formula in the plane (see, for example [8, 9]), and, as consequence, an explicit formula for a solution $f \in W^{1,2}(D)$ of the equation $\bar\partial f = u$ where D is a pseudo-convex domain with a smooth (infinitely differentiable) boundary, and u is a $\bar\partial$-closed $(0,1)$-form with coefficients in $W^{1,2}(D)$.

A. M. Kytmanov [11] used Theorem 0.1 to study the $\bar\partial$-Neumann problem for functions, and to prove a criterion for the holomorphic extension from ∂D to the domain D.

The Green integrals (see, for example, [18]) associated to systems of linear differential equations with injective symbols are natural analogues of the Martinelli-Bochner integral.

Within this more general context in the present paper the possibilities to prove a similar result to the theorem of Romanov is discussed. In particular, an answer to this question is provided for matrix factorizations of the Laplace operator in $\boldsymbol{R}^n$ and the Green integrals (associated to these systems), which are

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constructed (like the Martinelli-Bochner integral) by means of the standard fundamental solution of the Laplace operator.

1. Let $X \subset \mathbf{R}^n$ be an open set, $E = X \times \mathbf{C}^k$ and $F = X \times \mathbf{C}^l$ be (trivial) vector bundles over X, and $do_p(E \to F)$ be the vector space of all differential operators of type $(E \to F)$ and order $p \geqslant 1$. Sections of E and F of a class $\mathfrak{C}$ on an open set $\sigma \subset X$ can be interpreted as columns of functions from $\mathfrak{C}(\sigma)$, that is, $\mathfrak{C}(E|_\sigma) \cong [\mathfrak{C}(\sigma)]^k$, and similarly for F. Then for $P \in do_p(E \to F)$: $P(x, D) = \sum_{|\alpha| \leqslant p} P_\alpha(x) D^\alpha$ where $P_\alpha(x)$ are $(l \times k)$ matrices of smooth functions on X.

Let E^* be the conjugate bundle of E, and let $(\cdot, \cdot)_x$ be a hermitian metric on E. Then $*_E : E \to E^*$ is defined by $\langle *_E g, f \rangle_x = (f, g)_x$ (where f, g are sections of E and $\langle \cdot, \cdot \rangle_x$ is the natural pairing of E and E^*). Let Λ^q be the bundle of all complex valued exterior forms of degree q $(q = 1, 2, \dots)$ over X, and let dx be the volume form on X.

As usual, $P' \in do_p(F^* \to E^*)$ is the transposed operator, and $P^* = (*_{\bar{E}}^{-1} P' *_F) \in$ $\in do_p(F \to E)$ is the (formal) adjoint operator for $P \in do_p(E \to F)$.

Definition 1.1. A bidifferential operator $G_P(\cdot, \cdot) \in do_{p-1}((F^*, E) \to \Lambda^{n-1})$ is said to be a Green operator for $P \in do_p(E \to F)$ if for any open subset $U \subset X$, and any $g \in \mathcal{E}(F^*|_U)$, $f \in \mathcal{E}(E|_U)$ the following formula holds: $dG_P(g, f) = \langle g, Pf \rangle_x dx -$ $- \langle P'g, f \rangle_x dx$ $(x \in U)$ where $\mathcal{E}$ stands for the vector space of smooth sections.

A Green operator G_P can be written in the form (see [18, p. 82]):

$$G_P(g, f) = \sum_{|\beta + \gamma + 1_j| \leqslant p}' (-1)^\beta D^\beta (g P_{\beta + \gamma + 1_j}) D^\gamma f * dx_j .$$

For purposes of this paper it is convenient to write Green operators in another form.

Let D be a bounded domain in X with a boundary of class C^{p-1} of $p > 1$ (if $p = 1$, $\partial D \in C^1$), and let U be a neighbourhood of ∂D in X, and $F_j = U \times C^k$ $(0 \leqslant j \leqslant r < \infty)$ be (trivial) vector bundles over U. Fix a Dirichlet system $\{B_j\}_{j=0}^{p-1}$ of order $(p-1)$ on ∂D such that $b_j = j$ (for example, the system of the normal derivatives $\{\partial^j / \partial n^j\}_{j=0}^{p-1}$ with respect to ∂D). The following lemma was established in [19, p. 280, Lemma 28.3].

Lemma 1.2. *Let the hypersurface ∂D be non characteristic for $P \in do_p(E \to F)$. Then, if the neighbourhood U of ∂D, is sufficiently small, there exists a Green operator G_P such that*

$$G_P(g, f) = \sum_{j=0}^{p-1} \langle C_j g, B_j f \rangle_x \, ds + \frac{d\rho}{|d\rho|} \Lambda G_\nu(g, f)$$

where $\{C_j\}_{j=0}^{p-1}$ is a Dirichlet system of order $(p-1)$ on ∂D, $C_j \in do_{p-j-1}(F^*|_U \to F_j^*)$ $(0 \leqslant j \leqslant p-1)$, $G_\nu \in do_{p-1}((F^*, E)|_U \to \Lambda^{n-2})$, $g \in \mathcal{E}(F^*|_U)$, $f \in \mathcal{E}(E|_U)$.

If $W^{m,2}(E|_D)$ stands for the Sobolev space, denote by $S_P^{m,2}(D)$ the Hilbert space of all $W^{m,2}(E|_D)$-functions satisfying $Pf = 0$ in D.

Denote by Δ the differential operator $P^* P \in do_{2p}(E \to E)$, and assume that Δ is an elliptic operator which has a bilateral fundamental solution Φ on X. Then the symbol $\sigma(P)$ is necessarily injective, and P has the (left) fundamental solution $\mathfrak{J}(x, y) = P^{*\prime}(y, D)\Phi(x, y)$ on X. Set for $f \in W^{m,2}(E|_D)(m \geq p)$, $g \in W^{\tau,2}(F_{|D})(\tau \geq 0)$

$$
(1.1) \qquad \begin{cases} (Mf)(x) = -\displaystyle\int_{\partial D} G_P(P^{*\prime}(y, D)\Phi(x, y), f(y)), \\ (Tg)(x) = \displaystyle\int_D \langle P^{*\prime}(y, D)\Phi(x, y), g(y)\rangle_y dy . \end{cases}
$$

THEOREM 1.3. *If $m \geq p$, for any $f \in W^{m,2}(E|_D)$ the following formula holds:*

$$
(1.2) \qquad (Mf)(x) + (TPf)(x) = \begin{cases} f(x) & x \in D, \\ 0 & x \in X \backslash \overline{D} . \end{cases}
$$

PROOF. If $f \in C^p(E|_{\overline{D}})$ (that is, f is p times continuously differentiable in a neighbourhood of $\overline{D}$) then (1.2) follows from the Stokes' formula and the definition of Green operators. Since the boundary of D is sufficiently smooth, there exists a sequence of functions $\{f_N\}_{N=1}^{\infty} \subset C^p(E|_{\overline{D}})$ approximating f in $W^{m,2}(E|_D)$. Then for any number $N \in \mathbf{N}$

$$
(1.3) \qquad \int_D \langle \mathfrak{J}(x, y), Pf_N(y)\rangle_y dy - \int_{\partial D} G_P(\mathfrak{J}(x, y), f_N(y)) = \begin{cases} f_N(x) & x \in D, \\ 0 & x \in X \backslash \overline{D} . \end{cases}
$$

On the other hand, since the derivatives $D^\alpha f(|\alpha| \leq p-1)$ have natural boundary values $D^\alpha f|_{\partial D} \in W^{m-|\alpha|-1/2,2}(E_{|\partial D})$ (see [4, p.120]), it is easy to see from [18, Proposition 9.4] that the boundary integral in (1.2) does not depend on the choice of the Green operator G_P. Therefore choosing as G_P the Green operator provided by Lemma 1.2, and using the boundedness theorem for potential (co-boundary) operators on a manifold with boundary [13, pp. 161-165] one can conclude that the boundary integral in (1.2) defines a bounded linear operator from $W^{m,2}(E|_D)$ to $W^{m,2}(E|_D)$. Thus, to obtain (1.2) it suffices to make the limit passage for $N \to \infty$ in (1.3). □

REMARK 1.4. The boundary integral in the left hand side of (1.2) does not depend on the choice of the Green operator G_P.

PROPOSITION 1.5. *The integrals M and TP define linear bounded operators from $W^{m,2}(E|_D)$ to $W^{m,2}(E|_D)$ $(m \geq p)$.*

PROOF. Since M: $W^{m,2}(E|_D) \to W^{m,2}(E|_D)$ is bounded (this was shown while proving Theorem 1.3) then (1.2) implies that TP: $W^{m,2}(E|_D) \to W^{m,2}(E|_D)$ is bounded. □

This proposition implies that it is possible to consider iterations of the integrals M and TP in the Sobolev spaces $W^{m,2}(E|_D)$ $(m \geq p)$.

2. Let $\widetilde{M}$ and $\widetilde{TP}$ be the restrictions to the Hilbert space $S_{\Delta}^{m,2}(D)$ of the operators M and TP. Formula (1.2) and Proposition 1.5 imply that $\widetilde{M}\colon S_{\Delta}^{m,2}(D) \to S_{\Delta}^{m,2}(D)$, and $\widetilde{TP}\colon S_{\Delta}^{m,2}(D) \to S_{\Delta}^{m,2}(D)$, are linear bounded operators. Therefore the iterations of $\widetilde{M}$ and $\widetilde{TP}$ in $S_{\Delta}^{m,2}(D)$ are well defined.

To prove theorem on iterations it is sufficient to construct in the Hilbert space $S_{\Delta}^{m,2}(D)$ a scalar product $H_m^P(\cdot,\cdot)$ with the following properties:

(*I*) *For any* $f \in S_{\Delta}^{m,2}(D)$: $H_m^P(\widetilde{M}f, f) \geq 0$, $H_m^P(\widetilde{TP}f, f) \geq 0$.

(*II*) *The topologies induced in* $S_{\Delta}^{m,2}(D)$ *by* $H_m^P(\cdot,\cdot)$ *and by the standard scalar product of* $W^{m,2}(E|_D)$ *are equivalent.*

In section 3 we shall construct a scalar product, for which (*I*), (*II*) hold, for matrix factorizations of the Laplace operator.

Since the kernels $\ker \widetilde{M}$ and $\ker \widetilde{TP}$ of the operators $\widetilde{M}$ and $\widetilde{TP}$ are closed subspaces of $W^{m,2}(E|_D)$, they are Hilbert spaces (with the hermitian structure induced from $W^{m,2}(E|_D)$). Let $\Pi(S)$ be the orthogonal (with respect to $H_m^P(\cdot,\cdot)$) projection from $S_{\Delta}^{m,2}(D)$ to S, where S is a closed subspace of $S_{\Delta}^{m,2}(D)$.

THEOREM 2.1. *If in the space* $S_{\Delta}^{m,2}(D)$ *there exists a scalar product* $H_m^P(\cdot,\cdot)$ *for which* (*I*) *and* (*II*) *hold then*

$$\lim_{\nu \to \infty} \widetilde{M}^{\nu} = \Pi(\ker \widetilde{TP}), \qquad \lim_{\nu \to \infty} (\widetilde{TP})^{\nu} = \Pi(\ker \widetilde{M})$$

for the strong operator topology in $W^{m,2}(E|_D)$.

PROOF. Since $S_{\Delta}^{m,2}(D)$ is a complex (!) Hilbert space, (*I*) and (1.2) imply that the operators $\widetilde{M}$ and $\widetilde{TP}$ are self-adjoint in $S_{\Delta}^{m,2}(D)$ with respect to the scalar product $H_m^P(\cdot,\cdot)$, and that $0 \leq \widetilde{M} \leq Id$, $0 \leq \widetilde{TP} \leq Id$ (where Id stands for the identity operator on $W^{m,2}(E|_D)$.

On the other hand, (*II*) garantees that the space $S_{\Delta}^{m,2}(D)$ with the scalar product $H_m^P(\cdot,\cdot)$ is a Hilbert space too. At this point the spectral theorem for bounded self-adjoint operators yields

$$\widetilde{M}^{\nu} = \int_0^1 \lambda^{\nu} dE_{\lambda}, \quad (\widetilde{TP})^{\nu} = \int_0^1 (1-\lambda)^{\nu} dE_{\lambda} \tag{2.1}$$

where $\{E_{\lambda}\}_{0 \leq \lambda \leq 1}$ is a resolution of the identity in the Hilbert space $S_{\Delta}^{m,2}(D)$ corresponding to the operator $\widetilde{M}$ and the scalar product $H_m^P(\cdot,\cdot)$.

Passing to the limit in (2.1) one obtains

$$\lim_{\nu \to \infty} \widetilde{M}^{\nu} = \widetilde{E}_1, \qquad \lim_{\nu \to \infty} (\widetilde{TP})^{\nu} = \widetilde{E}_0$$

where $\widetilde{E}_0$, $\widetilde{E}_1$ are the orthogonal projections from $S_{\Delta}^{m,2}(D)$ onto the subspaces $V(0)$, $V(1)$ corresponding to the eigenvalues 0 and 1 for the operator $\widetilde{M}$. Finally, (1.2) implies that $V(0) = \ker \widetilde{M}$, $V(1) = \ker \widetilde{TP}$, which was to be proved. $\square$

LEMMA 2.2. *For any* $f \in W^{m,2}(E|_D)$ *there exists a (unique) function* $\psi(f) \in S^{m,2}_{\Delta}(D)$ *such that* $\partial^j \psi(f)/\partial n^j = \partial^j f/\partial n^j$ *on* ∂D $(0 \leq j \leq p-1)$ *where* $\{\partial^j/\partial n^j\}_{j=0}^{p-1}$ *is the system of the normal derivatives with respect to* ∂D.

PROOF. In fact we need to prove that the Dirichlet problem for the operator Δ, the domain D, and the Dirichlet data $\{\partial^j f/\partial n^j|_{\partial D}\}_{j=0}^{p-1}$ is solvable in $W^{m,2}(E|_D)$.

Let $\psi \in S^{m,2}_{\Delta}(D)$ be such that $\partial^j \psi/\partial n^j = 0$ on ∂D $(0 \leq j \leq p-1)$. Then using Stokes' formula and Definition 1.1 one can see that

$$0 = \int_D (\psi, \Delta\psi)_x \, dx = \int_D (P\psi, P\psi)_x \, dx - \int_{\partial D} G_P(*_F P\psi, \psi) =$$

$$= \int_D (P\psi, P\psi)_x \, dx - \int_{\partial D} \sum_{j=0}^{p-1} \left\langle C_j *_F P\psi, \frac{\partial^j \psi}{\partial n^j} \right\rangle_x ds = \int_D (P\psi, P\psi)_x \, dx .$$

Hence $\psi(f) \in S^{m,2}_P(D)$. Since the operator M does not depend on the choice of the Green operator G_P, without loss of generality we can set $B_j = \partial^j/\partial n^j$. Then Theorem 1.3 implies that $\psi(f) = M\psi(f) = 0$ in the domain D. That is, if there exists a solution of the above Dirichlet problem, the solution is unique.

On the other hand, since $\Delta^* = (P^*P)^* = \Delta$, and for $f \in W^{m,2}(E|_D)$ $(m \geq p)$ the restrictions to ∂D of the derivatives $\partial^j f/\partial n^j$ $(0 \leq j \leq p-1)$ belong the spaces $W^{m-j-1/2,2}(E|_{\partial D})$ respectively (see [4, p. 120]), the results of Lions and Magenes [12] (see also [16, pp. 136-137]) imply that the Dirichlet problem is solvable. □

Let $N^{m,2}(D)$ be the closed subspace of $W^{m,2}(E|_D)$ consisting of functions $f \in W^{m,2}(E|_D)$ such that $D^\alpha f|_{\partial D} = 0$ for all $|\alpha| \leq p-1$. It is clear that the difference $(f - \psi(f))$ belongs to $N^{m,2}(D)$. Then Lemma 2.2 implies the direct sum decomposition $W^{m,2}(E|_D) = S^{m,2}_{\Delta}(D) \oplus N^{m,2}(D)$. Denote by $\tilde{\Pi}(S^{m,2}_{\Delta}(D))$ and $\tilde{\Pi}(N^{m,2}(D))$ the (bounded) projectors corresponding to this decomposition (*i.e.* $\psi(f) = \tilde{\Pi}(S^{m,2}_{\Delta}(D))f$ and $(f - \psi(f)) = \tilde{\Pi}(N^{m,2}(D))f$).

COROLLARY 2.3. *Under the hypotheses of Theorem 2.1,*

$$\begin{cases} \lim\limits_{\nu \to \infty} M^\nu = \Pi(\ker \widetilde{TP})\, \tilde{\Pi}(S^{m,2}_{\Delta}(D)), \\ \lim\limits_{\nu \to \infty} (TP)^\nu = \Pi(\ker \tilde{M})\, \tilde{\Pi}(S^{m,2}_{\Delta}(D)) + \tilde{\Pi}(N^{m,2}(D)) \end{cases} \tag{2.2}$$

for the strong operator topology in $W^{m,2}(E|_D)$.

PROOF. By Lemma 2.2, for any $f \in W^{m,2}(E|_D)$ there exists a function $\psi(f) \in S^{m,2}_{\Delta}(D)$ such that $Mf = M\psi(f)$. It is clear that, if $\tilde{\psi} \in S^{m,2}_{\Delta}(D)$ such that $Mf = M\tilde{\psi}$ then $M\psi(f) = M\tilde{\psi}$, and $\lim\limits_{\nu \to \infty} \tilde{M}^\nu \tilde{\psi} = \lim\limits_{\nu \to \infty} \tilde{M}^\nu \psi(f)$. Therefore there exists the limit of the iterations $(\lim\limits_{\nu \to \infty} M^\nu f)$, and in the $W^{m,2}(E|_D)$-norm $\lim\limits_{\nu \to \infty} M^\nu f = \lim\limits_{\nu \to \infty} \tilde{M}^\nu \psi(f) = \Pi(\ker \widetilde{TP})\, \psi(f)$ for any f.

It is clear that $N^{m,2}(D) \subset \ker M$, therefore, for any $f \in W^{m,2}(E|_D)$

$$(2.3) \qquad (TP)^\nu f = (TP)^\nu (f - \psi(f)) + (TP)^\nu \psi(f) =$$
$$= (I - M)^\nu (f - \psi(f)) + (\widetilde{TP})^\nu \psi(f) = (f - \psi(f)) + (\widetilde{TP})^\nu \psi(f).$$

Passing to the limit in (2.3) we conclude that (2.2) holds. □

COROLLARY 2.4. *Under the hypotheses of Theorem 2.1, for any $f \in W^{m,2}(E|_D)$ ($m \geqslant p$) the following formula holds:*

$$(2.4) \qquad f = \lim_{\nu \to \infty} M^\nu f + \sum_{\mu=0}^{\infty} M^\mu (TPf)$$

where the series converges in the $W^{m,2}(E|_D)$-norm.

PROOF. Formula (1.2) implies that for any $\nu \in \boldsymbol{N}$

$$(2.5) \qquad f = M^\nu f + \sum_{\mu=0}^{\nu-1} M^\mu (TPf).$$

Passing to the limit in (2.5), and using Corollary 2.3 one sees that (2.4) holds. □

Let the set A consist of all distributions $u \in D'(F|_D)$ such that $(Tu) \in W^{m,2}(E|_D)$ and the series $\sum_{\mu=0}^{\infty} M^\mu Tu$ converges in $W^{m,2}(E|_D)$-norm. In the following proposition R stands for the operator $\left(\sum_{\mu=0}^{\infty} M^\mu T\right): A \to W^{m,2}(E|_D)$, and $R(A)$ indicates the range of A by the mapping R.

PROPOSITION 2.5. *Under hypotheses of Theorem 2.1, $R(A) = N^{m,2}(D) \oplus (\ker \widetilde{TP})^\perp$ where $(\ker \widetilde{TP})^\perp$ is the orthogonal (with respect to $H^P_m(\cdot,\cdot)$!) complement of $\ker \widetilde{TP}$ in $S^{m,2}_\Delta(D)$.*

PROOF. If $u \in A$ then $Ru \in W^{m,2}(E|_D)$, and, since M is continuous (see Proposition 1.5), $MRu = M \lim_{\nu\to\infty} \sum_{\mu=0}^{\nu} M^\mu Tu = Ru - Tu$. Therefore $M^\nu Ru = Ru - \sum_{\mu=0}^{\nu-1} M^\mu Tu$. Passing to the limit in the last equality and using Corollary 2.3 one sees that $\Pi(\ker \widetilde{TP})\tilde{\Pi}(S^{m,2}_\Delta(D))Ru = \lim_{\nu\to\infty} M^\nu Ru = Ru - Ru = 0$. Hence $R(A) \subset N^{m,2}(D) \oplus \oplus (\ker \widetilde{TP})^\perp$.

Conversely, if $f \in N^{m,2}(D) \oplus (\ker \widetilde{TP})^\perp$ then (2.4) implies that $f = RPf$. However, by Proposition 1.5, $Pf \in A$. Therefore $N^{m,2}(D) \oplus (\ker \widetilde{TP})^\perp \subset R(A)$. □

PROPOSITION 2.6. *Under the hypotheses of Theorem 2.1, $\ker R = 0$ if and only if $PR = Id$, if and only if $\ker T|_A = 0$.*

3. Let $X = \boldsymbol{R}^n$ ($n \geqslant 3$), $E = \boldsymbol{R}^n \times \boldsymbol{C}^i$, $F = \boldsymbol{R}^n \times \boldsymbol{C}^l$ ($l \geqslant i$) and $(\cdot,\cdot)_x$ is the usual hermitian metric on E.

DEFINITION 3.1. A differential operator $P \in do_1(E \to F)$ is said to be a matrix factorization of the Laplace operator in $\boldsymbol{R}^n$ if $\Delta = P^* P = -\Delta_n I_i$ where I_i is the identity $(i \times i)$-matrix, and Δ_n stands for the Laplace operator in $\boldsymbol{R}^n$.

Let P be a matrix factorization of the Laplace operator in $\boldsymbol{R}^n$, $\varphi_n(x-y)$ be the standard fundamental solution of the Laplace operator in $\boldsymbol{R}^n$, $D \Subset \boldsymbol{R}^n$, and $\partial D \in C^\infty$.

Denote by $S_\Delta^{m,2}(\boldsymbol{R}^n \setminus \overline{D})$ the closed subspace of $W^{m,2}(E|_{\boldsymbol{R}^n \setminus D})$ consisting of all (vector-valued) functions f which are harmonic in $\boldsymbol{R}^n \setminus \overline{D}$ and for which $\lim\limits_{|x| \to \infty} f(x) = 0$. Then the restriction operators from $S_\Delta^{m,2}(D)$ and $S_\Delta^{m,2}(\boldsymbol{R}^n \setminus \overline{D})$ onto $W^{m-1/2,2}(E|_{\partial D})$ $(m \geqslant 1)$ are linear topological isomorphisms (see [4, p. 126]). In particular, since the Dirichlet problem (see proof of Lemma 2.2) is solvable, for any $f \in W^{m,2}(E|_D)$ there exists a (unique) function $S(f) \in S_\Delta^{m,2}(\boldsymbol{R}^n \setminus \overline{D})$ such that $S(f)|_{\partial D} = f|_{\partial D}$. Consider now, for $f, g \in W^{m,2}(E|_D)$ $(m \geqslant 1)$, the hermitian form

$$H_1^P(f,g) = \int_D (Pf, Pg)_x \, dx + \int_{\boldsymbol{R}^n \setminus D} (PS(f), PS(g))_x \, dx . \tag{3.1}$$

PROPOSITION 3.2. *The hermitian form* (3.1) *defines a scalar product in* $W^{m,2}(E|_D)$.

PROOF. Since $P^* P = -\Delta_n I_i$, the coefficients of P are bounded, and, therefore, $PS(f) \in W^{m-1,2}(E|_{\boldsymbol{R}^n \setminus D})$. Then, since $(\cdot,\cdot)_x$ is a hermitian metric, to prove the statement it is sufficient to prove that $H_1^P(f,f) = 0$ implies $f \equiv 0$ in D.

If $H_1^P(f,f) = 0$ then $f \in S_P^{m,2}(D)$ and $S(f) \in S_P^{m,2}(\boldsymbol{R}^n \setminus \overline{D})$, and, by definition $f|_{\partial D} = S(f)|_{\partial D}$. Then Theorem 3.2 of [19] implies that there exists a section $\mathcal{F} \in S_P(\boldsymbol{R}^n)$ such that $\mathcal{F}|_D = f$, $\mathcal{F}|_{\boldsymbol{R}^n \setminus \overline{D}} = S(f)$. It is clear that $\mathcal{F}$ is a harmonic in $\boldsymbol{R}^n$ (vector-valued) function for which $\lim\limits_{|x| \to \infty} \mathcal{F}(x) = 0$. Therefore $\mathcal{F} \equiv 0$ in $\boldsymbol{R}^n$, and $f \equiv 0$ in D. □

In the following two lemmata the operator $P \in do_p(E \to F)$ satisfies the assumption of the first section (see (1.2))

LEMMA 3.3. *For any* $f \in S_\Delta^{m,2}(D)$ $(m \geqslant p)$,

$$(\widetilde{T}Pf)(x) = \sum_{j=0}^{p-1} \int_{\partial D} \langle (*_{F_j} B_j *_E^{-1}) \Phi(x,y), (*_{F_j}^{-1} C_j *_F) Pf \rangle_y \, ds (x \in X \setminus \partial D).$$

PROOF. Since the symbol $\sigma(P)$ is injective and $C_j \in do_{p-1-j}(F^* | U \to F_j^*)$ $(0 \leqslant \leqslant j \leqslant p-1)$, the boundary system $\{B_j, (*_{F_j}^{-1} C_j *_F) P\}_{j=0}^{p-1}$ is a Dirichlet system of order $(2p-1)$. Then Theorem 4.4 of [17] garanties that for any $f \in S_\Delta^{m,2}(D)$ there exist (weak) boundary values $((*_{F_j}^{-1} C_j *_F) Pf)|_{\partial D} \in W^{m+j-2p+1/2,2}(F_j|_{\partial D})$.

On the other hand, Lemma 1.2 implies that

$$\sum_{j=0}^{p-1} \int_{\partial D} \langle (*_{F_j} B_j *_E^{-1}) \Phi(x,y), (*_{F_j}^{-1} C_j *_F Pf) \rangle_y \, ds = \tag{3.2}$$

$$= \overline{\sum_{j=0}^{p-1} \int_{\partial D} \langle (C_j *_F Pf), B_j *_E^{-1} \Phi(x,y) \rangle_y \, ds} = \overline{\int_{\partial D} G_P(*_F Pf, *_E^{-1} \Phi)} .$$

Then Stokes' formula and Definition 1.1 yield the conclusion.

REMARK 3.4. If P is a matrix factorization of the Laplace operator in $\boldsymbol{R}^n$, and if $\Phi = I_i \varphi_n$ then $\widetilde{TP}$ is a single layer potential.

For $\psi \in W^{m+j-2p+1/2,2}(F_j|_{\partial D})(0 \leqslant j \leqslant p-1)$ denote by $\mathcal{G}(\oplus\psi_j)$ the following integral:

$$\mathcal{G}(\oplus\psi_j)(x) = \sum_{j=0}^{p-1} \int_{\partial D} \langle (*_{F_j} B_j *_E^{-1}) \Phi(x,y), \psi_j \rangle_y \, ds (x \in X \backslash \partial D).$$

And let $\mathcal{G}(\oplus\psi_j)^- = \mathcal{G}(\oplus\psi_j)|_D$, $\mathcal{G}(\oplus\psi_j)^+ = \mathcal{G}(\oplus\psi_j)|_{X\backslash\overline{D}}$.

LEMMA 3.5. *Let $\{B_j\}_{j=0}^{2p-1}$ be an extension of the Dirichlet system $\{B_j\}_{j=0}^{p-1}$ to a Dirichlet system of order $(2p-1)$ an ∂D (as above $b_j = j$). Then*

$$(3.3) \qquad (B_j \mathcal{G}(\oplus\psi_j)^-)|_{\partial D} - (B_j \mathcal{G}(\oplus\psi_j)^+)|_{\partial D} = \begin{cases} 0, & 0 \leqslant j \leqslant p-1, \\ \psi_{2p-j-1}, & p \leqslant j \leqslant 2p-1. \end{cases}$$

PROOF. This follows from [17, Lemma 2.7]. □

PROPOSITION 3.6. *If P satisfies Definition 3.1, and if $\Phi = I_i \varphi_n$, then for any $f, g \in S_\Delta^{m,2}(D)$ $(m \geqslant 1)$*

$$H_1^P(\widetilde{M}f, g) = \int_{\boldsymbol{R}^n \backslash D} (PS(f), PS(g))_x \, dx, \qquad H_1^P(\widetilde{TP}f, g) = \int_D (Pf, Pg)_x \, dx.$$

PROOF. Let $B_0 = I_i$ and let C_0 be the Dirichlet boundary operator corresponding to B_0 by Lemma 1.2. Then, similarly to (3.2),

$$(3.4) \qquad \int_{\partial D} \langle *_{F_0} B_0 g, (*_{F_0}^{-1} C_0 *_F) Pf \rangle_x \, ds = \overline{\int_{\partial D} G_P(*_F Pf, g)} = \int_{\partial D} (Pf, Pg)_x \, dx.$$

Since $\lim_{|x| \to \infty} S(f)(x) = 0$ we obtain a similar formula for $S(f)$:

$$(3.5) \qquad \int_{\partial D} \langle (*_{F_0} B_0) S(g), (*_{F_0}^{-1} C_0 *_F) PS(f) \rangle_x \, ds = \int_{\boldsymbol{R}^n \backslash D} (PS(f), PS(g))_x \, dx.$$

By definition $f|_{\partial D} = S(f)|_{\partial D}$ then (3.4), (3.5) imply that

$$(3.6) \qquad H_1^P(f,g) = \int_{\partial D} \langle (*_{F_0} B_0) g, (*_{F_0}^{-1} C_0 *_F) Pf - (*_{F_0}^{-1} C_0 *_F) PS(f) \rangle_y \, ds.$$

Set $(\widetilde{TP}f)^+ = (\widetilde{TP}f)|_{\boldsymbol{R}^n \backslash \overline{D}}$, $(\widetilde{TP}f)^- = (\widetilde{TP}f)|_D$, and introduce similar notations for $\widetilde{M}f$. Since $\Phi = \varphi_n I_i$ then $n \geqslant 3$ and $(\widetilde{M}f)^+ \in W^{m,2}(E|_{\boldsymbol{R}^n \backslash D})$ and $\lim_{|x| \to \infty} (\widetilde{M}f)^+(x) = 0$. It easy to see from (1.2) that $(\widetilde{M}f)^+ = (\widetilde{TP}f)^+$, and therefore $(\widetilde{TP}f)^+ \in S_\Delta^{m,2}(\boldsymbol{R}^n \backslash \overline{D})$. Then Lemmata 3.3 and 3.5 imply that $(\widetilde{TP}f)^+ = (\widetilde{TP}f)^-$ on ∂D, that is, $(\widetilde{TP}f)^+ = S(\widetilde{TP}f)$.

Since $\{B_0, *_{F_0}^{-1} C_0 *_F P\}$ is a Dirichlet system of the first order then (3.6), (3.3) and (3.4) yield

$$(3.7)\qquad H_1^P(\widetilde{TP}f, g) = \int_{\partial D} \langle *_{F_0} B_0 g, *_{F_0}^{-1} C_0 *_F P(\widetilde{TP}f)^- - *_{F_0}^{-1} C_0 *_F P(\widetilde{TP}f)^+ \rangle_y \, ds =$$

$$= \int_{\partial D} \langle (*_{F_0} B_0 g), (*_{F_0}^{-1} C_0 *_F) Pf \rangle_y ds = \int_{\partial D} (Pf, Pg)_y \, dy .$$

Finally, (1.2), (3.7) and (3.5) imply that

$$H_1^P(\widetilde{M}f, g) = H_1^P(f - \widetilde{TP}f, g) = \int_{\mathbf{R}^n \setminus D} (PS(f), PS(g))_y \, dy . \qquad \square$$

For $f \in S_\Delta^{m,2}(D)$ and for P satisfying Definition 3.1 set

$$\widetilde{TP}\, S(f)(x) = \int_{\partial D} \langle \Phi(x,y), *_{F_0}^{-1} C_0 *_F) PS(f) \rangle_y \, ds ,$$

$$\widetilde{M}S(f)(x) = - \int_{\partial D} \langle C_0 P^{*\prime} \Phi(x,y), S(f) \rangle_y \, ds .$$

LEMMA 3.7. *If $\Phi = I_i \varphi_n$ then for any $f \in S_\Delta^{m,2}(D)$ $(m \geq 1)$*

$$(\widetilde{TP}f)(x) - (\widetilde{TP}\, S(f))(x) = \begin{cases} f(x), & x \in D, \\ S(f)(x), & x \in \mathbf{R}^n \setminus \overline{D}. \end{cases}$$

PROOF. First note that, as in the proof of Lemma 3.3, $(*_{F_0}^{-1} C_0 *_{F_0} PS(f))|_{\partial D} \in W^{m-3/2}(F_0|_{\partial D})$. Therefore the integrals $\widetilde{TP}\, S(f)(x)$, $\widetilde{M}S(f)(x)$ are well defined.

Proposition 9.5 of [19] implies that $\widetilde{TP}\, S(f) + \widetilde{M}S(f) = - \int_{\partial D} G_P(P^{*\prime}\Phi, S(f)) - \int_{\partial D} G_P(\Phi, PS(f)) = - \int_{\partial D} G_\Delta(\Phi, S(f))$. Hence by Stokes formula, Definition 1.1, and because $\lim_{|x| \to \infty} S(f)(x) = 0$ we obtain

$$(3.8)\qquad -(\widetilde{TP}\, S(f))(x) - (\widetilde{M}\, S(f))(x) = \begin{cases} 0, & x \in D, \\ S(f)(x), & x \in \mathbf{R}^n \setminus \overline{D}. \end{cases}$$

Since $f|_{\partial D} = S(f)|_{\partial D}$ by definition then $\widetilde{M}f = \widetilde{M}S(f)$. Now adding (1.2) and (3.8) we obtain the statement. $\square$

In the following lemma $G = \mathbf{R}^n \times \mathbf{C}^N$ $(N \geq i)$.

LEMMA 3.8. *If $P \in do_1(E \to F)$, $Q \in do_1(E \to G)$ are matrix factorizations of the Laplace operators in $\mathbf{R}^n$ then, for any $f, g \in S_\Delta^{m,2}(D)$ $(m \geq 1)$, $H_1^P(f,g) = H_1^Q(f,g)$.*

PROOF. If $\widetilde{C}_0$ is the boundary operator corresponding to $B_0 = I_i$ and Q by Lemma

1.2 then Lemmata 3.3, 3.5, and 3.7 imply that

$$(3.9)\quad (*_{F_0}^{-1}\widetilde{C}_0 *_G Qf)|_{\partial D} - (*_{F_0}^{-1}\widetilde{C}_0 *_G QS(f))|_{\partial D} =$$

$$= (*_{F_0}^{-1}\widetilde{C}_0 *_G Q[(\widetilde{TP}f)^- - (\widetilde{TP}S(f))^-])|_{\partial D} -$$

$$-(*_{F_0}^{-1}\widetilde{C}_0 *_G Q[(\widetilde{TP}f)^+ - (\widetilde{TP}S(f))^+])|_{\partial D} =$$

$$= (*_{F_0}^{-1}\widetilde{C}_0 *_G Q[(\widetilde{TP}f)^- - (\widetilde{TP}f)^+])|_{\partial D} -$$

$$-(*_{F_0}^{-1}\widetilde{C}_0 *_G Q[(\widetilde{TP}S(f))^- - (\widetilde{TP}S(f))^+])|_{\partial D} =$$

$$= (*_{F_0}^{-1}C_0 *_F Pf)|_{\partial D} - (*_{F_0}^{-1}C_0 *_F PS(f))|_{\partial D}\,.$$

Now (3.9) and (3.6) yield $H_1^P(f,g) = H_1^Q(f,g)$. □

PROPOSITION 3.9. *The topologies induced in $S_\Delta^{1,2}(D)$ by $H_1^P(\cdot,\cdot)$ and by the standard scalar product of $W^{1,2}(E|_D)$ are equivalent.*

PROOF. Since $P^*P = -\Delta_n I_i$, there are constants $c_1,\ c_2 > 0$ such that for any $f \in S_\Delta^{1,2}(D)$

$$(Pf, Pf)_x \leq c_1 \sum_{|\alpha|\leq 1} (D^\alpha f, D^\alpha f)_x,\qquad (PS(f), PS(f))_x \leq c_2 \sum_{|\alpha|\leq 1} (D^\alpha S(f), D^\alpha S(f))_x.$$

On the other hand, the topological isomorphisms between $S_\Delta^{1,2}(D)$, $S_\Delta^{1,2}(\boldsymbol{R}^n\backslash\overline{D})$, and $W^{1/2,2}(F_0|_{\partial D})$ (see [8, p. 126]) imply that there exists a constant c_3 such that, for any $f \in S_\Delta^{1,2}(D)$, $\|S(f)\|^2_{S_\Delta^{1,2}(\boldsymbol{R}^n\backslash\overline{D})} \leq c_3\|f\|^2_{S_\Delta^{1,2}(D)}$. Hence $H_1^P(f,f) \leq (c_1 + c_2 c_3)\|f\|^2_{S_\Delta^{1,2}(D)}$.

Conversely, since the gradient operator Gr in $\boldsymbol{R}^n$ is a factorization of the Laplace operator, Lemma 3.8 (with $Q = Gr \otimes I_i$) implies that for $f,\ g \in S_\Delta^{1,2}(D)$

$$H_1^P(f,g) = \sum_{|\alpha|=1} \int_D (D^\alpha f, D^\alpha g)_x dx + \sum_{|\alpha|=1} \int_{\boldsymbol{R}^n\backslash D} (D^\alpha S(f), D^\alpha S(g))_x\, dx\,.$$

Therefore one can conclude that

$$\sum_{|\alpha|=1} \int_{\boldsymbol{R}^n\backslash D} (D^\alpha S(f), D^\alpha S(f))_x\, dx \leq H_1^P(f,f),\qquad \int_{\boldsymbol{R}^n\backslash D} (S(f), S(f))_x\, dx \leq c_4 H_1^P(f,f),$$

where c_4 is a constant which does not depend on f.

Thus, to complete the proof it suffices to note that there is a constant $c_5 > 0$ such that, for any $f \in S_\Delta^{1,2}(D)$, $\|f\|_{S_\Delta^{1,2}(D)} \leq c_5\|S(f)\|_{S_\Delta^{1,2}(\boldsymbol{R}^n\backslash D)}$ (see [8, p.126]). □

THEOREM 3.10. *If P is a matrix factorization of the Laplace operator and if $\Phi = \varphi_n I_i$ then in the strong operator topology in $W^{1,2}(E|_D)$*

$$\lim_{\nu\to\infty} M^\nu = \Pi(S_P^{1,2}(D))\widetilde{\Pi}(S_\Delta^{1,2}(D)),$$

$$\lim_{\nu\to\infty} (TP)^\nu = \Pi(S_P^{1,2}(\boldsymbol{R}^n\backslash D))\widetilde{\Pi}(S_\Delta^{1,2}(D)) + \widetilde{\Pi}(N^{1,2}(D))\,.$$

PROOF. Propositions 3.6 and 3.9 imply that (*I*) and (*II*) hold for $H_1^P(\cdot,\cdot)$. Proposition 3.6 imply that $\ker \widetilde{TP} = S_P^{1,2}(D)$. Proposition 3.6 and Lemma 3.7 imply that $\widetilde{M}f =$

$= 0$ if and only if $S(f) \in S_P^{1,2}(\boldsymbol{R}^n \backslash D)$. Finally, since $S(f)|_{\partial D} = f|_{\partial D}$ by definition, $\ker \widetilde{M} = S_P^{1,2}(\boldsymbol{R}^n \backslash D)$. Hence the theorem follows from Corollary 2.3. □

4. We consider now some examples and applications.

EXAMPLE 4.1. In [15] A. V. Romanov obtained Theorem 3.10 for

$$P = 2\begin{pmatrix} \partial/\partial\bar{z}_1 \\ \cdots \\ \partial/\partial\bar{z}_n \end{pmatrix}$$

in $\boldsymbol{C}^n$ $(n \geq 2)$ $(P^*P = -\Delta_{2n})$. In this case, if ∂D is connected, the theorem on removable compact singularities of holomorphic functions implies that $S_P^{m,2}(\boldsymbol{C}^n \backslash \overline{D}) = \{0\}$ $(m \geq 1)$.

EXAMPLE 4.2. If P be the gradient operator in $\boldsymbol{R}^n$ $(n \geq 3)$ then $P^*P = -\Delta_n$ and, if ∂D is connected, $S_P^{m,2}(\boldsymbol{R}^n \backslash \overline{D}) = \{0\}$ $(m \geq 1)$.

EXAMPLE 4.3. Let $x \in \boldsymbol{R}^{4n}$ $(n \geq 1)$, $q_j = x_j + \sqrt{-1}x_{j+2n}$, $\partial/\partial q_j = (\partial/\partial x_j - \sqrt{-1}\,\partial/\partial x_{j+n})/2$, $\partial/\partial\bar{q}_j = (\partial/\partial x_j + \sqrt{-1}\,\partial/\partial x_{j+n})/2$ $(1 \leq j \leq 2n)$ and let

$$Q_j = 2\begin{pmatrix} \partial/\partial q_j & \partial/\partial q_{j+n} \\ -\partial/\partial\bar{q}_{j+n} & \partial/\partial\bar{q}_j \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 \\ \cdots \\ Q_n \end{pmatrix}.$$

Then $Q^*Q = -I_2\Delta_{4n}$. In this case, for $n = 1$, the operator $\widetilde{M}$ is already the orthogonal projector onto $S_Q^{m,2}(D)$ $(m \geq 1)$.

EXAMPLE 4.4. Let Λ^q be the bundle of (complex valued) exterior forms of degree q over $\boldsymbol{R}^n$ ($\Lambda^q \neq 0$ only for $0 \leq q \leq n$); let $d_q \in do_1(\Lambda^q \to \Lambda^{q+1})$ be the exterior derivative operator, and $d_q^* \in do_1(\Lambda^{q+1} \to \Lambda^q)$ be the formal adjoint operator of d_q. Then for the «laplacians» of the de Rham complex $(d_q^* d_q + d_{q-1} d_{q-1}^*) \in do_2(\Lambda^q \to \Lambda^q)$ we have $(d_q^* d_q + d_{q-1} d_{q-1}^*) = I_{i(q)}\Delta_n$ (see [18, p. 85]). Therefore the operators

$$P_q = \begin{pmatrix} d_q \\ d_{q-1}^* \end{pmatrix} \in do_1(\Lambda^q \to (\Lambda^{q+1}, \Lambda^{q-1}))$$

are matrix factorizations of the Laplace operator in $\boldsymbol{R}^n$. The space $S_{P_q^* P_q}^{m,2}(D)$ is the space of the differential forms of degree q whose coefficients are harmonic $W^{m,2}(E|_D)$-functions.

EXAMPLE 4.5. Let $\Lambda^{t,q}$ be the bundle of (complex valued) exterior forms of bidegree (t, q) over $\boldsymbol{C}^n$, $\Lambda^{t,q} \neq 0$ only for $0 \leq t \leq n$, $0 \leq q \leq n$. Let $\bar{\partial}_{t,q} \in do_1(\Lambda^{t,q} \to \Lambda^{t,q+1})$ be the Cauchy-Riemann operator extended to forms of bidegree (t, q), and let $\bar{\partial}_{t,q}^* \in do_1(\Lambda^{t,q+1} \to \Lambda^{t,q})$ be the formal adjoint operator of $\bar{\partial}_{t,q}$. Then for the «laplacians» of the Dolbeault complex $(\bar{\partial}_{t,q}^* \bar{\partial}_{t,q} + \bar{\partial}_{t,q-1} \bar{\partial}_{t,q-1}^*) \in do_2(\Lambda^{t,q} \to \Lambda^{t,q})$ we have

$4(\bar\partial^*_{t,q}d_q + \bar\partial_{t,q-1}\bar\partial^*_{t,q-1}) = I_{i(t,q)}\Delta_{2n}$ (see [18, p. 88]). Therefore the operators

$$P_{t,q} = 2\begin{pmatrix} \bar\partial_{t,q} \\ \bar\partial^*_{t,q-1} \end{pmatrix} \in do_1(\Lambda^{t,q} \to (\Lambda^{t,q+1}, \Lambda^{t,q-1}))$$

are matrix factorizations of the Laplace operator in $\boldsymbol{R}^{2n}$. The space $S^{m,2}_{P^*_{t,q}P_{t,q}}(D)$ is the space of the differential forms of bidegree (t,q) whose coefficients are harmonic $W^{m,2}(E|_D)$-functions.

Consider now some applications of Theorem 3.10. The following corollary in the case $P = \bar\partial$ in $\boldsymbol{C}^n$ was obtained by A. M. Kytmanov (see [11, p. 170]).

COROLLARY 4.6. *Let P be a matrix factorization of the Laplace operator in $\boldsymbol{R}^n$, $\Phi(x,y) = I_i\varphi_n(x-y)$, and $f \in W^{m,2}(E|_D)$ $(m \geqslant 1)$. Then $f \in S^{m,2}_P(D)$ if and only if $Mf = f$.*

PROOF. Theorem 1.3 implies that $Mf = f$ for $f \in S^{m,2}_P(D)$. Conversely, if $Mf = f$, then $f = \lim_{\nu\to\infty} M^\nu f$ in the $W^{m,2}(E|_D)$-norm. Then the statement follows from Theorem 3.10. □

Theorem 3.10 implies that, for any function $f \in W^{1,2}(E|_D)$ and any matrix factorization P of the Laplace operator in $\boldsymbol{R}^n$, decomposition (2.4) holds. For $P = \bar\partial$ this decomposition was obtained by A. V. Romanov [15]. In this case it is a higher dimensional analogue of the Cauchy-Green formula in the plane (see [8, 9]). Earlier some multi-dimensional analogues of the Cauchy-Green formula were obtained by constructing, for holomorphic functions, special integral representations with holomorphic kernels (see [2, 3, 7]).

Decomposition (2.4) has interesting application to elliptic differential complexes: in particular, to the de Rham and Dolbeault complexes.

In the following theorem D is a bounded domain in $\boldsymbol{C}^n$ $(n > 1)$, and $M_{t,q}$, $T_{t,q}$ are the integrals defined by (1.2) for $P = P_{t,q}$ and $\Phi = I_{i(t,q)}\varphi_{2n}$.

THEOREM 4.7. *Let D be a strictly pseudo-convex domain with a boundary $\partial D \in C^\infty$ (or a pseudo-convex domain with a real analytic boundary). Then for any $\bar\partial$-closed form $u \in W^{1,2}(\Lambda^{t,q+1}_{|D})$ the series*

$$f = 2\sum_{\mu=0}^{\infty} M^{\mu}_{t,q} T_{t,q}\begin{pmatrix} u \\ 0 \end{pmatrix}$$

converges in the $W^{1,2}(\Lambda^{t,q}_{|D})$-norm, and

$$\bar\partial_{t,q} f = u, \quad \bar\partial^*_{t,q-1} f = 0 \tag{4.1}$$

where $\begin{pmatrix} u \\ 0 \end{pmatrix} \in W^{1,2}((\Lambda^{t,q+1}|_D, \Lambda^{t,q-1}|_D))$.

PROOF. In view of the hypotheses on the domain D, results established in [5, 10] (see also [7]) imply that for any $\bar\partial$-closed form $u \in W^{1,2}(\Lambda^{t,q+1}|_D)$ there exists a unique solution $Nu \in W^{2,2}(\Lambda^{t,q+1}|_D)$ of the $\bar\partial$-Neumann problem, and

$\bar{\partial}_{t,q}(\bar{\partial}^*_{t,q}Nu) = u$ in D. It is clear that $(\bar{\partial}^*_{t,q}Nu) \in W^{1,2}(\Lambda^{t,q+1}|_D)$, and $P_{t,q}(\bar{\partial}^*_{t,q}Nu) = \begin{pmatrix} u \\ 0 \end{pmatrix}$. Then Corollary 2.4 implies that

$$(\bar{\partial}^*_{t,q}Nu) = \lim_{\nu \to \infty} M^{\nu}_{t,q}(\bar{\partial}^*_{t,q}Nu) + \sum_{\mu=0}^{\infty} M^{\mu}_{t,q} T_{t,q}\begin{pmatrix} u \\ 0 \end{pmatrix}$$

and the series f converges in the $W^{1,2}(\Lambda^{t,q}|_D)$-norm. Therefore

$$f/2 = (\bar{\partial}^*_{t,q}Nu) - \lim_{\nu \to \infty} M^{\nu}_{t,q}(\bar{\partial}^*_{t,q}Nu) \quad \text{and} \quad P_{t,q}f/2 = P_{t,q}(\bar{\partial}^*_{t,q}Nu) = \begin{pmatrix} u \\ 0 \end{pmatrix}. \qquad \square$$

Certainly, conditions on the domain D and the form u in Theorem 4.7 are sufficient. From the proof one can see that the statement holds if for the form $u \in W^{0,2}(\Lambda^{t,q+1}|_D)$ there exists a form $\mathcal{F} \in W^{1,2}(\Lambda^{t,q}|_D)$ such that $\bar{\partial}_{t,q}\mathcal{F} = u$, $\bar{\partial}^*_{t,q-1}\mathcal{F} = 0$.

REMARK 4.8. Proposition 2.5 implies that the series f is the unique solution of the $\bar{\partial}$-equation which belongs to $N^{1,2}(D) \oplus (S^{1,2}_{P_{t,q}}(D))^{\perp}$.

In the case when u is a $(0,1)$-form Theorem 4.7 was obtained by A. V. Romanov [15]. In this case the theorem holds for a pseudo-convex domain D with $\partial D \in C^{\infty}$.

Earlier explicit formulae for solutions of equation (0.2) (D is a strictly pseudo-convex domain with $\partial D \in C^2$ and u is a $(0,q)$-form with continuous in $\bar{D}$ coefficients) were obtained in [3, 8] (see also [7]).

In [6] the explicit formula for the operator N was obtained in the case where D is the unit open euclidean ball in $\mathbf{C}^n$.

Proposition 2.6 implies that convergence of the series f yields (4.1) if this series defines an injective operator from $W^{0,2}(\Lambda^{t,q+1}|_D)$ to $W^{1,2}(\Lambda^{t,q}|_D)$. Romanov (see [15, Theorem 3 and Lemma 5]) proved that, if D is a bounded domain in $\mathbf{C}^2$ with a connected boundary $\partial D \in C^1$ and $u \in W^{1,2}(\Lambda^{0,1}|_D)$ then the convergence of the series f implies $\bar{\partial}_{0,0}f = u$ in D.

Similar results can be stated for the de Rham complex and for a convex domain D.

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