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On glueing curves on surfaces and zero cycles

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Matematica. — *On glueing curves on surfaces and zero cycles.* Nota (*) di Hursit Önsiper, presentata dal Socio E. Vesentini.

Abstract. — The structure of the group $H^2(X, K_2)$ of a surface X with prescribed singularities is investigated.

Key words: Albanese variety; Algebraic group; Neron-Severi group; Lefschetz pencil.

Riassunto. — *Incollamento di curve su superfici e cicli nulli.* Si studia il gruppo $H^2(X, K_2)$ di una superficie X con singolarità assegnate.

This paper concerns the structure of $H^2(X, K_2)$ of surfaces with prescribed singularities. More precisely, for a projective smooth surface X' over $\mathbf{C}$ and an effective divisor m on X' we consider a pushout situation

$$\begin{array}{ccc} m & \longrightarrow & X' \\ \downarrow & & \downarrow \pi \\ S & \longrightarrow & X \end{array}$$

where $m \to S$ is a finite surjective map between reduced curves. Relevant to this situation we have two algebraic groups, one is the generalized albanese variety G_{um} of X' with modulus m [5] and the other is the 1-motive $J^2(X) = H^3(X, \mathbf{C})/F^2H^3(X, \mathbf{C}) + H^3(X, Z)$. Motivated by the analogy with the pushout of curves and by the results in [2] we study the relation between G_{um}, $J^2(X)$ and $H^2(X, K_2)$.

The rest of our notation is as follows: $C_m(X')$ is the idéle class group of X' with modulus m [3]; $H_c^\bullet(\,,\,)$ denotes cohomology with compact support; Ω_G^{inv} = the space of invariant differentials on the algebraic group G; $NS(\;)$ = Néron-Severi group. We first prove:

Proposition 1. (*i*) *We have a surjective homomorphism* $G_{um}(\mathbf{C}) \to J^2(X)$ *which is an isomorphism if* S *is integral.*

(*ii*) *If* $H^2(X', \mathcal{O}_{X'}) = 0$ *then* $J^2(X)$ *is an extension of* $Alb_{X'}$ *by a torus of dimension* $d = \operatorname{rank}(NS(m)/(NS(S) + NS(X')))$.

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PROOF. We let $U = X' - m = X - S$ and consider the diagram

$$(*)\qquad \begin{array}{ccccccccc} H^2(X,C) & \longrightarrow & H^2(S,C) & \longrightarrow & H_c^3(U,C) & \longrightarrow & H^3(X,C) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \downarrow \pi^* & & \\ H^2(X',C) & \longrightarrow & H^2(m,C) & \longrightarrow & H_c^3(U,C) & \longrightarrow & H^3(X',C) & \longrightarrow & 0 \end{array}$$

As the nonzero Hodge numbers h^{pq} of $H^3(X, C)$ satisfy $1 \leqslant p,\ q \leqslant 2$, we get successively $0 = F^2H^3(X,C) \cap W^0H^3(X,C) = F^2H^3(X,C) \cap W^1H^3(X,C) = F^2H^3(X,C) \cap W^2H^3(X,C)$. Therefore, since kernel $(\pi^*) = W^2H^3(X,C)$ we see that $F^2H^3(X,C) = F^2H^3(X',C) = H^1(X',\Omega^2)$. On the other hand the map $H_c^3(U,C) \to H^3(X',C)$ is the dual of $0 \to H^1(X',C) \to H^1(U,C)$ by Poincaré duality. This last sequence gives [1] $H^1(U,C) \approx \Omega_{G_{um}}^{\text{inv}} \oplus H^1(X',\mathcal{O}_{X'}) = \Omega_{G_{um}}^{\text{inv}} \oplus H^1(X',\Omega^2)$.

Therefore, we get $\Omega_{G_{um}}^{\text{inv}*} \to H^3(X,C)/F^2H^3(X,C) \to 0$ and taking quotients by $H_c^3(U,Z)$ and $H^3(X,Z)$ respectively using $H_c^3(U,Z) \to H^3(X,Z) \to 0$ and that $H_c^3(U,Z) \approx H_1(U,Z)$ we obtain the first half of statement (*i*).

For the second part of this statement we observe that when S is integral $H^2(S,C) \approx C$ and as X is projective $H^2(X,C) \to H^2(S,C)$ is not the zero map. Therefore $H_c^3(U,C) \approx H^3(X,C)$ and the result follows.

For statement (*ii*) we consider the exact sequence

$$0 \to \frac{H^2(m,C)}{H^2(S,C) + H^2(X',C)} \to \frac{H_c^3(U,C)}{H^2(S,C) + H^1(X',\mathcal{O}_{X'})} \to \frac{H^3(X',C)}{F^2H^3(X',C)} \to 0.$$

The first row of $(*)$ shows that

$$\frac{H_c^3(U,C)}{H^2(S,C) + H^1(X',\mathcal{O}_{X'})} \approx \frac{H^3(X,C)}{F^2H^3(X,C)}$$

and when $H^2(X',\mathcal{O}_{X'}) = 0$ the exponential sequence yields $\dim H^2(X',C) = \operatorname{rank}(NS(X'))$. As $\dim H^2(m,C)$ (resp. $\dim H^2(S,C)$) $= \operatorname{rank}(NS(m))$, (resp. $\operatorname{rank}(NS(S))$), this completes the proof. □

As to the relation of G_{um} and $J^2(X)$ to $H^2(X,K_2)$ we first recall that $G_{um}(C) \approx H^0(X',\Omega_{X'}(m))_{d=0}^*/H_1(U,Z)$ and fixing some $x_0 \in U$ the generalized albanese map $\alpha_{um}: U \to G_{um}(C)$ is given by

$$\alpha_{um}(x) = \left(\int_\gamma w_1, \ldots, \int_\gamma w_n\right)$$

modulo periods, for any path γ joining x_0 to x, where $w_1, \ldots, w_n$ is a basis for $H^0(X',\Omega_{X'}(m))_{d=0} =$ the differentials of the third kind with residues on m only [5, Corrigendum]. Extending by linearity we obtain a homomorphism

$$\alpha\colon \bigoplus_{x \in U} Z \to G_{um}(C).$$

Now we can prove

PROPOSITION 2. *α induces a surjective homomorphism $\tilde{\alpha}$: $H^2(X, K_2)_0 \to J^2(X)$ where $H^2(X, K_2)_0$ is the group of zero cycles of degree zero.*

PROOF. Since $\alpha_{um}(U)$ generates G_{um}, α is clearly surjective. Therefore, by the explicit description of $H^2(X, K_2)$ given in [4] it suffices to show that if $Y \subset X$ is a curve, no component of which is in S and if $f \in K(Y)$ is unit at each $p \in Y \cap S$, then $\alpha(\mathrm{div}(f)) \in \mathrm{Kernel}(G_{um}(C) \to J^2(X))$. To see this we must check that for $\mathrm{div}(f) = \sum n_x x$ and for each w_i we have

$$\sum n_x \int w_i \in \mathrm{Image}(H^2(S, C) \to H_c^3(U, C)).$$

This however is a consequence of the following standard calculation for which we may clearly assume that Y is integral.

Writing

$$S = \bigcup_{i=1}^{k} S_i, \qquad m = \bigcup_{i=1} \bigcup_{j=1} D_{ij}$$

as union of irreducible components where $\pi(D_{ij}) = S_i$ for $i = 1, \ldots, k$ we see that $H^2(S, C) \to H_c^3(U, C)$ is the composite map $H^2(S, C) \to H^2(m, C) \to H_c^3(U, C)$ where the second arrows is the dual of

$$\begin{aligned} \mathrm{res}\colon H^0(X', \Omega_{X'}(m))_{d=0} &\longrightarrow H^2(m, C) \approx \oplus_i \oplus_j H^2(D_{ij}, C) \\ w &\longrightarrow (\ldots, \mathrm{res}_{D_{ij}}(w), \ldots) \end{aligned}$$

and the first arrow is simply the diagonal map $H^2(S_i, C) \to \oplus_j H^2(D_{ij}, C)$ for each $i = 1, \ldots, k$. Therefore, $\mathrm{Image}(H^2(S, C) \to H_c^3(U, C))$ is spanned by $\theta_i \in H^0(X', \Omega_{X'}(m))^*_{d=0}$ where

$$\theta_i(w_l) = \sum_{j=1}^{r_i} \mathrm{res}_{D_{ij}}(w_l).$$

On the other hand letting $Y' \to \pi^{-1}(Y)$ be the normalization and φ: $Y' \to X$ be the composite map, we see that $\varphi^*(w_i)$ has poles only at $p \in \varphi^{-1}(Y \cap S)$ and these are simple poles.

Given $f \in k(Y)$, we consider ψ: $Y' \to P^1$ defined by $C(f) \subset k(Y)$. We choose a chain c in P^1 missing $\psi(\varphi^{-1}(Y \cap S))$ with $\partial c = 0 - \infty$, so that $\partial \psi^{-1}(c) = \psi^{-1}(\partial c) = \mathrm{div}(f)$. Since $\varphi^*(w_i)$ has simple poles and since for each

$$q \in P^1, \qquad \mathrm{res}_q(Tr(\varphi^*(w_i))) = \sum_{\psi(p)=q} \mathrm{res}_p(\varphi^*(w_i)),$$

the equality

$$\int_{\psi^{-1}(c)} \varphi^*(w) = \int_c Tr(\varphi^*(w))$$

completes the proof. □

Finally, via the following lemma we show that for any m and X' as above we can always realize $G_{um}(\boldsymbol{C})$ as a homomorphic image of $H^2(X, K_2)_0$ for a suitable pushout X.

Lemma 3. *Given any divisor m (not necessarily reduced) on X', we can find a Lefschetz pencil $X'' \xrightarrow{\alpha} P^1$ such that*

1) *m is flat over P^1 and G_{um} for $X'' = G_{um}$ for X',*

2) *the pushout*

$$\begin{array}{ccc} m & \xrightarrow{i} & X'' \\ \alpha\downarrow & & \downarrow \\ P^1 & \longrightarrow & X \end{array}$$

exists,

3) *for reduced m, we have a canonical surjective homomorphism*

$$H^2(X, K_2)_0 \to G_{um}(\boldsymbol{C}).$$

Proof. 1) Since $\dim(m) = 1$, we have a dense open set V in the Grassmannian $G_r(n-2, P^n)$ of the ambient projective space P^n such that for $L \in V$, $L \cap m = \phi$. We identify V with an open subset of $G_r(1, P^{n'})$ where $P^{n'}$ is the dual projective space, via the isomorphism

$$\begin{aligned} G_r(1, P^{n'}) &\longrightarrow G_r(n-2, P^n), \\ p = \{H_t\}_{t \in P^1} &\longmapsto \Delta_p = H_0 \cdot H_\infty . \end{aligned}$$

Then taking $p \in V \cap V_L$ (V_L is the open subset consisting of Lefschetz pencils) we get a pencil $X'' \xrightarrow{\alpha} P^1$, $f\colon X'' \to X'$ by blowing up $(\Delta_p \cap X') \subset X' - m$. Clearly $m \subset X''$ is flat over P^1

That G_{um} for $X'' = G_{um}$ for X' is a immediate consequence of the fact that $C_m(X'') = C_m(X')$ [3, Chapter II, Lemma 5].

2) We check that the hypothesis of [2, Proposition 4.3] is satisfied. To see this, for $t \in P^1$ we take $t' = 0$ or ∞, $t' \neq t$. As X'' is a blow-up of X' along $\Delta \cap X' \subset X' - m$, for $H_{t'}$ the hyperplane through t' we have $f^{-1}(X' - H_{t'}) \simeq X' - H_{t'}$ which is affine and clearly $i^{-1}(f^{-1}(X' - H_{t'})) = \alpha^{-1}(P^1 - t')$.

3) This follows from Proposition 1 (i) and Proposition 2. □

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