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Rigidity of holomorphic isometries


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Geometria. — Rigidity of holomorphic isometries. Nota(*) del Socio EDOARDO VESENTINI.

ABSTRACT. — A rigidity theorem for holomorphic families of holomorphic isometries acting on Cartan domains is proved.

KEY WORDS: Cartan factors; Carathéodory distance; Holomorphic isometry; Extreme point.

RIASSUNTO. — Rigidità di isometrie olomorfe. Si stabilisce un teorema di rigidità per famiglie di isometrie olomorfe in domini di Cartan.

1. Let $D$ and $D'$ be bounded domains in two complex Banach spaces $\mathcal{E}$ and $\mathcal{E}'$, and let $\text{Iso}(D, D')$ be the family of all holomorphic maps of $D$ into $D'$ which are isometries for the Carathéodory distances $c_D$ and $c_{D'}$, in $D$ and $D'$. Denoting by $A$ a domain in $\mathbb{C}$, let $f$ be a holomorphic map of $A \times D$ into $D'$. According to Lemma 2.3 of [6], if, for every pair points $x, y$ in $D$, there is $\zeta \in A$ such that $c_D(f(\zeta, x), f(\zeta, y)) = c_D(x, y)$, then $f(\zeta, \cdot) \in \text{Iso}(D, D')$ for all $\zeta \in A$. As a consequence, the following proposition holds:

**Proposition 1.** If there is a point $\zeta_0 \in A$ such that $f(\zeta_0, \cdot) \in \text{Iso}(D, D')$, then, $f(\zeta, \cdot) \in \text{Iso}(D, D')$ for all $\zeta \in A$.

Let $D = D'$ (in which case $\text{Iso}D$ will stand for $\text{Iso}(D, D')$) and let $\text{Aut}D \subset \text{Iso}D$ be the group of all holomorphic automorphisms of $D$. According to Proposition V.1.10 of [1], if $f(\zeta_0, \cdot) \in \text{Aut}D$ for some $\zeta_0 \in A$, then $f(\zeta, \cdot)$ is independent of $\zeta \in A$, i.e.

$$f(\zeta_0, \cdot) = f(\zeta, \cdot) \quad \text{for all } \zeta \in A. \quad (1)$$

Under which conditions on $D$ and $D'$ does this latter conclusion hold when $\text{Aut}D$ is replaced by $\text{Iso}(D, D')$?

It was shown in [9] that, if $D$ is the open unit ball $B$ of $\mathcal{E}$, and if $\mathcal{E}$ is a complex Hilbert space, the fact that $f(\zeta_0, \cdot) \in \text{Iso}B$ for some $\zeta_0 \in A$ implies (1).

Let $\mathcal{E}$ be the $C^*$ algebra $\mathcal{E} = \mathcal{L}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Starting from any infinite dimensional $\mathcal{H}$, an example was constructed in [5] of a non-trivial holomorphic family of holomorphic isometries of the open unit ball $B$ of $\mathcal{E}$, i.e. a holomorphic map $f: A \times B \rightarrow B$ such that $f(\zeta, \cdot) \in \text{Iso}B$ depends effectively on $\zeta$.

The $C^*$ algebra $\mathcal{L}(\mathcal{H})$ belongs to the class of $J^*$-algebras: in L. A. Harris’ termino-
logy [2], it is a special kind of Cartan factor of type one. It was also shown in [5] that the same conclusion holds when $\mathcal{E}$ is any infinite dimensional Cartan factor of type two or three.

The investigation will be pursued in this Note by considering all Cartan domains of type four and a class of Cartan domains of type one. It will be shown that – in contrast with the results established in [5] – no non-trivial holomorphic families of holomorphic isometries exist in these cases. More specifically, let $B$ and $B'$ be the open unit balls of $\mathcal{E}$ and $\mathcal{E}'$, and let $f \in \text{Hol}(A \times B, B')$ (the set of all holomorphic maps of $A \times B$ into $B'$) be such that $f(\zeta_0, \cdot) \in \text{Iso}(B, B')$ for some $\zeta_0 \in A$. The purpose of this Note is that of proving the following.

**Theorem.** If $\mathcal{E}$ and $\mathcal{E}'$ are both Cartan factors of type four, or if $\mathcal{E} = \mathcal{E}(X, X), \mathcal{E}' = \mathcal{E}(X, X')$, where $X, X$ and $X'$ are complex Hilbert spaces and $\dim_C X < \infty$, then $f$ is independent of $\zeta \in A$.

This theorem extends a similar result which was previously established by the author when $\mathcal{E} = \mathcal{E}'$ and $f(\zeta, \cdot)$ is a holomorphic isometry for all $\zeta \in A$. A similar question to the one posed at the beginning can be formulated in the case in which $D$ and $D'$ are hyperbolic domains and the Carathéodory distances are replaced by the Kobayashi distances. This question is obviously answered by the above theorem in the case when $D = B$, $D' = B'$, because then Carathéodory’s and Kobayashi’s distances coincide. If $\mathcal{E}'$ has finite dimension (and therefore $\dim_C \mathcal{E} \leq \dim_C \mathcal{E}'$) and if the domains $D$ and $D'$ are bounded, the same question can be posed in terms of the Bergman metrics on $D$ and $D'$. This question seems to be open, also in the particular case in which $D$ and $D'$ are the euclidean open unit balls of $\mathcal{E}$ and $\mathcal{E}'$.

2. This section will be devoted to some preliminaries. Let $A$ be a connected open neighbourhood of 0 in $\mathbb{C}$. If $f \in \text{Hol}(A \times B, B')$, for $\zeta \in A, X \in B$, $d_1 f(\zeta, X) \in \mathcal{E}'$ and $d_2 f(\zeta, X) \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ will indicate the partial Fréchet differentials of $f$ with respect to the first and the second variable, evaluated at the point $(\zeta, X)$.

Suppose that:

(i) $f(0, 0) = 0$;

(ii) $d_2 f(0, 0) \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ is a linear isometry of $\mathcal{E}$ onto a closed linear subspace $\mathcal{F}'$ of $\mathcal{E}'$;

(iii) there is a projector $P'$ in $\mathcal{E}'$ such that

$$P'(B') = B' \cap \mathcal{F}' .$$

Note that $\|P'\| \leq 1$.

As a consequence of (ii), there is a map $L \in \mathcal{L}(\mathcal{F}', \mathcal{E})$ which is a linear isometry of $\mathcal{F}'$ onto $\mathcal{E}$, for which $L \circ d_2 f(0, 0)$ is the identity on $\mathcal{E}$. Let $\tilde{P}' \in \mathcal{L}(\mathcal{E}', \mathcal{F}')$ be the map induced by $P'$, and let $g \in \text{Hol}(A \times B, B)$ be the map defined by $g = L \circ \tilde{P}' \circ f$. 

Then $d_2g(\zeta, X) = L \circ \bar{P}' \circ d_2 f(\zeta, X)$, and therefore $d_2g(0, 0) = L \circ \bar{P}' \circ d_2 f(0, 0) = I$ the identity on $E$. Thus, by H. Cartan's uniqueness theorem [1], $g(0, X) = X$ for all $X \in B$, and, by Proposition V.1.10 of [1] $g(\zeta, X)$ is independent of $\zeta \in A$, i.e.

(3) \hspace{1cm} g(\zeta, X) = X \hspace{1cm} \text{for all } X \in B \text{ and all } \zeta \in A.

Let $f(\zeta, X) = Q_0(\zeta) + Q_1(\zeta, X) + Q_2(\zeta, X) + \ldots$, be the power series expansion of $f(\zeta, \cdot)$ in $B$, where $Q_\nu(\zeta, \cdot)$ is a continuous homogeneous polynomial $E \to E'$ of degree $\nu = 0, 1, \ldots$, expressed, for $\zeta \in A, X \in B$, by the integral

(4) \hspace{1cm} Q_\nu(\zeta, X) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-i\nu \theta) f(\zeta, X) d\theta,

and where $Q_1(\zeta, X) = d_2 f(\zeta, 0) X$.

Equation (3) implies that, for all $\zeta \in A, X \in B$,

(5) \hspace{1cm} L \circ \bar{P}' \circ Q_1(\zeta, X) = X,

(6) \hspace{1cm} \bar{P}' \circ Q_\nu(\zeta, X) = 0 \hspace{1cm} \text{for } \nu = 0, 2, 3, \ldots.

Since, by (4), $\|Q_1(\zeta, \cdot)\| \leq 1$, (5) yields $\|X\| = \|L \circ \bar{P}' \circ Q_1(\zeta, X)\| \leq \|L\| \|P'\| \|Q_1(\zeta, X)\| \leq \|Q_1(\zeta, X)\| \leq \|X\|$, whence $\|Q_1(\zeta, X)\| = \|X\|$ for all $X \in E$.

Thus, $Q_1(\zeta, \cdot)$ is a linear isometry of $E$ into $E'$ for all $\zeta \in A$.

Example (3.1) of p. 301 of [5] shows that $Q_1(\zeta, \cdot)$ can depend on $\zeta \in A$. However, the following result holds.

Let $H$ and $H'$ be the sets of all real extreme points of the closures $\overline{B}$ and $\overline{B'}$ of $B$ and $B'$.

**Lemma 2.** If $f$ satisfies conditions (i)-(iii), if

(7) \hspace{1cm} d_2 f(0, 0) H \subset H',

and if $E$ is reflexive, then

(8) \hspace{1cm} Q_1(\zeta, \cdot) = d_2 f(0, 0) \hspace{1cm} \text{for all } \zeta \in A.

**Proof.** If $d_2 f(0, 0) Y$ is a complex extreme point of $\overline{B'}$, the strong maximum principle [1] yields $Q_1(\zeta, Y) = Q_1(0, Y) = d_2 f(0, 0) Y$ for all $\zeta \in A$. By (7), these equalities hold for all $Y \in H$. Let $X \in B$. For any continuous linear form $\lambda'$ on $E'$ and for any $\varepsilon > 0$, there is a finite convex combination $\sum a^i X_i$ of points $X_i \in H$ such that $|\lambda' \circ Q_1(\zeta, X - \sum a^i X_i)| < \varepsilon/2$, $|\lambda' \circ d_2 f(0, 0)(X - \sum a^i X_i)| < \varepsilon/2$.

Since $Q_1(\zeta, X_i) = d_2 f(0, 0) X_i$, then $|\lambda' \circ (Q_1(\zeta, X) - d_2 f(0, 0) X)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

The fact that $\lambda'$ and $\varepsilon$ are arbitrary, and the Hahn-Banach theorem, imply then that $Q_1(\zeta, X) = d_2 f(0, 0) X$ for all $\zeta \in A$ and all $X \in E$. Q.E.D.

3. If $\mathcal{X}$ and $\mathcal{X}'$ are two complex Hilbert spaces, the space $\mathcal{L}(\mathcal{X}, \mathcal{X}')$ of all bounded linear maps from $\mathcal{X}$ to $\mathcal{X}'$ is a complex Banach space with respect to the uniform operator norm $\|\|$.
It will be assumed henceforth that \( n = \dim \mathcal{K} < \infty \).

If \( e_1, \ldots, e_n \) is an orthonormal basis of \( \mathcal{X} \), for any \( X' \in \mathcal{L}(\mathcal{X}, \mathcal{X}') \) let \( X'_j = X' e_j \).

Then, for \( x = \sum_{j=1}^{n} a^j e_j \in \mathcal{X} \) (\( a^j \in \mathbb{C} \)), \( X' x = \sum_{j=1}^{n} a^j X'_j \), and, denoting by the same symbols (\( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \)) inner products and norms in \( \mathcal{X} \) and \( \mathcal{X}' \),

\[
\| X' x \|^2 = \sum_{j=1}^{n} |a^j|^2 \| X'_j \|^2 + 2 \Re \sum_{j<k}^{n} a^j a^k \langle X'_j, X'_k \rangle \leq \leq n \sum_{j=1}^{n} |a^j|^2 \| X'_j \|^2 \leq n (\Max\{ \| X'_j \| : j = 1, \ldots, n \})^2 \sum_{j=1}^{n} |a^j|^2 = n (\Max\{ \| X'_j \| : j = 1, \ldots, n \})^2 \| x \|^2,
\]

whence

(9) \( \| X' \| \leq \sqrt{n} \Max\{ \| X'_j \| : j = 1, \ldots, n \} \).

Let \( \tilde{X}' \) be the vector \((X'_1, \ldots, X'_n)\) in the Hilbert space direct sum \( \oplus_1^n \mathcal{X}' \) of \( n \) copies of \( \mathcal{X}' \). Then, by (9), the norm \( \| X' \| \) of \( X' \) is estimated by \( \| X' \|^2 \leq n \| \tilde{X}' \|^2 \).

Hence, the bijective linear map \( X' \to \tilde{X}' \) of \( \mathcal{L}(\mathcal{X}, \mathcal{X}') \) into \( \oplus_1^n \mathcal{X}' \) is bi-continuous.

That shows that, if \( \dim \mathcal{K} < \infty \), the Banach space \( \mathcal{L}(\mathcal{X}, \mathcal{X}') \) is reflexive.

Let \( \{ f'_\mu : \mu \in M \} \) be an orthonormal basis of \( \mathcal{X}' \), indexed by a set \( M \). Every \( X' \in \mathcal{L}(\mathcal{X}, \mathcal{X}') \) is expressed by

\[
X' = \sum_{\mu \in M} \left( \sum_{\nu = 1}^{n} (X'_\nu | f'_\mu)(f'_\mu \otimes e^*_\nu) \right),
\]

where the right-hand side (is summable and) converges to \( X' \) in the norm of \( \mathcal{L}(\mathcal{X}, \mathcal{X}') \) [7, Lemma 5]. For \( x \in \mathcal{X} \), \( f'_\mu \otimes e^*_\nu (x) = (x | e_\nu) f'_\mu \), and therefore

(10) \( \| X' x \|^2 = \sum_{\mu \in M} \left\| \sum_{\nu = 1}^{n} (X'_\nu | f'_\mu)(x | e_\nu) \right\|^2 \).

Let \( M_0 \) be a non-empty subset of \( M \) and let \( P' \) be the projector acting on \( \mathcal{L}(\mathcal{X}, \mathcal{X}') \), defined on \( X' \) by \( P' X' = \sum_{\mu \in M} \left( \sum_{\nu = 1}^{n} (X'_\nu | f'_\mu)(f'_\mu \otimes e^*_\nu) \right) \).

Since, by (10), \( \| P' X' x \| \leq \| X' x \| \) for all \( x \in \mathcal{X} \), then \( \| P' X' \| \leq \| X' \| \) for all \( X' \in \mathcal{L}(\mathcal{X}, \mathcal{X}') \) and therefore the norm \( \| P' \| \) of \( P' \) is

(11) \( \| P' \| \leq 1 \).

Furthermore, \( I - P' = 0 \) if \( M_0 = M \) while, if \( M_0 \neq M \), then

\[
(I - P') X' = \sum_{\mu \in M \setminus M_0} \left( \sum_{\nu = 1}^{n} (X'_\nu | f'_\mu)(f'_\mu \otimes e^*_\nu) \right),
\]

and, by the same argument as before, \( \| I - P' \| \leq 1 \).
For all $x \in \mathcal{X}$
\[(P'X'x | (I - P')X'x) = \sum_{\mu_1 \in M_0} \sum_{\mu_2 \in M \setminus M_0} \sum_{v_1, v_2 = 1}^n (X' e_{v_1} | f'_{\mu_1} (X' e_{v_2} | f'_{\mu_2}) \cdot (x | e_{v_1} (x | e_{v_2} (f'_{\mu_1} | f'_{\mu_2}) = 0\]
and therefore
\[(12) \quad \|X'x\|^2 = \|P'X'x\|^2 + \|(I - P')X'x\|^2 .\]

Let $\mathcal{E}$ be another complex Hilbert space and let $B$ and $B'$ be the open unit balls of $\mathcal{E} = \mathcal{E}(\mathcal{X}, \mathcal{X})$ and of $\mathcal{E}' = \mathcal{E}(\mathcal{X}, \mathcal{X}')$. If $f \in \text{Hol}(A \times B, B')$ is such that $f(\zeta_0, \cdot) \in \text{Iso}(B, B')$ for some $\zeta_0 \in A$, then $f(\zeta, \cdot) \in \text{Iso}(B, B')$ for all $\zeta \in A$, and, in particular, for $\zeta = 0 \in A$.

Since $B'$ is homogeneous [2], there is no restriction in assuming $f(0, 0) = 0$. Since the Carathéodory differential metric is the derivative of the Carathéodory distance (14); cf. also, e.g. [9]), and since the Carathéodory differential metrics of $B$ and $B'$ at the center 0 coincide with the norms in $\mathcal{E}$ and in $\mathcal{E}'$, then $d_2 f(0, 0)$ is a linear isometry of $\mathcal{E}$ into $\mathcal{E}'$. According to Theorem I of [7], there exists a unitary operator $V$ on $\mathcal{X}$ and a linear isometry $U$ of $\mathcal{X}$ into $\mathcal{X}'$ such that
\[(13) \quad d_2 f(0, 0) x = U \circ X \circ V \quad \text{for all } x \in \mathcal{E}(\mathcal{X}, \mathcal{X}').\]

Remark. Theorem I was established in [7] when $\mathcal{E} = \mathcal{E}'$, but the proof holds, with only purely formal changes, in the more general context considered here.

Given an orthonormal basis in $\mathcal{X}$, its image by $V$ is an orthonormal basis $\{e_1, \ldots, e_n\}$ in $\mathcal{X}$. On the other hand, the image by $U$ of an orthonormal basis in $\mathcal{E}$ is an orthonormal set in $\mathcal{E}'$, which, by a standard orthogonalization process, can be identified with a subset, $\{f'_\mu\}_{\mu \in M_0}$ of an orthonormal basis $\{f_\mu\}_{\mu \in M}$ of $\mathcal{E}'(M_0 \subset M)$. Since the closed linear span of $\{f'_\mu \otimes e_\nu^*: \nu = 1, \ldots, n; \mu \in M_0\}$ is the space $\mathcal{F}' = = d_2 f(0, 0) \mathcal{E}$, the above considerations show that there exists a projector $P'$ in $\mathcal{E}'$ with range $\mathcal{F}'$, satisfying (11) and therefore (2).

Hence, all the hypotheses of Lemma 2 are satisfied, and (8) holds.

Since, by (6),
\[Q_0(x) + \sum_{\nu = 2}^\infty Q_\nu(x) = (I - P') f(x, X),\]
then (12) yields, for all $x \in \mathcal{X}$,
\[(14) \quad \|x\|^2 \geq \|f(\zeta, X)x\|^2 = \|d_2 f(0, 0) X x\|^2 + \left(\left(Q_0(\zeta) + \sum_{\nu = 2}^\infty Q_\nu(\zeta, X)\right) x\right)^2 .\]

If $X: \mathcal{X} \to \mathcal{E}$ is a linear isometry, (13) implies that $d_2 f(0, 0) X: \mathcal{X} \to \mathcal{E}'$ is a linear isometry. For $0 < t < 1$, $tX \in B$ and (14) yields
\[t^2 \|x\|^2 + \left(\left(Q_0(\zeta) + \sum_{\nu = 2}^\infty Q_\nu(\zeta, tX)\right) x\right)^2 \leq \|x\|^2 .\]
for all \( x \in \mathcal{X} \), whence

\[
\left\| Q_0(X) + \sum_{v=2}^{+\infty} Q_v(\xi, tX) \right\| = (1 - t^2)^{1/2},
\]

for all linear isometries \( X : \mathcal{X} \to \mathcal{X} \). Since \( 0 < t < 1 \), the function \( Z \to Q_0(\xi) + \sum_{v=2}^{+\infty} Q_v(\xi, tZ) \) is holomorphic in a neighbourhood of \( \overline{B} \). By Proposition 2 of [7] and Proposition 2 of [2], the set of all linear isometries \( \mathcal{X} \to \mathcal{X} \) is stable. Thus, Harris' maximum principle [2, Theorem 9] entails that (15) holds for all \( X \in \overline{B} \) and all \( t \in (0, 1) \), implying that

\[
Q_0(\xi) + \sum_{v=2}^{+\infty} Q_v(\xi, X) = 0,
\]

and therefore

\[
f(\xi, X) = d \frac{1}{2} f(0, 0) X
\]

for all \( \xi \in A \) and all \( X \in B \).

That proves the part of the theorem stated in n. 1, concerning the Cartan factors of type one \( \mathcal{E} = \mathcal{L}(\mathcal{X}, \mathcal{X}) \) and \( \mathcal{E}' = \mathcal{L}(\mathcal{X}', \mathcal{X'}) \).

It is easily seen that the conclusion of the theorem does not always hold when \( \mathcal{E}' = \mathcal{L}(\mathcal{X}', \mathcal{X}) \) and \( \mathcal{E} = \mathcal{L}(\mathcal{X}, \mathcal{X}') \), \( \mathcal{X}' \) being a finite dimensional Hilbert space with \( \dim_{\mathbb{C}} \mathcal{X} \leq \dim_{\mathbb{C}} \mathcal{X}' \). A simple example is given by \( \mathcal{X} = \mathcal{X} = \mathbb{C} \), \( \mathcal{X}' = \mathbb{C}^2 \) (endowed with the euclidean metric). Let \( X_1', X_2' \) be two vectors in \( \mathcal{X}' \), with \( \| X_1' \| = \| X_2' \| = 1 \), \( (X_1' \mid X_2') = 0 \), and choosing \( A \) to be the open unit disc \( \Delta \) of \( \mathbb{C} \) – let \( f \in \text{Hol}(\Delta \times \Delta, B') \) be the function whose value at \( (\xi, z) \in \Delta \times \Delta \) is the linear map \( z(x^1 X_1' + \xi x^2 X_2') \) of \( \mathbb{C} \) into \( \mathcal{X}' \). For every \( \xi \in \Delta \), \( f(\xi, \cdot) \in \text{Hol}(\Delta, B') \) is a complex geodesic for \( c_{B'} \), and therefore \( f(\xi, \cdot) \), which depends effectively on \( \xi \in \Delta \), is a holomorphic isometry of \( \Delta \) into \( B' \).

4. Given a complex Hilbert space \( \mathcal{X}' \) with \( \dim_{\mathbb{C}} \mathcal{X}' > 1 \), consider the complex Banach space \( \mathcal{L}(\mathcal{X}') \) of all bounded linear operators on \( \mathcal{X}' \), and let a closed linear subspace \( \mathcal{E}' \) of \( \mathcal{L}(\mathcal{X}') \) be a Cartan factor of type four \([2,8]\). Let \( B' \) be the open unit ball of \( \mathcal{E}' \). This latter space is endowed with a Hilbert space structure defined by a positive-definite inner product \( (\mid \mid) \) whose associated norm \( \| \| \) is equivalent to the uniform operator norm \( \| \| \). More specifically \([2]\)

\[
(1/2) \| X' \|_2^2 \leq \| X' \|_2 \leq \| X' \|_2^2 \quad \text{for all } X' \in \mathcal{E}'.
\]

A complete spin system \( H' = \{ U'_\mu : \mu \in M \} \) in \( \mathcal{E}' \) is an orthonormal basis of \( \mathcal{E}' \), whose elements \( U'_\mu \) are self-adjoint, unitary operators on \( \mathcal{X}' \) – called spin-operators on \( \mathcal{X}' \) – such that \( U'_\mu \circ U'_{\mu_2} + U'_{\mu_2} \circ U'_\mu = 2 \delta_{\mu_1\mu_2} I \) (\( \mu_1, \mu_2 \in M \)).

Every \( X' \in \mathcal{E}' \) is represented, in terms of \( H' \), by the Fourier series expansion

\[
X' = \sum_{\mu \in M} (X' | U'_\mu) U'_\mu.
\]
If \( \emptyset \neq M_0 \subset M \), the map \( P' : \mathcal{E}' \to \mathcal{E}' \) defined by
\[
P' X' = \sum_{\mu \in M_0} (X'|U'_\mu) U'_\mu
\]
is an orthogonal projector on the Hilbert space \( \mathcal{E}' \).

The set \( H' \) is the family of all real (= complex) extreme points of \( \bar{B}' \). Since \( \mathcal{E}' \) is reflexive, the norm of \( P' \) as a linear operator in the Banach space \( (\mathcal{E}', \|\|) \), satisfies (11).

Let \( \mathcal{H} \) be a complex Hilbert space and let a closed linear subspace \( \mathcal{E} \) of \( \mathcal{E}(\mathcal{H}) \) be a Cartan factor of type four. Let \( B \) be the open unit ball of \( \mathcal{E} \) and let \( f \in \text{Hol}(A \times B, B') \) be such that \( f(\zeta_0, \cdot) \in \text{Iso}(B, B') \) at some \( \zeta_0 \in A \) and therefore – by Proposition 1 – at all \( \zeta \in A \). Since \( B' \) is homogeneous, there is no restriction in assuming \( f(0, 0) = 0 \). As before, that implies that \( d_2 f(0, 0) \) is a linear isometry of \( \mathcal{E} \) into \( \mathcal{E}' \) for the norm \( \|\| \).

By theorem 1 and by the Remark in [8], there exists a constant \( a \in \mathbb{C} \), with \( |a| = 1 \), such that \( ad_2 f(0, 0) \) is a real linear isometry of the Hilbert space \( \mathcal{E} \) into the Hilbert space \( \mathcal{E}' \). Thus, if \( \{U_\mu : \mu \in M_0\} \) is a complete spin system in \( \mathcal{E} \), then \( \{ad_2 f(0, 0) U_\mu : \mu \in M_0\} \) is a spin system in \( \mathcal{E}' \). Thus there is a complete spin system \( \{U'_\mu : \mu \in M\} \) containing \( \{ad_2 f(0, 0) U_\mu : \mu \in M_0\} \) as a subset. The closed linear span of this subset is the image \( \mathcal{E}' = d_2 f(0, 0) \mathcal{E} \). Thus the above argument shows that there is a projector \( P' \) whose range is \( \mathcal{E}' \) and which satisfies (11) and therefore also (2). Hence all the hypotheses of Lemma 2 are fulfilled, and (8) holds.

Since the orthogonal projectors \( P' \) and \( I - P' \) are orthogonal to each other with respect to the Hilbert space structure of \( \mathcal{E}' \), then (6) yields
\[
(I - P') Q_\nu(\zeta, X) = Q_\nu(\zeta, X) \quad (\zeta \in A, \ X \in B, \ \nu = 0, 2, 3, \ldots)
\]

By (18), \( \|f(\zeta, X)\| \leq 1 \), and that is equivalent to
\[
\|d_2 f(0, 0) X\|^2 + \left\| Q_0(\zeta) + \sum_{\nu = 2}^{+\infty} Q_\nu(\zeta, X) \right\|^2 \leq 1,
\]
because, by (19), \( d_2 f(0, 0) = P' d_2 f(0, 0) \) is orthogonal to \( Q_\nu(\zeta, X) \) for \( \nu = 0, 2, 3, \ldots \). Since \( \|d_2 f(0, 0) X\| = \|X\| \) for all \( X \in \mathcal{E} \), (20) yields
\[
\left\| Q_0(\zeta) + \sum_{\nu = 2}^{+\infty} Q_\nu(\zeta, X) \right\|^2 \leq 1 - \|X\|^2
\]
for all \( X \in B \) and all \( \zeta \in A \). This latter inequality is satisfied when \( X = t Z \), where \( 0 < t < 1 \) and \( Z \) is any spin-operator on \( \mathcal{E} \). Since the set of all spin-operators coincides with the set \( H \) of all real (= complex) extreme points of \( \bar{B} \) and the set \( H \) is stable, L. A. Harris' maximum principle implies, as at the end of n. 3, that (16) and (17) hold. That completes the proof of the theorem stated in n. 1.

References


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