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Rigidity of holomorphic isometries

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Geometria. — *Rigidity of holomorphic isometries.* Nota (*) del Socio EDOARDO VESENTINI.

ABSTRACT. — A rigidity theorem for holomorphic families of holomorphic isometries acting on Cartan domains is proved.

KEY WORDS: Cartan factors; Carathéodory distance; Holomorphic isometry; Extreme point.

RIASSUNTO. — *Rigidità di isometrie olomorfe.* Si stabilisce un teorema di rigidità per famiglie di isometrie olomorfe in domini di Cartan.

1. Let D and D' be bounded domains in two complex Banach spaces $\mathcal{E}$ and $\mathcal{E}'$, and let $\mathrm{Iso}\,(D, D')$ be the family of all holomorphic maps of D into D' which are isometries for the Carathéodory distances c_D and $c_{D'}$, in D and D'. Denoting by A a domain in $\mathbb{C}$, let f be a holomorphic map of $A \times D$ into D'. According to Lemma 2.3 of [6], if, for every pair points x, y in D, there is $\zeta \in A$ such that $c_{D'}(f(\zeta, x), f(\zeta, y)) = c_D(x, y)$, then $f(\zeta, \cdot) \in \mathrm{Iso}\,(D, D')$ for all $\zeta \in A$ (1). As a consequence, the following proposition holds:

PROPOSITION 1. *If there is a point* $\zeta_0 \in A$ *such that* $f(\zeta_0, \cdot) \in \mathrm{Iso}\,(D, D')$, *then,* $f(\zeta, \cdot) \in \mathrm{Iso}\,(D, D')$ *for all* $\zeta \in A$.

Let $D = D'$ (in which case $\mathrm{Iso}\,D$ will stand for $\mathrm{Iso}\,(D, D')$) and let $\mathrm{Aut}\,D \subset \mathrm{Iso}\,D$ be the group of all holomorphic automorphisms of D. According to Proposition V.1.10 of [1], if $f(\zeta_0, \cdot) \in \mathrm{Aut}\,D$ for some $\zeta_0 \in A$, then $f(\zeta, \cdot)$ is independent of $\zeta \in A$, *i.e.*

$$f(\zeta_0, \cdot) = f(\zeta, \cdot) \quad \text{for all } \zeta \in A. \tag{1}$$

Under which conditions on D and D' does this latter conclusion hold when $\mathrm{Aut}\,D$ is replaced by $\mathrm{Iso}\,(D, D')$?

It was shown in [9] that, if D is the open unit ball B of $\mathcal{E}$, and if $\mathcal{E}$ is a complex Hilbert space, the fact that $f(\zeta_0, \cdot) \in \mathrm{Iso}\,B$ for some $\zeta_0 \in A$ implies (1).

Let $\mathcal{E}$ be the C^* algebra $\mathcal{E} = \mathcal{L}(\mathcal{H})$ of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Starting from any infinite dimensional $\mathcal{H}$, an example was constructed in [5] of a non-trivial holomorphic family of holomorphic isometries of the open unit ball B of $\mathcal{E}$, *i.e.* a holomorphic map f: $A \times B \to B$ such that $f(\zeta, \cdot) \in \mathrm{Iso}\,B$ depends effectively on ζ.

The C^* algebra $\mathcal{L}(\mathcal{H})$ belongs to the class of J^*-algebras: in L. A. Harris' termino-

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(1) This lemma was established in [6] when $D = D'$. However, the proof carries over, with only minor formal changes, to the more general case considered here.

logy [2], it is a special kind of Cartan factor of type one. It was also shown in [5] that the same conclusion holds when $\mathcal{E}$ is any infinite dimensional Cartan factor of type two or three.

The investigation will be pursued in this *Note* by considering all Cartan domains of type four and a class of Cartan domains of type one. It will be shown that – in contrast with the results established in [5] – no non-trivial holomorphic families of holomorphic isometries exist in these cases. More specifically, let B and B' be the open unit balls of $\mathcal{E}$ and $\mathcal{E}'$, and let $f \in \mathrm{Hol}(A \times B, B')$ (the set of all holomorphic maps of $A \times B$ into B') be such that $f(\zeta_0, \cdot) \in \mathrm{Iso}(B, B')$ for some $\zeta_0 \in A$. The purpose of this *Note* is that of proving the following.

THEOREM. *If $\mathcal{E}$ and $\mathcal{E}'$ are both Cartan factors of type four, or if $\mathcal{E} = \mathcal{L}(\mathcal{K}, \mathcal{H})$, $\mathcal{E}' = \mathcal{L}(\mathcal{K}, \mathcal{H}')$, where $\mathcal{K}$, $\mathcal{H}$ and $\mathcal{H}'$ are complex Hilbert spaces and* $\dim_{\mathbb{C}} \mathcal{K} < \infty$, *then f is independent of $\zeta \in A$.*

This theorem extends a similar result which was previously established by the author when $\mathcal{E} = \mathcal{E}'$ and $f(\zeta, \cdot)$ is a holomorphic isometry for all $\zeta \in A$. A similar question to the one posed at the beginning can be formulated in the case in which D and D' are hyperbolic domains and the Carathéodory distances are replaced by the Kobayashi distances. This question is obviously answered by the above theorem in the case when $D = B$, $D' = B'$, because then Carathéodory's and Kobayashi's distances coincide. If $\mathcal{E}'$ has finite dimension (and therefore $\dim_{\mathbb{C}} \mathcal{E} \leq \dim_{\mathbb{C}} \mathcal{E}'$) and if the domains D and D' are bounded, the same question can be posed in terms of the Bergman metrics on D and D'. This question seems to be open, also in the particular case in which D and D' are the euclidean open unit balls of $\mathcal{E}$ and $\mathcal{E}'$.

2. This section will be devoted to some preliminaries. Let A be a connected open neighbourhood of 0 in $\mathbb{C}$. If $f \in \mathrm{Hol}(A \times B, B')$, for $\zeta \in A$, $X \in B$, $d_1 f(\zeta, X) \in \mathcal{E}'$ and $d_2 f(\zeta, X) \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ will indicate the partial Fréchet differentials of f with respect to the first and the second variable, evaluated at the point (ζ, X).

Suppose that:

(i) $f(0, 0) = 0$;

(ii) $d_2 f(0, 0) \in \mathcal{L}(\mathcal{E}, \mathcal{E}')$ is a linear isometry of $\mathcal{E}$ onto a closed linear subspace $\mathcal{F}'$ of $\mathcal{E}'$;

(iii) there is a projector P' in $\mathcal{E}'$ such that

$$P'(B') = B' \cap \mathcal{F}' . \tag{2}$$

Note that $\|P'\| \leq 1$.

As a consequence of (ii), there is a map $L \in \mathcal{L}(\mathcal{F}', \mathcal{E})$ which is a linear isometry of $\mathcal{F}'$ onto $\mathcal{E}$, for which $L \circ d_2 f(0, 0)$ is the identity on $\mathcal{E}$. Let $\widetilde{P}' \in \mathcal{L}(\mathcal{E}', \mathcal{F}')$ be the map induced by P', and let $g \in \mathrm{Hol}(A \times B, B)$ be the map defined by $g = L \circ \widetilde{P}' \circ f$.

Then $d_2 g(\zeta, X) = L \circ \tilde{P}' \circ d_2 f(\zeta, X)$, and therefore $d_2 g(0, 0) = L \circ \tilde{P}' \circ d_2 f(0, 0) = I$ the identity on $\mathcal{E}$. Thus, by H. Cartan's uniqueness theorem [1], $g(0, X) = X$ for all $X \in B$, and, by Proposition V.1.10 of [1] $g(\zeta, X)$ is independent of $\zeta \in A$, *i.e.*

(3) $$g(\zeta, X) = X \quad \text{for all } X \in B \text{ and all } \zeta \in A .$$

Let $f(\zeta, X) = Q_0(\zeta) + Q_1(\zeta, X) + Q_2(\zeta, X) + \ldots$, be the power series expansion of $f(\zeta, \cdot)$ in B, where $Q_\nu(\zeta, \cdot)$ is a continuous homogeneous polynomial $\mathcal{E} \to \mathcal{E}'$ of degree $\nu = 0, 1, \ldots$, expressed, for $\zeta \in A, X \in B$, by the integral

(4) $$Q_\nu(\zeta, X) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-i\nu\theta) f(\zeta, X) \, d\theta ,$$

and where $Q_1(\zeta, X) = d_2 f(\zeta, 0) X$.

Equation (3) implies that, for all $\zeta \in A$, $X \in B$,

(5) $$L \circ \tilde{P}' \circ Q_1(\zeta, X) = X ,$$

(6) $$\tilde{P}' \circ Q_\nu(\zeta, X) = 0 \quad \text{for } \nu = 0, 2, 3, \ldots .$$

Since, by (4), $\|Q_1(\zeta, \cdot)\| \leq 1$, (5) yields $\|X\| = \|L \circ \tilde{P}' \circ Q_1(\zeta, X)\| \leq \|L\| \|P'\| \|Q_1(\zeta, X)\| \leq \|Q_1(\zeta, X)\| \leq \|X\|$, whence $\|Q_1(\zeta, X)\| = \|X\|$ for all $X \in \mathcal{E}$.

Thus, $Q_1(\zeta, \cdot)$ is a linear isometry of $\mathcal{E}$ into $\mathcal{E}'$ for all $\zeta \in A$.

Example (3.1) of p. 301 of [5] shows that $Q_1(\zeta, \cdot)$ can depend on $\zeta \in A$. However, the following result holds.

Let H and H' be the sets of all real extreme points of the closures $\bar{B}$ and $\overline{B'}$ of B and B'.

Lemma 2. *If f satisfies conditions (i)-(iii), if*

(7) $$d_2 f(0, 0) H \subset H' ,$$

and if $\mathcal{E}$ is reflexive, then

(8) $$Q_1(\zeta, \cdot) = d_2 f(0, 0) \quad \textit{for all } \zeta \in A .$$

Proof. If $d_2 f(0, 0) Y$ is a complex extreme point of $\overline{B'}$, the strong maximum principle [1] yields $Q_1(\zeta, Y) = Q_1(0, Y) = d_2 f(0, 0) Y$ for all $\zeta \in A$. By (7), these equalities hold for all $Y \in H$. Let $X \in B$. For any continuous linear form λ' on $\mathcal{E}'$ and for any $\varepsilon > 0$, there is a finite convex combination $\Sigma a^i X_i$ of points $X_i \in H$ such that $|\lambda' \circ Q_1(\zeta, X - \Sigma a^i X_i)| < \varepsilon/2$, $|\lambda' \circ d_2 f(0, 0)(X - \Sigma a^i X_i)| < \varepsilon/2$.

Since $Q_1(\zeta, X_i) = d_2 f(0, 0) X_i$, then $|\lambda' \circ (Q_1(\zeta, X) - d_2 f(0, 0) X)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

The fact that λ' and ε are arbitrary, and the Hahn-Banach theorem, imply then that $Q_1(\zeta, X) = d_2 f(0, 0) X$ for all $\zeta \in A$ and all $X \in \mathcal{E}$. Q.E.D.

3. If $\mathcal{H}$ and $\mathcal{H}'$ are two complex Hilbert spaces, the space $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ of all bounded linear maps from $\mathcal{H}$ to $\mathcal{H}'$ is a complex Banach space with respect to the uniform operator norm $||| \; |||$.

It will be assumed henceforth that $n = \dim_{\mathbf{C}} \mathcal{K} < \infty$.

If $e_1, \ldots, e_n$ is an orthonormal basis of $\mathcal{K}$, for any $X' \in \mathcal{L}(\mathcal{K}, \mathcal{H}')$ let $X_j' = X' e_j$. Then, for $x = \sum_{j=1}^{n} a^j e_j \in \mathcal{K}$ $(a^j \in \mathbb{C})$, $X'x = \sum_{j=1}^{n} a^j X_j'$, and, denoting by the same symbols $(\,|\,)$ and $\|\ \|$ inner products and norms in $\mathcal{K}$ and $\mathcal{H}'$,

$$\|X'x\|^2 = \sum_{j=1}^{n} |a^j|^2 \|X_j'\|^2 + 2\,\mathrm{Re} \sum_{j<k}^{1\ldots n} a^j \bar{a}^k (X_j' \,|\, X_k') \leqslant$$

$$\leqslant n \sum_{j=1}^{n} |a^j|^2 \|X_j'\|^2 \leqslant n(\mathrm{Max}\{\|X_j'\| : j = 1, \ldots, n\})^2 \sum_{j=1}^{n} |a^j|^2 =$$

$$= n(\mathrm{Max}\{\|X_j'\| : j = 1, \ldots, n\})^2 \|x\|^2 ,$$

whence

$$\|\!|X'|\!\| \leqslant \sqrt{n}\,\mathrm{Max}\{\|X_j'\| : j = 1, \ldots, n\}. \tag{9}$$

Let $\tilde{X}'$ be the vector $(X_1', \ldots, X_n')$ in the Hilbert space direct sum $\bigoplus_1^n \mathcal{H}'$ of n copies of $\mathcal{H}'$. Then, by (9), the norm $\|\!|X'|\!\|$ of X' is estimated by $\|\!|X'|\!\|^2 \leqslant n \|\tilde{X}'\|^2$.

Hence, the bi-jective linear map $X' \to \tilde{X}'$ of $\mathcal{L}(\mathcal{K}, \mathcal{H}')$ into $\bigoplus_1^n \mathcal{H}'$ is bi-continuous. That shows that, if $\dim_{\mathbf{C}} \mathcal{K} < \infty$, the Banach space $\mathcal{L}(\mathcal{K}, \mathcal{H}')$ is reflexive.

Let $\{f_\mu' : \mu \in M\}$ be an orthonormal basis of $\mathcal{H}'$, indexed by a set M. Every $X' \in \mathcal{L}(\mathcal{K}, \mathcal{H}')$ is expressed by

$$X' = \sum_{\mu \in M} \left(\sum_{\nu=1}^{n} (X' e_\nu \,|\, f_\mu')(f_\mu' \otimes e_\nu^*) \right),$$

where the right-hand side (is summable and) converges to X' in the norm of $\mathcal{L}(\mathcal{K}, \mathcal{H}')$ [7, Lemma 5]. For $x \in \mathcal{K}$, $f_\mu' \otimes e_\nu^*(x) = (x \,|\, e_\nu) f_\mu'$, and therefore

$$\|X'x\|^2 = \sum_{\mu \in M} \left| \sum_{\nu=1}^{n} (X' e_\nu \,|\, f_\mu')(x \,|\, e_\nu) \right|^2 . \tag{10}$$

Let M_0 be a non-empty subset of M and let P' be the projector acting on $\mathcal{L}(\mathcal{K}, \mathcal{H}')$, defined on X' by $P'X' = \sum_{\mu \in M_0} \left(\sum_{\nu=1}^{n} (X' e_\nu \,|\, f_\mu')(f_\mu' \otimes e_\nu^*) \right)$.

Since, by (10), $\|P'X'x\| \leqslant \|X'x\|$ for all $x \in \mathcal{K}$, then $\|\!|P'X'|\!\| \leqslant \|\!|X'|\!\|$ for all $X' \in \mathcal{L}(\mathcal{K}, \mathcal{H}')$ and therefore the norm $\|P'\|$ of P' is

$$\|P'\| \leqslant 1 . \tag{11}$$

Furthermore, $I - P' = 0$ if $M_0 = M$ while, if $M_0 \neq M$, then

$$(I - P')X' = \sum_{\mu \in M \setminus M_0} \left(\sum_{\nu=1}^{n} (X' e_\nu \,|\, f_\mu')(f_\mu' \otimes e_\nu^*) \right),$$

and, by the same argument as before, $\|I - P'\| \leqslant 1$.

For all $x \in \mathcal{K}$

$$(P'X'x \mid (I-P')X'x) = \sum_{\mu_1 \in M_0} \sum_{\mu_2 \in M \setminus M_0} \sum_{\nu_1, \nu_2 = 1}^{n} (X'e_{\nu_1} \mid f'_{\mu_1}) \overline{(X'e_{\nu_2} \mid f'_{\mu_2})} \cdot$$
$$\cdot (x \mid e_{\nu_1}) \overline{(x \mid e_{\nu_2})} (f'_{\mu_1} \mid f'_{\mu_2}) = 0$$

and therefore

$$\|X'x\|^2 = \|P'X'x\|^2 + \|(I-P')X'x\|^2 . \tag{12}$$

Let $\mathcal{H}$ be an another complex Hilbert space and let B and B' be the open unit balls of $\mathcal{E} = \mathcal{L}(\mathcal{K}, \mathcal{H})$ and of $\mathcal{E}' = \mathcal{L}(\mathcal{K}, \mathcal{H}')$. If $f \in \mathrm{Hol}(A \times B, B')$ is such that $f(\zeta_0, \cdot) \in \mathrm{Iso}(B, B')$ for some $\zeta_0 \in A$, then $f(\zeta, \cdot) \in \mathrm{Iso}(B, B')$ for all $\zeta \in A$, and, in particular, for $\zeta = 0 \in A$.

Since B' is homogeneous [2], there is no restriction in assuming $f(0, 0) = 0$. Since the Carathéodory differential metric is the derivative of the Carathéodory distance ([4]; cf. also, *e.g.* [9]), and since the Carathéodory differential metrics of B and B' at the center 0 coincide with the norms in $\mathcal{E}$ and in $\mathcal{E}'$, then $d_2 f(0, 0)$ is a linear isometry of $\mathcal{E}$ into $\mathcal{E}'$. According to Theorem I of [7], there exists a unitary operator V on $\mathcal{K}$ and a linear isometry U of $\mathcal{H}$ into $\mathcal{H}'$ such that

$$d_2 f(0, 0) X = U \circ X \circ V \quad \text{for all } X \in \mathcal{L}(\mathcal{K}, \mathcal{H}). \tag{13}$$

REMARK. Theorem I was established in [7] when $\mathcal{H} = \mathcal{H}'$, but the proof holds, with only purely formal changes, in the more general context considered here.

Given an orthonormal basis in $\mathcal{K}$, its image by V is an orthonormal basis $\{e_1, \dots, e_n\}$ in $\mathcal{K}$. On the other hand, the image by U of an orthonormal basis in $\mathcal{H}$ is an orthonormal set in $\mathcal{H}'$, which, by a standard orthogonalization process, can be identified with a subset, $\{f'_\mu\}_{\mu \in M_0}$ of an orthonormal basis $\{f'_\mu\}_{\mu \in M}$ of $\mathcal{H}'$ ($M_0 \subset M$). Since the closed linear span of $\{f'_\mu \otimes e_\nu{}^* : \nu = 1, \dots, n; \mu \in M_0\}$ is the space $\mathcal{F}' = d_2 f(0, 0)\, \mathcal{E}$, the above considerations show that there exists a projector P' in $\mathcal{E}'$ with range $\mathcal{F}'$, satisfying (11) and therefore (2).

Hence, all the hypotheses of Lemma 2 are satisfied, and (8) holds.

Since, by (6),

$$Q_0(X) + \sum_{\nu=2}^{+\infty} Q_\nu(\zeta, X) = (I-P')\, f(\zeta, X),$$

then (12) yields, for all $x \in \mathcal{K}$,

$$\|x\|^2 \geq \|f(\zeta, X)x\|^2 = \|d_2 f(0, 0)Xx\|^2 + \left\| \left(Q_0(\zeta) + \sum_{\nu=2}^{+\infty} Q_\nu(\zeta, X) \right) x \right\|^2 . \tag{14}$$

If $X: \mathcal{K} \to \mathcal{H}$ is a linear isometry, (13) implies that $d_2 f(0, 0)X: \mathcal{K} \to \mathcal{H}'$ is a linear isometry. For $0 < t < 1$, $tX \in B$ and (14) yields

$$t^2 \|x\|^2 + \left\| \left(Q_0(\zeta) + \sum_{\nu=2}^{+\infty} Q_\nu(\zeta, tX) \right) x \right\|^2 \leq \|x\|^2$$

for all $x \in \mathcal{K}$, whence

$$\left|\left|\left| Q_0(X) + \sum_{\nu=2}^{+\infty} Q_\nu(\zeta, tX) \right|\right|\right| \leq (1 - t^2)^{1/2}, \tag{15}$$

for all linear isometries $X: \mathcal{K} \to \mathcal{H}$. Since $0 < t < 1$, the function $Z \to Q_0(\zeta) + \sum_{\nu=2}^{+\infty} Q_\nu(\zeta, tZ)$ is holomorphic in a neighbourhood of $\bar{B}$. By Proposition 2 of [7] and Proposition 2 of [2], the set of all linear isometries $\mathcal{K} \to \mathcal{H}$ is stable. Thus, Harris' maximum principle [2, Theorem 9] entails that (15) holds for all $X \in \bar{B}$ and all $t \in (0, 1)$, implying that

$$Q_0(\zeta) + \sum_{\nu=2}^{+\infty} Q_\nu(\zeta, X) = 0, \tag{16}$$

and therefore

$$f(\zeta, X) = d_2 f(0, 0) X \tag{17}$$

for all $\zeta \in A$ and all $X \in B$.

That proves the part of the theorem stated in n. 1, concerning the Cartan factors of type one $\mathcal{E} = \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $\mathcal{E}' = \mathcal{L}(\mathcal{K}, \mathcal{H}')$.

It is easily seen that the conclusion of the theorem does not always hold when $\mathcal{E} = \mathcal{L}(\mathcal{K}', \mathcal{H})$ and $\mathcal{E}' = \mathcal{L}(\mathcal{K}, \mathcal{H}')$, $\mathcal{K}'$ being a finite dimensional Hilbert space with $\dim_{\mathbb{C}} \mathcal{K} \leq \dim_{\mathbb{C}} \mathcal{K}'$. A simple example is given by $\mathcal{K} = \mathcal{H} = \mathbb{C}$, $\mathcal{K}' = \mathbb{C}^2$ (endowed with the euclidean metric). Let X_1', X_2' be two vectors in $\mathcal{H}'$, with $\|X_1'\| = \|X_2'\| = 1$, $(X_1' | X_2') = 0$, and – choosing A to be the open unit disc Δ of $\mathbb{C}$ – let $f \in \mathrm{Hol}(\Delta \times \Delta, B')$ be the function whose value at $(\zeta, z) \in \Delta \times \Delta$ is the linear map $(x^1, x^2) \to z(x^1 X_1' + \zeta x^2 X_2')$ of $\mathbb{C}^2$ into $\mathcal{H}'$. For every $\zeta \in \Delta$, $f(\zeta, \cdot) \in \mathrm{Hol}(\Delta, B')$ is a complex geodesic for $c_{B'}$, and therefore $f(\zeta, \cdot)$, which depends effectively on $\zeta \in \Delta$, is a holomorphic isometry of Δ into B'.

4. Given a complex Hilbert space $\mathcal{H}'$ with $\dim_{\mathbb{C}} \mathcal{H}' > 1$, consider the complex Banach space $\mathcal{L}(\mathcal{H}')$ of all bounded linear operators on $\mathcal{H}'$, and let a closed linear subspace $\mathcal{E}'$ of $\mathcal{L}(\mathcal{H}')$ be a Cartan factor of type four [2, 8]. Let B' be the open unit ball of $\mathcal{E}'$. This latter space is endowed with a Hilbert space structure defined by a positive-definite inner product $(\,|\,)$ whose associated norm $\|\ \|$ is equivalent to the uniform operator norm $|||\ |||$. More specifically [2]

$$(1/2)\, |||X'|||^2 \leq \|X'\|^2 \leq |||X'|||^2 \quad \text{for all } X' \in \mathcal{E}'. \tag{18}$$

A complete spin system $H' = \{U_\mu' : \mu \in M\}$ in $\mathcal{E}'$ is an orthonormal basis of $\mathcal{E}'$, whose elements U_μ' are self-adjoint, unitary operators on $\mathcal{H}'$ – called spin-operators on $\mathcal{H}'$ – such that $U_{\mu_1}' \circ U_{\mu_2}' + U_{\mu_2}' \circ U_{\mu_1}' = 2\delta_{\mu_1 \mu_2} I$ $(\mu_1, \mu_2 \in M)$.

Every $X' \in \mathcal{E}'$ is represented, in terms of H', by the Fourier series expansion

$$X' = \sum_{\mu \in M} (X' | U_\mu') U_\mu'.$$

If $\emptyset \neq M_0 \subset M$, the map $P'\colon \mathcal{E}' \to \mathcal{E}'$ defined by

$$P'X' = \sum_{\mu \in M_0} (X' \mid U'_\mu)\, U'_\mu$$

is an orthogonal projector on the Hilbert space $\mathcal{E}'$.

The set H' is the family of all real (= complex) extreme points of $\bar{B}'$. Since $\mathcal{E}'$ is reflexive, the norm of P' as a linear operator in the Banach space ($\mathcal{E}'$, $|||\ |||$), satisfies (11).

Let $\mathcal{H}$ be a complex Hilbert space and let a closed linear subspace $\mathcal{E}$ of $\mathcal{L}(\mathcal{H})$ be a Cartan factor of type four. Let B be the open unit ball of $\mathcal{E}$ and let $f \in \mathrm{Hol}(A \times B, B')$ be such that $f(\zeta_0, \cdot) \in \mathrm{Iso}(B, B')$ at some $\zeta_0 \in A$ and therefore – by Proposition 1 – at all $\zeta \in A$. Since B' is homogeneous, there is no restriction in assuming $f(0, 0) = 0$. As before, that implies that $d_2 f(0, 0)$ is a linear isometry of $\mathcal{E}$ into $\mathcal{E}'$ for the norm $|||\ |||$. By theorem I and by the Remark in [8], there exists a constant $a \in \mathbb{C}$, with $|a| = 1$, such that $a d_2 f(0, 0)$ is a real linear isometry of the Hilbert space $\mathcal{E}$ into the Hilbert space $\mathcal{E}'$. Thus, if $\{U_\mu : \mu \in M_0)$ is a complete spin system in $\mathcal{E}$, then $\{a d_2 f(0, 0)\, U_\mu : \mu \in M\}$ is a spin system in $\mathcal{E}'$. Thus there is a complete spin system $\{U'_\mu : \mu \in M\}$ containing $\{a d_2 f(0, 0)\, U_\mu : \mu \in M_0\}$ as a subset. The closed linear span of this subset is the image $\mathcal{F}' = d_2 f(0, 0)\, \mathcal{E}$. Thus the above argument shows that there is a projector P' whose range is $\mathcal{F}'$ and which satisfies (11) and therefore also (2). Hence all the hypotheses of Lemma 2 are fulfilled, and (8) holds.

Since the orthogonal projectors P' and $I - P'$ are orthogonal to each other with respect to the Hilbert space structure of $\mathcal{E}'$, then (6) yields

(19) $$(I - P')\, Q_\nu(\zeta, X) = Q_\nu(\zeta, X) \qquad (\zeta \in A\,,\ \ X \in B\,,\ \ \nu = 0, 2, 3, \ldots)\,.$$

By (18), $\|f(\zeta, X)\| \leqslant 1$, and that is equivalent to

(20) $$\|d_2 f(0, 0) X\|^2 + \left\| Q_0(\zeta) + \sum_{\nu = 2}^{+\infty} Q_\nu(\zeta, X) \right\|^2 \leqslant 1\,,$$

because, by (19), $d_2 f(0, 0) = P' d_2 f(0, 0)$ is orthogonal to $Q_\nu(\zeta, X)$ for $\nu = 0, 2, 3, \ldots$. Since $\|d_2 f(0, 0) X\| = \|X\|$ for all $X \in \mathcal{E}$, (20) yields

$$\left\| Q_0(\zeta) + \sum_{\nu = 2}^{+\infty} Q_\nu(\zeta, X) \right\|^2 \leqslant 1 - \|X\|^2$$

for all $X \in B$ and all $\zeta \in A$. This latter inequality is satisfied when $X = tZ$, where $0 < t < 1$ and Z is any spin-operator on $\mathcal{E}$. Since the set of all spin-operators coincides with the set H of all real (= complex) extreme points of $\bar{B}$ and the set H is stable, L. A. Harris' maximum principle implies, as at the end of n. 3, that (16) and (17) hold. That completes the proof of the theorem stated in n. 1.

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