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Meccanica dei solidi. — *The role of deviatoric and volumetric non-associativities on strain localization.* Nota di AHMED BENALLAL e CLAUDIA COMI, presentata (*) dal Corrisp. G. Maier.

ABSTRACT. — A homogeneous solid subject to quasi-static loading in the small strain range is considered. The material model assumed is rate-independent, non-associative and incrementally bilinear. The strain localization conditions are analytically solved using a geometric method. The expressions of the critical hardening moduli, their domains of validity and the form of the strain rate discontinuity are obtained. Finally these results, and in particular the role of hydrostatic and deviatoric non-normality, are discussed with reference to an elastic-plastic model for rock-like materials.

KEY WORDS: Localization; Strain discontinuity; Non-associative plasticity.

RIASSUNTO. — *Il ruolo della non associatività deviatorica e volumetrica nella localizzazione delle deformazioni*. Si considera un solido omogeneo soggetto ad azioni variabili staticamente, in regime di piccole deformazioni; si assume che il comportamento del materiale sia descritto da una legge incrementalmente bilineare, genericamente non associata. Le condizioni di localizzazione delle deformazioni vengono risolte analiticamente facendo uso di un metodo geometrico. Si ottengono le espressioni dei moduli di incrudimento critici, i loro domini di validità e la forma della discontinuità delle velocità di deformazione. Tali risultati ed in particolare il ruolo della non associatività sia volumetrica che deviatorica sono poi discussi con riferimento ad un modello elastoplastico per materiali lapidei.

NOTATION: Throughout the paper tensor notation is used. The symbol $\otimes$ denotes the tensor product, the symbol : denotes the doubly contracted product and the symbol $\cdot$ denotes the scalar product.

1. INTRODUCTION

Localization phenomena observed in ductile materials as well as in cohesive-brittle materials can be interpreted as an instability manifestation in the inelastic behaviour. The condition of strain localization which corresponds to the loss of ellipticity of the governing equations has been formulated by Rudnicki and Rice [1], Rice and Rudnicki [2], Borrè and Maier [3].

Several studies on this subject have shown the key role of the non-associativity of the material model on localization [1, 4]. All these results concern elasto-plastic models in which the direction of plastic flow differs from the normal direction to the yield surface by a vector parallel to the hydrostatic axis (volumetric non-associativity).

In [5] the localization analysis is performed for a Mohr material model which is non-associative both on the volumetric and deviatoric parts, but with the two non-associativities governed by the same parameter.

In this *Note* localization phenomena are investigated as for the role of a complete

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non-associativity of the model (volumetric non-associativity and deviatoric non-associativity). To this purpose the localization conditions for rate-independent materials obeying the constitutive equations (1) are solved. The solution method leads to closed-form expressions for the critical hardening modulus and the normal to the localization plane and rests on a geometric interpretation of the localization conditions in the Mohr plane associated to an arbitrary tensor $\boldsymbol{c}$, coaxial to the second-order coaxial tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which show up in equations (1). The result obtained herein extend those of Rudnicki and Rice [1], generalize those of Perrin and Leblond [4] and refine those of Bigoni and Hueckel [6]. The geometric approach to localization analysis in a simpler case was first proposed in [7] and extensively studied in [8] in a quite different form.

In sect. 4 the theoretical results established in sect. 3 are applied to non-associative models for concrete-like materials. When the particular case of uniaxial compression is considered, it is observed that, while volumetric non-associativity is essentially destabilizing, deviatoric non-associativity can have both stabilizing or destabilizing effects on localization.

2. Formulation

The general class of rate constitutive laws here considered can be expressed in the form:

$$(1) \qquad \dot{\boldsymbol{\sigma}} = \boldsymbol{L} : \dot{\boldsymbol{\varepsilon}}, \qquad \boldsymbol{L} = \begin{cases} \boldsymbol{E} & \text{if } f < 0 \text{ or } f = 0 \text{ and } \dot{f} < 0\,, \\ \boldsymbol{H} = \boldsymbol{E} - \dfrac{\boldsymbol{\alpha} \otimes \boldsymbol{\beta}}{H} & \text{if } f = 0 \text{ and } \dot{f} = 0\,, \end{cases}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ the strain tensor, $\boldsymbol{E}$ is the isotropic elastic tensor (with Lamé constants λ and μ), $\boldsymbol{L}$ is the tangent tensor, f is the yield function and H is a scalar such that $(H - \boldsymbol{\alpha}\colon \boldsymbol{E}^{-1} \colon \boldsymbol{\beta})$ is the hardening modulus h. In eq. (1) $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are second order tensors which can be decomposed in their deviatoric and hydrostatic parts: $\boldsymbol{\alpha} = \boldsymbol{a} + p\mathbf{1}$ $(p = \mathrm{tr}\,(\boldsymbol{\alpha})/3)$ and $\boldsymbol{\beta} = \boldsymbol{b} + q\mathbf{1}$ $(q = \mathrm{tr}\,(\boldsymbol{\beta})/3)$, $\mathbf{1}$ denoting the second order identity tensor. $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are assumed to be coaxial, but otherwise arbitrary in order to represent generically non-associative behaviours ($p \neq q$ corresponding to hydrostatic non-normality and $\boldsymbol{a} \neq \boldsymbol{b}$ to deviatoric non-normality).

Let us consider a homogeneous unbounded body subject to quasi-static loading in the small strain range so that the stress (and strain) field is initially homogeneous. The necessary and sufficient conditions under which non-uniqueness of the rate problem in the form of strain localization into a planar band of normal $\boldsymbol{n}$ can occur are expressed in the form (see [1-3]):

$$(2) \qquad \det(\boldsymbol{n} \cdot \boldsymbol{H} \cdot \boldsymbol{n}) \leqslant 0\,, \qquad \boldsymbol{\alpha} \cdot \boldsymbol{n} \neq \mathbf{0}\,, \qquad \boldsymbol{\beta} \cdot \boldsymbol{n} \neq \mathbf{0}$$

where $\boldsymbol{n} \cdot \boldsymbol{H} \cdot \boldsymbol{n}$ is the acoustic tensor. The equality sign in (2) corresponds to *continuous localization i.e.* to «inelastic yielding» (*loading*) on both sides of the discontinuity surface, while the strict inequality corresponds to *discontinuous localization, i.e.* to *loading* on one side of the discontinuity surface and *elastic unloading* on the other.

The strain rate discontinuity satisfies the Maxwell compatibility conditions:

$$[\![\dot{\boldsymbol{\varepsilon}}]\!] = (\boldsymbol{g} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{g})/2 \tag{3}$$

where $[\![\ ]\!]$ indicates the discontinuity or «jump» and $\boldsymbol{g}$ is the vector which defines the jump in the velocity gradient.

3. Localization analysis

Using eq. (1), the acoustic tensor is expressed in the form:

$$\boldsymbol{n}\cdot\boldsymbol{H}\cdot\boldsymbol{n} = \mu\mathbf{1} + \left(\lambda + \mu - \frac{pq}{H}\right)\boldsymbol{n}\otimes\boldsymbol{n} - \frac{1}{H}(\boldsymbol{n}\cdot\boldsymbol{a}\otimes\boldsymbol{b}\cdot\boldsymbol{n} + p\,\boldsymbol{n}\otimes\boldsymbol{b}\cdot\boldsymbol{n} + q\,\boldsymbol{n}\cdot\boldsymbol{a}\otimes\boldsymbol{n}) \tag{4}$$

and the localization condition (2) becomes:

$$\det(\boldsymbol{n}\cdot\boldsymbol{H}\cdot\boldsymbol{n}) = \mu\Big\{-(\lambda + 2\mu)\frac{1}{H}(\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\boldsymbol{n}) + (\lambda + \mu)\frac{1}{H}(\boldsymbol{n}\cdot\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{b}\cdot\boldsymbol{n}) - \\ - \mu\left[\frac{q}{H}(\boldsymbol{n}\cdot\boldsymbol{a}\cdot\boldsymbol{n}) + \frac{p}{H}(\boldsymbol{n}\cdot\boldsymbol{b}\cdot\boldsymbol{n})\right] + \mu\left(\lambda + 2\mu - \frac{pq}{H}\right)\Big\} \leqslant 0\,. \tag{5}$$

The vector $\boldsymbol{g}$ is the right eigenvector of the acoustic tensor associated to the zero eigenvalue and can be expressed in the form:

$$\boldsymbol{g} = \boldsymbol{a}\cdot\boldsymbol{n} + \frac{\mu}{\lambda + 2\mu}p\,\boldsymbol{n} - \frac{\lambda + \mu}{\lambda + 2\mu}(\boldsymbol{n}\cdot\boldsymbol{a}\cdot\boldsymbol{n})\,\boldsymbol{n}\,. \tag{6}$$

Taking into account eq. (3), the normal and tangential components of the strain discontinuity are:

$$[\![\dot{\varepsilon}]\!]_n = \boldsymbol{n}\cdot[\![\dot{\boldsymbol{\varepsilon}}]\!]\cdot\boldsymbol{n} = \boldsymbol{g}\cdot\boldsymbol{n}\,, \qquad [\![\dot{\varepsilon}]\!]_{nt} = [([\![\dot{\boldsymbol{\varepsilon}}]\!]\cdot\boldsymbol{n})([\![\dot{\boldsymbol{\varepsilon}}]\!]\cdot\boldsymbol{n}) - [\![\dot{\varepsilon}]\!]_n^2]^{1/2}\,. \tag{7}$$

Localization takes place in the form of shear band, when $\boldsymbol{g}$ is orthogonal to $\boldsymbol{n}$, namely, according to eq. (6), when:

$$\boldsymbol{n}\cdot\boldsymbol{a}\cdot\boldsymbol{n} = -p\,. \tag{8}$$

A pure split mode occurs when $[\![\dot{\varepsilon}]\!]_{nt}$ vanishes. This happens if $\boldsymbol{g}$ is parallel to $\boldsymbol{n}$, *i.e.* if:

$$(\boldsymbol{a}\cdot\boldsymbol{n})(\boldsymbol{a}\cdot\boldsymbol{n}) - (\boldsymbol{n}\cdot\boldsymbol{a}\cdot\boldsymbol{n})^2 = 0\,. \tag{9}$$

Let us introduce a second order tensor $\boldsymbol{c}$ coaxial to $\boldsymbol{a}$ and $\boldsymbol{b}$ and such that $\operatorname{tr}(\boldsymbol{c}) = 0$. Denote by c_1, c_2 and c_3 its eigenvalues, assumed to be distinct, and define the square magnitude T and the normal component Σ of the vector $\boldsymbol{c}\cdot\boldsymbol{n}$ as:

$$\Sigma = \boldsymbol{n}\cdot\boldsymbol{c}\cdot\boldsymbol{n}, \qquad T = (\boldsymbol{c}\cdot\boldsymbol{n})(\boldsymbol{c}\cdot\boldsymbol{n})\,. \tag{10}$$

From now on we denote by i, j, k a triplet of distinct number belonging to the set $(1, 2, 3)$. In the common principal frame of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ the normal components may be defined by using the classical Mohr formulas:

$$n_i^2 = \frac{T + c_i\Sigma + c_jc_k}{(c_i - c_j)(c_i - c_k)}\,, \qquad 0 \leqslant n_i^2 \leqslant 1\,. \tag{11}$$

By substituting eq. (11) into the localization condition (5) one obtains:

$$(12) \quad H\mu(\lambda + 2\mu) \leq (\lambda + 2\mu) \cdot$$
$$\cdot \left[a_1 b_1 \frac{T + c_1 \Sigma + c_2 c_3}{(c_1 - c_2)(c_1 - c_3)} + a_2 b_2 \frac{T + c_2 \Sigma + c_1 c_3}{(c_2 - c_1)(c_2 - c_3)} + a_3 b_3 \frac{T + c_3 \Sigma + c_1 c_2}{(c_3 - c_1)(c_3 - c_2)} \right] +$$
$$-(\lambda + \mu) \left[a_1 \frac{T + c_1 \Sigma + c_2 c_3}{(c_1 - c_2)(c_1 - c_3)} + a_2 \frac{T + c_2 \Sigma + c_1 c_3}{(c_2 - c_1)(c_2 - c_3)} + a_3 \frac{T + c_3 \Sigma + c_1 c_2}{(c_3 - c_1)(c_3 - c_2)} \right] \cdot$$
$$\cdot \left[b_1 \frac{T + c_1 \Sigma + c_2 c_3}{(c_1 - c_2)(c_1 - c_3)} + b_2 \frac{T + c_2 \Sigma + c_1 c_3}{(c_2 - c_1)(c_2 - c_3)} + b_3 \frac{T + c_3 \Sigma + c_1 c_2}{(c_3 - c_1)(c_3 - c_2)} \right] +$$
$$+\mu \left\{ q \left[a_1 \frac{T + c_1 \Sigma + c_2 c_3}{(c_1 - c_2)(c_1 - c_3)} + a_2 \frac{T + c_2 \Sigma + c_1 c_3}{(c_2 - c_1)(c_2 - c_3)} + a_3 \frac{T + c_3 \Sigma + c_1 c_2}{(c_3 - c_1)(c_3 - c_2)} \right] + \right.$$
$$\left. + p \left[b_1 \frac{T + c_1 \Sigma + c_2 c_3}{(c_1 - c_2)(c_1 - c_3)} + b_2 \frac{T + c_2 \Sigma + c_1 c_3}{(c_2 - c_1)(c_2 - c_3)} + b_3 \frac{T + c_3 \Sigma + c_1 c_2}{(c_3 - c_1)(c_3 - c_2)} \right] \right\} + \mu p q \,.$$

With the equality sign inequality (12) represents, in the (Σ, T) plane, a conical curve $\mathcal{F}$ changing in position and/or «size» with parameter H. By computing its discriminant one can easily realize that this curve is either a hyperbola or a parabola. In the same plane the set of a points which satisfy inequalities (11*b*) is represented by a triangle $\mathcal{J}$ of vertices A: (c_3, c_3^2), B: (c_2, c_2^2) and C: (c_1, c_1^2) («admissible region», see fig. 1). This admissible region in the (Σ, T) plane can be conceived as obtained from the traditional Mohr's diagram in the $(\Sigma, S = \sqrt{T - \Sigma^2})$ plane through the mapping: $T = S^2 + \Sigma^2$.

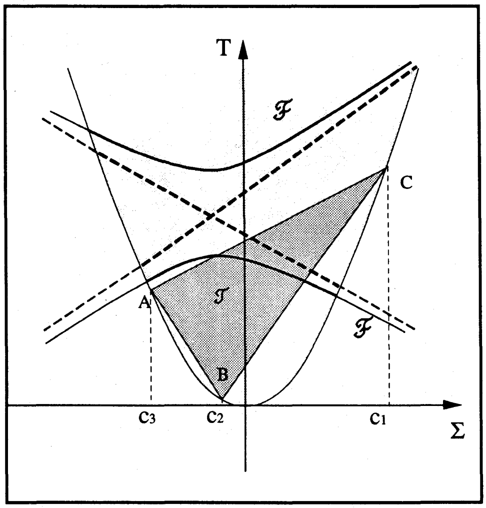

Fig. 1. – Geometrical interpretation of the localization condition in the (Σ, T), plane. The admissible domain is the triangular area $\mathcal{J}$ and the domain representing the localization condition is bounded by the hyperbola $\mathcal{F}$.

If at the beginning of the inelastic process there is a point (Σ, T) belonging to the triangle $\mathfrak{J}$ and to the region bounded by the hyperbola (12), localization occurs immediately. If such points do not exist, continuous localization will occur first either when the conical curve passes through a vertex or when it becomes tangent to one of the sides of the triangle, say L_{ij} defined by the equation: $T + c_k\Sigma + c_i c_j = 0$.

In the case where localization occurs at a vertex V_k $(k = 1, 2, 3)$, we have $n_k = 1$, $n_i = n_j = 0$, $\Sigma = c_k$ and $T = c_k^2$ and the critical value of H at localization is easily derived from (12):

$$H_k = (a_k + p)(b_k + q)/(\lambda + 2\mu) = \alpha_k \beta_k/(\lambda + 2\mu). \tag{13}$$

In eq. (13) and in what follows the index of H indicates the axis coinciding with the normal to the corresponding localization plane; two indices of H indicate the plane containing the corresponding normal.

Consider now the case when localization occurs for $\boldsymbol{n}$ different from a principal direction. The condition that the hyperbola $\mathcal{F}$ be tangent to the side L_{ij} is obtained by requiring that the normal to the conic curve be orthogonal to L_{ij}: $(\partial\mathcal{F}/\partial T)\, c_k = \partial\mathcal{F}/\partial\Sigma$. This condition reads:

$$\begin{aligned} 2(\lambda + \mu)\Sigma(a_i - a_j)(b_i - b_j) &= (\lambda + 2\mu)(c_i - c_j)(a_i b_i - a_j b_j) + (\lambda + \mu)\cdot \\ &\cdot[(a_i - a_j)(c_j b_i - c_i b_j) + (b_i - b_j)(c_j a_i - c_i a_j)] + \mu(c_i - c_j)[q(a_i - a_j) + p(b_i - b_j)]. \end{aligned} \tag{14}$$

If $A_{ij} = (a_i - a_j)(b_i - b_j) \neq 0$, the corresponding value of H is obtained by substituting (14) and the equation of the segment L_{ij} into (12):

$$\begin{aligned} H_{ij} = \frac{1}{16\mu(\lambda + \mu)(a_i - a_j)(b_i - b_j)} \Big\{ &\mu[(2p - a_k)(b_i - b_j) + (2q - b_k)(a_i - a_j)]^2 + \\ + 4(\lambda + \mu)[(a_i - a_j)(b_i - b_j)]^2 - &\frac{\mu(\lambda + \mu)}{\lambda + 2\mu}[(2p - a_k)(b_i - b_j) - (2q - b_k)(a_i - a_j)]^2 \Big\}. \end{aligned} \tag{15}$$

This expression is valid if Σ given by eq. (14) is between c_i and c_j, *i.e.* if the following inequalities are fulfilled:

$$B_{ij} \leqslant 0 \quad \text{and} \quad B_{ji} \leqslant 0, \quad B_{ij} = \frac{1 - 2\nu}{2A_{ij}}[(2p - a_k)(b_i - b_j) + (2q - b_k)(a_i - a_j)] - 1 \tag{16}$$

ν being the Poisson ratio.

From (11*a*) and (14) the normal to the localization plane turns out to have the following components: $n_k = 0$ and

$$\begin{aligned} n_i^2 &= \frac{\Sigma - c_j}{c_i - c_j} = \\ &= \frac{(\lambda + 2\mu)(a_i b_i - a_j b_j) - (\lambda + \mu)[(a_i - a_j)b_j + (b_i - b_j)a_j] + \mu[q(a_i - a_j) + p(b_i - b_j)]}{2(\lambda + \mu)(a_i - a_j)(b_i - b_j)}, \end{aligned} \tag{17}$$

$$\begin{aligned} n_j^2 &= -\frac{\Sigma - c_i}{c_i - c_j} = \\ &= -\frac{(\lambda + 2\mu)(a_i b_i - a_j b_j) - (\lambda + \mu)[(a_i - a_j)b_i + (b_i - b_j)a_i] + \mu[q(a_i - a_j) + p(b_i - b_j)]}{2(\lambda + \mu)(a_i - a_j)(b_i - b_j)}. \end{aligned} \tag{18}$$

If a_k and b_k are double eigenvalues, *i.e.* if $\boldsymbol{a}$ and $\boldsymbol{b}$ are axisymmetric with the same axis of symmetry $\boldsymbol{e}_i$ or $(\boldsymbol{e}_j)$, only the $\boldsymbol{n}$ component corresponding to the simple eigenvalue, n_i (or n_j), is uniquely defined, while the other two are indeterminate. Therefore the normal to the localization plane describes a cone of axis $\boldsymbol{e}_i$ (or $\boldsymbol{e}_j$).

Let us now consider the case $A_{ij} = (a_i - a_j)(b_i - b_j) = 0$. When the right hand side of (14) is non-zero, the critical hardening modulus can never be H_{ij}. If the right hand side of (14) is zero, *i.e.* if $a_i = a_j$ and $b_i = b_j$, eq. (14) is fulfilled for every Σ belonging to L_{ij} including the vertices V_i and V_j. The critical H is $H_i = H_j$, the expression of which is given by eq. (13). For this value of H the hyperbola degenerates in two straight lines, one of these lines coinciding with L_{ij}. In this case the set of normals to the localization plane is the plane Π_{ij} orthogonal to the direction $\boldsymbol{e}_k$.

Note that the results obtained are, as expected, independent of the chosen tensor $\boldsymbol{c}$.

In order to find the critical value of H, one should compute the maximum of the H_i and of the admissible $H_{ij}\,(i, j = 1, 2, 3)$, keeping in mind that H_{ij} is valid only if conditions (16) are complied with. From (13) and (15) one can compute the following differences (for more details see [8]):

$$(19) \qquad H_{ij} - H_i = \frac{A_{ij}(\lambda + \mu)}{4\mu(\lambda + 2\mu)} B_{ij}^2 , \qquad H_{ij} - H_j = \frac{A_{ij}(\lambda + \mu)}{4\mu(\lambda + 2\mu)} B_{ji}^2 ,$$

$$(20a) \qquad H_{ij} - H_{ik} = \frac{\lambda + \mu}{4\mu(\lambda + 2\mu)} (A_{ij} B_{ij}^2 - A_{ik} B_{ik}^2) ,$$

$$(20b) \qquad H_{ij} - H_k = \frac{\lambda + \mu}{4\mu(\lambda + 2\mu)} [A_{ij} B_{ij}^2 + 4A_{ik}(B_{ik} + 1)] ,$$

$$(21) \qquad H_i - H_j = \frac{(2p - a_k)(b_i - b_j) + (2q - b_k)(a_i - a_j)}{2(\lambda + 2\mu)} .$$

As a consequence of eqs. (19)-(21), the following statements are easily seen to hold for (i, j, k) distinct and belonging to the set (1,2,3):

(a) the critical value H_c is H_{ij} if the following inequalities are simultaneously satisfied:

$$(22) \qquad \begin{cases} A_{ij} > 0 , \quad B_{ij} \leqslant 0 , \quad B_{ji} \leqslant 0 , \\ A_{ij} B_{ij}^2 \geqslant A_{ik} B_{ik}^2 , \quad A_{ji} B_{ij}^2 \geqslant A_{jk} B_{jk}^2 , \quad A_{ij} B_{ij}^2 + 4A_{ik}(B_{ik} + 1) \geqslant 0 ; \end{cases}$$

(b) the critical value H_c is H_i if one of the following inequalities sets is fulfilled:

$$(23) \qquad A_{ij} \geqslant 0 , \quad A_{ik} \geqslant 0 , \quad B_{ij} \geqslant 0 , \quad B_{ik} \geqslant 0$$

or

$$A_{ij} \geqslant 0\,, \quad A_{ik} \leqslant 0\,, \quad B_{ij} \geqslant 0\,, \quad A_{ik}(B_{ik}+1) \geqslant 0 \tag{24}$$

or

$$A_{ij} \leqslant 0\,, \quad A_{ik} \leqslant 0\,, \quad B_{ij} \leqslant -1\,, \quad B_{ik} \leqslant -1\,, \quad A_{kj}B_{kj}^2 + 4A_{ki}(B_{ki}+1) \leqslant 0\,. \tag{25}$$

When $A_{rs} = 0$, B_{rs} is not defined and the inequalities involving it should be dropped from (22)-(25).

The material model, the values of its parameters and the loading histories will determine which one of the above inequality sets can be fulfilled; as a consequence one of eqs. (13) and (15) with $i, j \in (1, 2, 3)$ will yield the critical H. It is worth stressing that a priori each one of inequality sets (22)-(25) must be considered as a possible condition to satisfy.

4. Application to non-associative models for concrete-like materials

The above results are applied to a general set of constitutive relations used to describe the plastic fracturing behaviour of concrete-like materials. The Hsieh-Ting-Chen model, the Ottosen and the William-Wranke models are included. For details on these models, see *e.g.* chapter IV in Chen-Han's book [9].

The tangent modulus has the general form:

$$\boldsymbol{H} = \boldsymbol{E} - \frac{\left(\frac{\alpha}{\bar\sigma}\boldsymbol{S} + \beta\boldsymbol{s} + \gamma\bar\sigma\mathbf{1}\right) \otimes \left(\frac{\alpha'}{\bar\sigma}\boldsymbol{S} + \beta'\boldsymbol{s} + \gamma'\bar\sigma\mathbf{1}\right)}{h + \left[\frac{2}{27}\alpha\alpha' + \frac{2}{3}\beta\beta' + (\beta\alpha' + \alpha\beta')(3N^3 - N)\right]\frac{\bar\sigma^2}{2\mu} + \gamma\gamma'\frac{\bar\sigma^2}{K}} \tag{26}$$

where K is the bulk modulus, $\mathbf{s}$ is the stress deviator, $\bar\sigma = \sqrt{3/2\, \boldsymbol{s} : \boldsymbol{s}}$ is the equivalent stress, N is the intermediate eigenvalue of $\boldsymbol{s}/\bar\sigma$ (N_i denoting the i-th eigenvalue, $i = 1, 2, 3$) and $\boldsymbol{S}$ is the gradient of the third invariant $J_3 = 1/3\,(\boldsymbol{s}\cdot\boldsymbol{s}) : \boldsymbol{s}$ with respect to $\boldsymbol{\sigma}$, *i.e.* $\boldsymbol{S} = \dfrac{\partial J_3}{\partial \boldsymbol{\sigma}} = \boldsymbol{s}\cdot\boldsymbol{s} - \dfrac{1}{3}\,\mathrm{tr}\,(\boldsymbol{s}\cdot\boldsymbol{s})\,\mathbf{1}$. By setting $\alpha = \alpha' = 0$ and $\beta = \beta'$ the Drucker-Prager non-associative model treated in [1] is recovered.

Since in all real cases $\beta\beta' > 0$, one can divide the numerator and the denominator in (26) by $\beta\beta'$; by a suitable redefinition of the symbols α, α', γ, γ' and h, this is equivalent to setting $\beta = \beta' = 1$ in eq. (26), as it will be assumed in what follows. Therefore the model considered in this section (eq. (26)) is obtained from (1) by setting:

$$\begin{gathered} \boldsymbol{a} = \frac{\alpha}{\bar\sigma}\boldsymbol{S} + \boldsymbol{s}\,, \qquad \boldsymbol{b} = \frac{\alpha'}{\bar\sigma}\boldsymbol{S} + \boldsymbol{s}\,, \qquad p = \gamma\bar\sigma, \qquad q = \gamma'\bar\sigma\,, \\ H = h + \left[\frac{2}{27}\alpha\alpha' + \frac{2}{3} + (\alpha' + \alpha)(3N^3 - N)\right]\frac{\bar\sigma^2}{2\mu} + \gamma\gamma'\frac{\bar\sigma^2}{K}\,. \end{gathered} \tag{27}$$

By virtue of eqs. (27*a, b*) $\boldsymbol{a}$ and $\boldsymbol{b}$ turn out to be coaxial; $\alpha - \alpha'$ measures the deviatoric non-associativity and $\gamma - \gamma'$ the volumetric non-associativity. If h is computed from eq.

(27*e*), eqs. (13) and (15) acquire the form:

$$(28)\quad \frac{h_{ij}}{\mu} = \frac{\bar{\sigma}^2}{4\mu^2(1-\alpha N_k)(1-\alpha' N_k)}\left\{ -\frac{1+\nu}{2}\left[2N_k - 2\alpha\alpha'\left(N_k^2 - \frac{2}{9}\right)N_k - \right.\right.$$

$$\left. -\frac{2}{9}(\alpha+\alpha') + \frac{1-2\nu}{1+\nu}(\gamma+\gamma'-(\gamma\alpha'+\gamma'\alpha)N_k)\right]^2 + \frac{9}{4}\left[\left(N_k^2 - \frac{1}{9}\right)(\alpha-\alpha')\right]^2 +$$

$$\left. + \frac{(1-2\nu)^2}{1-\nu^2}\left[\frac{1+\nu}{2(1-2\nu)}\left(-N_k^2 + \frac{1}{9}\right)(\alpha-\alpha') - \gamma + \gamma' + (\gamma\alpha' - \gamma'\alpha)N_k\right]^2\right\},$$

$$(29)\quad \frac{h_k}{\mu} = \frac{\bar{\sigma}^2}{\mu(\lambda+2\mu)}\left[\left(N_k^2 - \frac{2}{9}\right)\alpha + N_k + \gamma\right]\left[\left(N_k^2 - \frac{2}{9}\right)\alpha' + N_k + \gamma'\right] -$$

$$-\left[\frac{2}{27}\alpha\alpha' + \frac{2}{3} + (\alpha'+\alpha)(3N^3 - N)\right]\frac{\bar{\sigma}^2}{2\mu^2} - \gamma\gamma'\frac{\bar{\sigma}^2}{K\mu} = [\text{if } (1-\alpha N_k)(1-\alpha' N_k)\neq 0] =$$

$$= \frac{\bar{\sigma}^2}{4\mu^2(1-\alpha N_k)(1-\alpha' N_k)}\left\{ -\frac{1+\nu}{1-\nu}\left[N_k - \alpha\alpha'\left(N_k^2 - \frac{2}{9}\right)N_k - \frac{1}{9}(\alpha+\alpha') - \right.\right.$$

$$\left. -\frac{1-2\nu}{1+\nu}(\gamma+\gamma'-(\gamma\alpha'+\gamma'\alpha)N_k)\right]^2 - [(1-\alpha N_k)(1-\alpha' N_k)]^2\left(\frac{4}{3} - 3N_k^2\right) +$$

$$\left. + \frac{(1-2\nu)^2}{1-\nu^2}\left[\frac{1+\nu}{1-2\nu}\left(N_k^2 - \frac{1}{9}\right)(\alpha-\alpha') - \gamma + \gamma' + (\gamma\alpha' - \gamma'\alpha)N_k\right]^2\right\}.$$

Equations (28) or (29*b*) provide the critical hardening modulus as algebraic sums of squares. This clearly shows that, as expected, the critical hardening modulus is never positive for associative constitutive behaviour. On the other hand, deviatoric and/or volumetric non-associatives may result into localization within the hardening regime.

The normalized critical hardening modulus $\frac{\mu h_c}{\bar{\sigma}^2}$ is represented in fig. 2 *vs.* N (note that this parameter takes values between 1/3 and $-$ 1/3 only) for the following fixed constitutive parameters: $\alpha = 0$, $\alpha' = 4$, $\gamma = 0.8$, $\gamma' = 0.2$, $\nu = 0.2$. In this figure, denoting by 1 the axis corresponding to $N_1 \geqslant N$, by 2 the axis corresponding to N and by 3 the axis corresponding to $N_3 \leqslant N$, we represent the six different possible values of the hardening modulus given by (28) and (29) (the h_i's by dashed lines and the h_{ij}'s by solid lines) and the critical value at localization (heavy solid line). For this particular choice of parameters the model is non-associative both on the deviatoric and on the volumetric part and localization is predicted to occur in the hardening regime ($h_c > 0$) for every loading condition. The critical hardening modulus is given by h_{13} for $N \leqslant 0.144$ and by

h_1 for $N \geqslant 0.144$. To obtain this optimal value, account has been taken of relations (22)-(25). Figure 2 shows also the straight line corresponding to $A_{23} = 0$ and the points corresponding to $B_{ij} = 0$, *i.e.* the points where $h_{ij} = h_i$ and the two curves are tangent. It is worth noting that the admissibility conditions play an important role: in fact the critical hardening modulus is not simply the maximum between the h_{ij} and the h_i (see fig. 2 for $N > 0$). The value $N = 1/3$ ($N = -1/3$) corresponds to axially symmetric compression (respectively tension); this symmetry entails that: $h_{13} = h_{23}$ and $h_2 = h_1$ ($h_{13} = h_{12}$), as one can note in fig. 2.

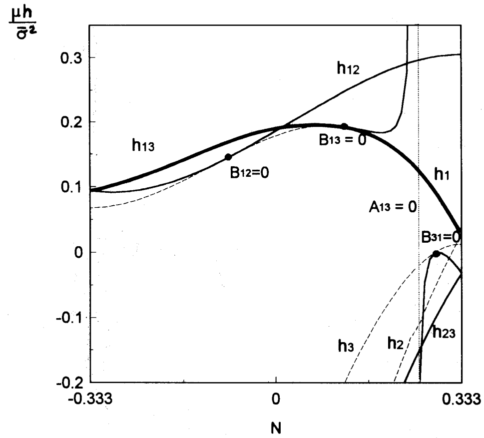

Fig. 2. – Normalized critical hardening modulus *vs.* N.

4.1. *Uniaxial compression*

Let us denote by $\boldsymbol{e}_1$ the axis of compression and by $\sigma \leqslant 0$ the compression stress. Being the loading condition axially symmetric, the other two principal axis are undetermined in the plane of normal $\boldsymbol{e}_1$ and one has: $a_2 = a_3$, $b_2 = b_3$, $A_{23} = 0$, $A_{12} = A_{13}$, $B_{12} = B_{13}$. Moreover, $N_1 = -2/3$, $N_2 = N_3 = 1/3$ and $\bar{\sigma} = -\sigma$ (note that the N_i are not ordered as in fig. 2). Substituting this values into (22)-(25), we can draw the following conclusions on the critical hardening modulus h_c:

(a) $h_c = h_{12} = h_{13}$ given by eq. (28) with $i = 1$ and $j = 2$ or 3 when the following conditions are fulfilled:

$$A_{12} = \left(1 - \frac{\alpha}{3}\right)\left(1 - \frac{\alpha'}{3}\right)\sigma^2 > 0\,, \tag{30a}$$

$$B_{12} = \frac{1 - 2\nu}{\left(1 - \frac{\alpha}{3}\right)\left(1 - \frac{\alpha'}{3}\right)} \cdot \left[\frac{\gamma\alpha' + \gamma'\alpha}{3} - (\gamma + \gamma') + \frac{\alpha\alpha'}{27} - \frac{\alpha + \alpha'}{9} + \frac{1}{3}\right] - 1 \leqslant 0\,, \tag{30b}$$

(30c) $$B_{21} = -B_{12} - 2 = -\frac{1-2\nu}{\left(1-\frac{\alpha}{3}\right)\left(1-\frac{\alpha'}{3}\right)} \cdot$$

$$\cdot\left[\frac{\gamma\alpha' + \gamma'\alpha}{3} - (\gamma+\gamma') + \frac{\alpha\alpha'}{27} - \frac{\alpha+\alpha'}{9} + \frac{1}{3}\right] - 1 \leqslant 0\,,$$

(b) $h_c = h_1$ given by eq. (29) with $k = 1$, when:

(31) $$A_{12} > 0 \text{ and } B_{12} \geqslant 0\,, \quad \text{or} \quad A_{12} \leqslant 0 \text{ and } B_{12} \leqslant -1\,,$$

(c) $h_c = h_2 = h_3$ given again by (29) with $k = 2$ or 3, when:

(32) $$A_{21} = A_{12} > 0 \text{ and } B_{21} \geqslant 0\,, \quad \text{or} \quad A_{21} \leqslant 0 \text{ and } B_{21} \leqslant -1\,.$$

Notice that $A_{12}=0$, $B_{12}=0$, $B_{21}=0$ and $B_{12}=-1$ represent in the plane ($\gamma-\gamma'$, $\alpha-\alpha'$) an horizontal line $\mathcal{L}_{A12}$ and three hyperbolas $\mathcal{C}_{B12}$, $\mathcal{C}_{B21}$ and $\mathcal{C}_{B12=-1}$; fig. 3 shows the branches of interest of these hyperbolas for fixed $\gamma+\gamma'=1$, $\alpha=0$ (plastic potential of Drucker-Prager) and $\nu=0.2$.

The domains of validity of h_1, $h_2 = h_3$ and $h_{12} = h_{13}$ are also shown in fig. 3, using eqs. (30)-(32). The potential normal to the localization plane: is the axis of compression if $h_c = h_1$; it belongs to the plane 2-3 if $h_c = h_2 = h_3$; it belongs to a cone of axis $\boldsymbol{e}_1$ if $h_c = h_{12} = h_{13}$.

By using eqs. (8) and (9) one can conclude that pure split modes of localization correspond to the vertex values h_1 and $h_2\,(=h_3)$, while for $h_{12}\,(=h_{13})$ mixed localization modes (shear modes superposed to split modes) are predicted. The pure shear band mode occurs for the constitutive parameters corresponding to the dashed line in fig. 3. The two dotted segments correspond to material models for which $\boldsymbol{\alpha}\cdot\boldsymbol{n}=\mathbf{0}$ or $\boldsymbol{\beta}\cdot\boldsymbol{n}=\mathbf{0}$. For these particular combinations of material parameters localization is a priori excluded.

In the same figure we also plot the isocurves for the critical hardening modulus h_c. This allows to visualize the role of non-associativity on localization. In the case of associative model or for «slightly» non-associative models (region surrounding the origin in the fig. 3), localization occurs in the softening regime. The isocurves are almost symmetric with respect to the axis $\gamma-\gamma'=0$ and this implies that the presence of volumetric non-associativity has basically a destabilizing effect. On the other hand deviatoric non-associativity has different effects on localization depending on the sign of $\alpha-\alpha'$. If $\alpha-\alpha'<0$ the deviation from normality in the deviatoric plane anticipates localization (h_c increases moving from $\alpha-\alpha'=0$ to $\alpha-\alpha'=-7$). If $\alpha-\alpha'>0$ there is an interaction between deviatoric and volumetric non-associativity, namely: an increase in deviatoric non-associativity anticipates localization for high volumetric non-associativity, while it postpones localization for low volumetric non-associativity.

This latter conclusion concerns only the compression case. Further study is needed in order to generalize this result.

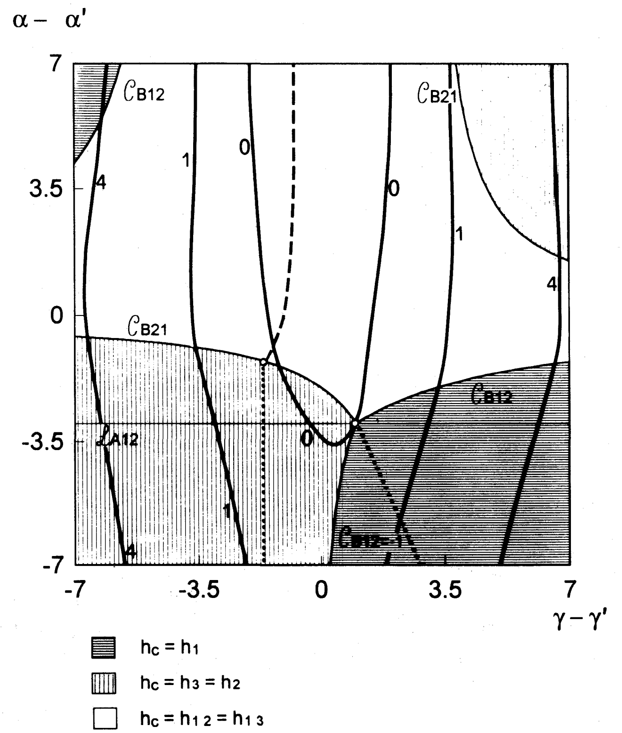

Fig. 3. – Isocurves of the normalized critical hardening modulus and domains of validity of its various expressions for uniaxial compression.

5. Conclusions

This *Note* focused on the influence of non-associativity on localization phenomena, namely on the destabilizing effects of the deviation from normality in material models. The findings of the present study can be summarized as follows.

1. At localization the critical hardening modulus is shown to have six different expressions in the general case, each of these expressions being valid in a specific range of the constitutive parameters for a given loading path. The admissibility conditions for each value are important and a simple maximization between the available expressions does not lead to the correct critical hardening modulus.

2. When the second order tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, defined as those which appear in the tangent tensor (1), have distinct eigenvalues, the normal to the localization plane is always contained in one of their principal planes. In some circumstances this normal coincides with one of the principal directions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. This conclusion holds also when $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have each a double eigenvalue corresponding to distinct principal planes.

3. When $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ have a double eigenvalue and this eigenvalue is associated to the same principal plane, the normal to the localization plane is either the principal direction associated to the simple eigenvalue, or an arbitrary direction in the principal

plane corresponding to the double eigenvalue, or a direction belonging to a cone the axis of which is the principal direction corresponding to the simple eigenvalue.

4. Both the deviatoric and volumetric non-associativities have primarily destabilizing effects: localization may occur in the hardening regime in the presence of deviatoric and/or volumetric non-associativity, while it can occur only in the softening branch for associative models. However, in some particular cases, a combination of the two non-associativities may postpone localization. This stabilizing effect never occurs for the simpler models considered in [1].

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