Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Mario Girardi, Michele Matzeu

Periodic Solutions of Second Order Nonautonomous Systems with the Potentials Changing Sign

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **4** (1993), n.4, p. 273–277.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1993_9_4_4_273_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1993.

Rend. Mat. Acc. Lincei
s. 9, v. 4:273-277 (1993)

Analisi funzionale. — *Periodic Solutions of Second Order Nonautonomous Systems with the Potentials Changing Sign.* Nota di Mario Girardi e Michele Matzeu, presentata (*) dal Corrisp. A. Ambrosetti.

Abstract. — Some existence and multiplicity results for periodic solutions of second order nonautonomous systems with the potentials changing sign are presented. The proofs of the existence results rely on the use of a linking theorem and the Mountain Pass theorem by Ambrosetti and Rabinowitz [2]. The multiplicity results are deduced by the study of constrained critical points of minimum or Mountain Pass type.

Key words: Periodic solutions; Potentials changing sign; Second order nonautonomous systems.

Riassunto. — *Soluzioni periodiche di sistemi non autonomi del secondo ordine con potenziali che cambiano segno.* Vengono presentati alcuni risultati di esistenza e di molteplicità per soluzioni periodiche di sistemi non autonomi del secondo ordine con potenziali che cambiano segno. Le prove dei risultati di esistenza si basano sull'uso di un teorema di «linking» e sul teorema del Passo Montano di Ambrosetti e Rabinowitz [2]. I risultati di molteplicità sono dedotti dallo studio di punti critici vincolati di tipo minimo o Passo Montano.

1. Existence results

Let us consider the following second order system in the space $\boldsymbol{R}^N$;

$$\ddot{x}(t) + b(t)\,V'(x(t)) = 0 \tag{V}$$

where b is a T-periodic continuous real function and $V \in C^2(\boldsymbol{R}^N; \boldsymbol{R})$ satisfies the following superquadratic growth assumptions:

$$V(x) \geqslant 0 \quad \forall x \in \boldsymbol{R}^N, \qquad V(0) = 0\,, \tag{V_1}$$

$$V(x) = o(|x|^2) \quad \text{as } x \to 0\,, \tag{V_2}$$

$$\exists \beta > 2\colon\ V'(x)x \geqslant \beta V(x) \quad \forall x \in \boldsymbol{R}^N, \quad |x| \geqslant R, \quad \text{with } R \text{ sufficiently large.} \tag{V_3}$$

Then one can state some existence results of T-periodic solutions to (V) in the case when either

$$\int_0^T b(t)\,dt > 0 \tag{1}$$

or

$$\int_0^T b(t)\,dt < 0\,, \tag{2}$$

by giving suitable assumptions connecting the number

$$B^- = \max_{t \in [0,T]} b^-(t), \qquad \text{where } b^-(t) = -\min\{b(t), 0\} \quad \forall t \in [0,T]$$

with the behavior of V, and suitably reinforcing the superquadratic growth conditions on V.

(*) Nella seduta del 18 giugno 1993.

THEOREM 1. *Let b a T-periodic continuous real function satisfying* (1), *let* $V \in C^2(\mathbf{R}^N; \mathbf{R})$ *satisfy* $(V_1), (V_2), (V_3)$ *and let there exist two numbers* $c \geq 0$, $d > 0$ *such that*

$$(3) \qquad B^-(V'(x)x - \beta V(x)) \leq c|x|^2 \qquad \forall x \in \mathbf{R}^N, \qquad |x| \geq R,$$

with R sufficiently large

$$(4) \qquad c < \frac{2(\beta-2)\pi^2}{(1+4\pi^2)} 1/T^2,$$

$$(5) \qquad B^-(|V''(x)| - d|x|^{\beta-2}) \leq 0 \qquad \forall x \in \mathbf{R}^N, \qquad |x| \geq R.$$

Then there exists a non-zero T-periodic solution of (V).

THEOREM 2. *Let b a T-periodic continuous real function satisfying* (2) *and*

$$(6) \qquad \exists t_0 \in [0, T]: \ b(t_0) > 0,$$

let $V \in C^2(\mathbf{R}^N)$ *satisfy* $(V_1), (V_2), (V_3)$ *and let there exist two numbers* $c \geq 0$, $d > 0$ *such that (3), (4), (5) hold. Moreover let V satisfy the further assumptions*

$$(V_4) \qquad \exists a_1 > 0, \quad \beta' > 2: \ V(x) \geq a_1 |x|^{\beta'} \qquad \forall x \in \mathbf{R}^N, \ |x| \leq r,$$

with r sufficiently small

$$(V_5) \qquad \exists a_2 > 0: \ |V'(x)| \leq a_2 |x|^{\beta'-1} \qquad \forall x \in \mathbf{R}^N, \ |x| \leq r.$$

Then there exists a non-zero T periodic solution of (V).

The proof of Theorems 1, 2 are based on the well known fact that the T-periodic solutions of (V) can be obtained by periodically extending to the whole real line the critical points of the functional

$$f(u) = 1/2 \int_0^T |\dot{u}|^2 - \int_0^T b(t)\, V(u)$$

on the space $H_T^1 = \{v \in H^1(0, T; \mathbf{R}^N): v(0) = v(T)\}$.

At this purpose, a basic lemma in the proofs of Theorems 1, 2 is given by the following result, which seems to be interesting by itself.

LEMMA 1. *Let b a T-periodic continuous real function satisfying either* (1) *or* (2), *let* $V \in C^2(\mathbf{R}^N; \mathbf{R})$, $V \geq 0$ *satisfy* (V_3) *and let there exist two numbers* $c \geq 0$, $d > 0$ *such that* (3), (4), (5) *hold. Then the functional f satisfies the Palais-Smale condition on* H_T^1.

At this point, in case that (1) holds, one checks that the functional f is *positive* on a sufficiently small sphere of the subspace $\tilde{H}_T^1$ of H_T^1 given by

$$\tilde{H}_T^1 = \left\{ u \in H_T^1 : \int_0^T u = 0 \right\}$$

and that f is negative on the constant functions of H_T^1: starting from these remarks, one is able to discover a geometrical structure of linking type for the functional f around the origin of H_T^1, so Lemma 1 enables to exhibit a critical point of linking type.

On the other side, when (2) holds, it is easy to see that f is positive on the constant functions of H_T^1 as well as on a suitably small sphere of $\tilde{H}_T^1$ and that f is negative somewhere in H_T^1.

Then a careful analysis of the different local behaviors of f on the constant functions and on $\tilde{H}_T^1$ and still Lemma 1 allow to get a critical point of Mountain Pass type.

Let us observe that, if $B^- = 0$, *i.e.* $b(t) \geqslant 0$ $\forall t \in [0, T]$, then conditions (3), (4), (5) are always satisfied with $c = 0$ and d arbitrary, so one gets a well known result due to Rabinowitz [10].

If $V'(x)x = \beta V(x)$ for $|x| \geqslant R$, that is V is homogeneous of degree $\beta > 2$, at least outside of a sphere of $\boldsymbol{R}^N$, then (3), (4) are still always verified with the choice $c = 0$.

In this latter case, for $R = 0$, that is in case that V is homogeneous on the whole space $\boldsymbol{R}^N$, some results were obtained by Lassoued in [9], under the further assumption that V is convex, and by Ben Naoum, Troestler and Willem in [3], in case that (2) holds, for homogeneous potentials V with any degree $\beta \neq 2$.

Finally we would point out the meaning of assumptions (3), (4), (5), when $B^- = 0$ or V is homogeneous. In this general situation, from one side, one requires that the function $g(x) = |x|^2/(V'(x)x - \beta V(x))$ has a positive lower bound l for $|x| \geqslant R$ on the other side B^- has to be sufficiently small with respect to l. In particular, condition (3) has been used in [1] for an analogous problem concerning elliptic equation and it plays a basic rule in the proof of Lemma 1.

2. The multiplicity results

In this section we deal with the problem of existence of multiple kT-periodic solutions of (V) with $k \in N$. We give two different results, assuming an evenness condition on V, that is

$$(V_6) \qquad V(-x) = V(x) \qquad \forall x \in \boldsymbol{R}^N$$

and a suitable symmetry condition on b, given by

$$(7) \qquad b(t) = b(T-t) \qquad \forall t \in [0, T/2].$$

Firstly, concerning with the case $k = 1$, one obtains a result assuming that the superquadratic growth conditions hold in the whole space $\boldsymbol{R}^N$ and that a suitable relation connecting V' and V'' is verified. Precisely one can prove the following

Theorem 3. *Let b a T-periodic continuous real function satisfying* (6), (7), *let* $V \in C^2(\boldsymbol{R}^N; \boldsymbol{R})$ *satisfy* (V_6) *and*

$$(V_7) \qquad \exists a_1 > 0, \quad \beta > 2\colon\ V(x) \geqslant a_1 |x|^\beta \qquad \forall x \in \boldsymbol{R}^N,$$

$$(V_8) \qquad \exists a_2 > 0\colon\ |V'(x)| \leqslant a_2 |x|^{\beta-1} \qquad \forall x \in \boldsymbol{R}^N,$$

$$(V_9) \qquad V'(x)x \geqslant \beta V(x) \qquad \forall x \in \boldsymbol{R}^N,$$

$$(V_{10}) \qquad V''(x)xx \geqslant (\beta - 1)\, V'(x)x \qquad \forall x \in \boldsymbol{R}^N,$$

and let there exist two numbers c', $d' \geq 0$ *such that*

(8) $$B^-(V'(x)x - \beta V(x)) \leq c'|x|^2 \quad \forall x \in \mathbf{R}^N,$$

(9) $$B^-(V''(x)xx - (\beta - 1)V'(x)x) \leq d'|x|^2 \quad \forall x \in \mathbf{R}^N,$$

(10) $$\max(2c', d') < 8(\beta - 2)/T^2$$

where $B^- = \max\{b^-(t), t \in [0, T]\}$, $b^-(t) = -\min(b(t), 0)$ $\forall t \in [0, T]$. *Then there exist infinitely many pairs of distinct T-periodic solutions* x *of* (V) *verifying the further property*

(11) $$x(t + T/2) = -x(T/2 - t) \quad \forall t \in [0, T/2].$$

Concerning with the problem of subharmonic solutions of (V), one can state the following result

THEOREM 4. *Let* b *a T-periodic continuous real function with minimal period* T *and satisfying* (6), (7), *let* $V \in C^2(\mathbf{R}^N, \mathbf{R})$ *be positively homogeneous of degree* $\beta > 2$, *that is*

(V_{11}) $$V(\lambda x) = |\lambda|^\beta V(x) \quad \forall x \in \mathbf{R}^N, \ \lambda \in \mathbf{R}$$

with $V(x) > 0$ *for* $x \neq 0$. *Moreover let* b *satisfy the further condition*

(12) *The set* $Z = \{t \in [0, T/2]: b(t) = 0$ *and* $\exists \delta = \delta(t) \in (0, \min(t, T/2 - t))$ *s.t.* $b(s_1)b(s_2) < 0$ $\forall s_1 \in (t - \delta, t)$, $\forall s_2 \in (t, t + \delta)\}$, *is finite.*

Then, for any $k \in \mathbf{N}$ *there exist infinitely many pairs of distinct kT-periodic solutions* $x^{(k)}$ *of* (V) *satisfying the further property*

(13) $$x^{(k)}(t + kT/2) = -x^{(t)}(kT/2 - t) \quad \forall t \in [0, kT/2].$$

Moreover, for any odd integer k, at least two distinct pairs of these solutions have minimal period kT.

The proofs of Theorems 3, 4 are based on the consideration of the Nehari's manifold M_k associated with the functional

$$f_k(u) = 1/2 \int_0^{kT/2} |\dot{u}|^2 - \int_0^{kT/2} b(t)V(u)$$

on the space $X_k = H_0^1(0, kT/2; \mathbf{R}^N)$ whose critical points yield, in a standard way, due to (V_6), the kT-periodic solutions of (V). The manifold M_k is defined precisely as $M_k = \{u \in X_k \setminus \{0\}: \langle f_k'(u), u \rangle = 0\}$ and the assumptions of Theorem 3 (so, in particular, the positivity and the homogeneity of V, required in Theorem 4) enable to state that M_k is a closed, regular manifold of X_k and that the critical points of f_k on X_k coincide with the critical points of $f_{k|M_k}$, the restriction of f_k on M_k.

Moreover, some minimization procedures and the application of a basic result of the Lusternik-Schrinelman theory enable to find infinitely many pairs of critical points of $f = f_1$ on $M = M_1$. So one proves Theorem 3, observing that (11) is a consequence of (7) and the way of constructing a solution in the H_0^1-framework in presence of even potentials.

As for Theorem 4, one is able to prove, by some delicate arguments with a lot of technicalities, that M_k is path connected in the homogeneous case, so one can construct a pair of minimum points of $f|_{M_k}$ (say, u_k and its opposite $-u_k$) as well as a critical point of Mountain Pass type, say $\tilde{u}_k$. It is easy to check, by a straightforward argument, the minimality of the period kT for the solution $x^{(k)}$ corresponding to u_k, while the same property for the solution $\tilde{x}^{(k)}$ corresponding to $\tilde{u}_k$ relies to an appropriate use of the well known Morse index estimates for critical points of Mountain Pass type due to Ekeland and Hofer (see [5, 6, 8]), suitably adapted to this «constrained» case.

Other multiplicity results have been obtained in [3, 4, 9], always related on the homogeneous case.

Some similar argument to those yielding the proof of Theorem 4 have been used in [7], where the potential V is positive.

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M. Girardi: Dipartimento di Matematica
Università degli Studi di Roma III
Via C. Segrè, 2/6 - 00146 Roma

M. Matzeu: Dipartimento di Matematica
Università degli Studi di Roma «Tor Vergata»
Via della Ricerca Scientifica - 00133 Roma