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Compact embedding theorems for generalized Sobolev spaces

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Analisi matematica. — *Compact embedding theorems for generalized Sobolev spaces.* Nota di MARIA MANFREDINI, presentata (*) dal Corrisp. B. Pini.

ABSTRACT. — In this *Note* we give some compact embedding theorems for Sobolev spaces, related to m-tuples of vectors fields of C^1 class on R^N.

KEY WORDS: Sobolev spaces; Compact embedding; Carathéodory-distance.

RIASSUNTO. — *Alcuni teoremi di immersione compatta per spazi di Sobolev generalizzati.* In questa *Nota* dimostriamo alcuni teoremi di immersione compatta per spazi di Sobolev, relativi a m-uple di campi vettoriali di classe C^1 su R^N.

1. INTRODUCTION

The aim of this *Note* is to establish some compact embedding theorems for Sobolev spaces related to a family of vector fields on an open subset of R^N.

More precisely, let $X = (X_1, \dots, X_m)$ be a m-tuple of vector fields, $X_j \in C^1(R^N, R^N)$, $j = 1, \dots, m$. Let Ω be an open, bounded or unbounded, subset of R^N and let $p \in [1, +\infty[$.

We denote by $\overset{\circ}{W}{}^p_X(\Omega)$ the subspace of $L^p(\Omega)$ obtained by completion of the space $C_0^1(\Omega)$ with respect to the norm:

$$\|u\|_{\overset{\circ}{W}{}^p_X(\Omega)} = \left(\int_\Omega \left(|u(x)|^p + \sum_{j=1}^m |X_j u(x)|^p\right) dx\right)^{1/p},$$

where we have identified the vector field $X_j = (b_{j1}, \dots, b_{jN})$ with the first order differential operator $\sum_{i=1}^N b_{ji}\partial_{x_i}$. We note that for every $u \in \overset{\circ}{W}{}^p_X(\Omega)$ there exists, in a weak sense, $X_j u \in L^p(\Omega)$ for every $j = 1, \dots, m$.

In this paper we give geometrical conditions on the open set Ω related to the vector fields $X_1, \dots, X_m$ and integral type inequalities which assure the compact embedding of $\overset{\circ}{W}{}^p_X(\Omega)$ in $L^p(\Omega)$.

This *Note* is organized as follows:

In section 2 we define the control distance associated to the vector fields $X_1, \dots, X_m$ and we prove some compact embedding theorems.

In section 3 we apply our results when:

(3*a*) The vector fields are invariant with respect to a group of translation. This example includes the Berger and Schechter's compactt embedding theorem, see [2], for the classical Sobolev space, ($X_j = \partial_{x_j}$, $j = 1, \dots, N$) and a theorem by Garofalo and Lanconelli, in [11], for the Sobolev space on the Heisenberg group.

(*) Nella seduta del 18 giugno 1993.

(3*b*) The vector fields are of Grushin type.

Moreover, we present an application of our results for a particular case when there are unbounded sets which have finite diameter with respect to the control distance.

2. Definitions and theorems

Let $X = (X_1, \dots, X_m)$ be a m-tuple of vector fields, $X_j \in C^1(R^N, R^N)$, $j = 1, \dots, m$.

Definition 2.1. *Let Ω be an open subset of R^N and let $p \in [1, +\infty[$. We say that Ω is in $\mathcal{K}_p(X)$ if $\overset{\circ}{W}{}^p_X(\Omega)$ is compactly embedded in $L^p(\Omega)$. If Ω_0 is an open subset of Ω we say that $\Omega_0 \in \mathcal{K}_p(X; \Omega)$ if the restriction $u \mapsto u|_{\Omega_0}$ is a compact operator from $\overset{\circ}{W}{}^p_X(\Omega_0)$ in $L^p(\Omega_0)$.*

In this *Note* we look for conditions on Ω to assure that Ω is in $\mathcal{K}_p(X)$.

The next notion of subunitary curve, introduced by Fefferman and Phong in the smooth case [4] and subsequently considered by Franchi and Lanconelli in [7], in non regular cases, is essential for our purposes.

Definition 2.2. *We say that a continuous curve γ: $[0, T] \to R^N$, with piecewise continuous first derivatives, is subunitary with respect to the vector fields $X_1, \dots, X_m$ if*

$$\dot{\gamma}(t) = \sum_{j=1}^{m} a_j(t) X_j(\gamma(t)) \quad \text{for almost every } t \in [0, T]$$

with a_j: $[0, T] \to R$, piecewise continuous and such that $|a_j(t)| \leq 1$ for every $j = 1, \dots, m$ and for every $t \in [0, T]$.

Let $x, y \in R^N$. If there exists (at least) a subunitary curve joining x to y, we define

$$d(x, y) = \inf\{T > 0; \text{ there exists } \gamma: [0, T] \to R^N, \text{ subunitary}, \ \gamma(0) = x, \ \gamma(T) = y\}.$$

If R^N is X-connected, that is for any $x, y \in R^N$ there exists a subunitary curve connecting x to y, then d is a distance on R^N. It is said *control-distance* or *Carathéodory-distance* associated to the vector fields $X_1, \dots, X_m$. We indicate by $B_d(x, r)$ the ball with center x and radius r with respect to the metric d:

$$B_d(x, r) = \{y \in R^N / d(x, y) < r\}.$$

Throughout this *Note* we shall suppose, about the distance d, the following covering property holds:

(C_r) *Let $r > 0$. There exists a positive integer $M = M(r)$ and a finite or countable family of d-balls $(B_d(x_i, r))_i$, which is a covering of R^N such that every point $x \in R^N$ is contained in at most M balls of the family $(B_d(x_i, 2r))_i$.*

All the results of this *Note* are consequences of the following key observation, which we call Main Lemma for reading convenience:

MAIN LEMMA. *Let Ω be an open subset of R^N. Suppose that, for a suitable fixed $r > 0$ the covering property* (C_r) *holds and for every $\varepsilon > 0$ there exists $\Omega_0 \in \mathcal{K}_p(X; \Omega)$, $\Omega_0 \subseteq \Omega$, such that for every $u \in C_0^1(\Omega)$ and for every d-balls $B_d(x, r)$ of the covering* (C_r):

$$(E_r) \qquad \int_{(\Omega \setminus \Omega_0) \cap B_d(x,r)} |u(y)|^p dy \leqslant \varepsilon \int_{B_d(x,2r)} \left(|u(y)|^p + \sum_{j=1}^m |X_j u(y)|^p \right) dy .$$

Then $\Omega \in \mathcal{K}_p(X)$.

NOTE. Here and in what follows, we denote by $C_0^1(\Omega)$ the linear space of the continuous differentiable functions on the whole R^N with compact support contained in Ω.

PROOF OF MAIN LEMMA. We begin by establishing the following statement which is a sufficient condition for precompactness of subsets of $L^p(\Omega)$.

CLAIM. *Let $T \subset L^p(\Omega)$ bounded. We suppose that for every $\varepsilon > 0$ there exist $\Omega_0, \Omega_1 \subset \Omega$ such that:*

(i) $\Omega = \Omega_0 \cup \Omega_1$;

(ii) the set $\{u|_{\Omega_0}, u \in T\}$ *is precompact in* $L^p(\Omega_0)$;

(iii) $\int_{\Omega_1} |u(y)|^p dy \leqslant \varepsilon$ *for every* $u \in T$.

Then T is precompact in $L^p(\Omega_0)$.

We assume this claim true for a moment.

Then, if $T \subset \overset{\circ}{W}{}^p_X(\Omega)$ is bounded, it is sufficient to prove that for any $\varepsilon > 0$ there exists a decomposition of Ω which satisfies (i)-(iii). Let ε be a fixed positive number and let $\Omega_0 \in \mathcal{K}_p(X; \Omega)$ be a open subset of Ω satisfying (E_r). If we prove that for every $u \in T$

$$(2.1) \qquad \int_{\Omega \setminus \Omega_0} |u(y)|^p dy \leqslant \varepsilon$$

then, since $\Omega_0 \in \mathcal{K}_p(X; \Omega)$, the decomposition $\{\Omega_0, \Omega \setminus \Omega_0\}$ of Ω satisfies (i)-(iii), so that T is precompact in $L^p(\Omega)$, hence $\Omega \in \mathcal{K}_p(X)$.

We prove (2.1). Let

$$T^* = \{u^* \in C_0^1(\Omega) / \|u - u^*\|_{\overset{\circ}{W}{}^p_X(\Omega)} \leqslant \varepsilon, \ \text{for some } u \in T\} .$$

If we show (2.1) for every $u^* \in T^*$ then (2.1) follows straightforwardly for every $u \in T$.

Let $\boldsymbol{B} = (B_r^{(i)}) = (B_d(x_i, r))$ denote a covering of R^N with balls of radius r, as in (C_r).

We have by hypothesis (E_r):

$$\int_{\Omega\setminus\Omega_0} |u^*(y)|^p dy = \int_{(\Omega\setminus\Omega_0)\cap\left(\bigcup_i B_r^{(i)}\right)} |u^*(y)|^p dy \leqslant \sum_i \int_{(\Omega\setminus\Omega_0)\cap B_r^{(i)}} |u^*(y)|^p dy \leqslant$$

$$\leqslant \varepsilon \sum_{i,\,(\Omega\setminus\Omega_0)\cap B_r^{(i)}\neq\emptyset} \int_{B_{2r}^{(i)}} \left(|u^*(y)|^p + \sum_{j=1}^m |X_j u^* y)|^p\right) dy\,.$$

Now since at most M of the sets $B_{2r}^{(i)}$ have nonempty intersection, we conclude that for every $u^* \in T^*$:

$$\int_{\Omega\setminus\Omega_0} |u^*(y)|^p dy \leqslant \varepsilon M \int_\Omega \left(|u^*(y)|^p + \sum_{j=1}^m |X_j u^*(y)|^p\right) dy\,,$$

which actually yields (2.1).

It remains to prove the claim.

Let $\varepsilon > 0$ fixed and let $\Omega_0, \Omega_1 \subset \Omega$ satisfying (i)-(iii). Now from (ii), there are $u_1, \ldots, u_m \in T$ such that for every $u \in T$ there is $i \in \{1, \ldots, m\}$ such that

$$\int_{\Omega_0} |u(y) - u_i(y)|^p dy \leqslant \varepsilon\,.$$

Then, from (i) and (iii)

$$\int_\Omega |u(y) - u_i(y)|^p dy \leqslant \int_{\Omega_0} |u(y) - u_i(y)|^p dy + \int_{\Omega_1} |u(y) - u_i(y)|^p dy \leqslant$$

$$\leqslant \varepsilon + 2^p \left(\int_{\Omega_1} |u(y)|^p dy + \int_{\Omega_1} |u_i(y)|^p dy\right) \leqslant 2^{p+2}\varepsilon\,.$$

This proves the claim and completes the proof of our Main Lemma.

To obtain more explicit compact embedding results, we first look for sufficient conditions for the covering property (C_r) holds.

The control distance d will be said verifying hypothesis (H_r) if:

(H_r) *The function* $(x, y) \mapsto d(x, y)$ *is continuous with respect to the Euclidean topology and there exists a constant* $D = D(r) > 0$ *such that*

(2.2) $$|B_d(x, 2r)| \leqslant D|B_d(x, r)| \quad \text{for every } x \in R^N\,.$$

$|\ |$ denotes the Lebesgue measure on R^N.

Covering lemma. *If* (H_r) *holds then covering hypothesis* (C_r) *is verified.*

Proof. The first part of the proof is essentially the same as in Lemma 3.2 in [11]. Since the distance d is continuous with respect to the Euclidean distance we can cover R^N with a finite or countable family of balls $\boldsymbol{B} = (B_d(x^i, r/3)) = (B_{r/3}^{(i)})$. We select from $\boldsymbol{B}$ a collection of disjoint balls $\boldsymbol{B}^* = (B_{r/3}^{(k_j)})$, such that $(B_r^{(k_j)})$ covers R^N, in the following way. We let $B_{r/3}^{(k_1)} = B_{r/3}^{(1)}$. Assume that $B_{r/3}^{(k_j)}$, $j = 1, \ldots, n$ have been chosen. If $B_{r/3}^{(i)} \cap$

$\cap B_{r/3}^{(k_j)} \neq \emptyset$ for every $i > k_n$ and for a suitable $j = 1, \dots, k_n$ then we have finished and in this case the family is finite. Otherwise we let

$$k_{n+1} = \min\{i; i > k_n, B_{r/3}^{(i)} \cap B_{r/3}^{(k_j)} = \emptyset \text{ for every } j = 1, \dots, k_n\}.$$

To show that the family $(B_r^{(k_j)})$ covers R^N we prove that each $B_{r/3}^{(i)}$ is contained in some $B_r^{(k_j)}$. We suppose that $B_{r/3}^{(i)}$ does not belong to $\boldsymbol{B}^*$. If $\boldsymbol{B}^*$ is infinite then $k_n \nearrow \infty$ so that there exists $n \in N$ such that $k_n \leqslant i < k_{n+1}$. But from definition of k_{n+1} there exists a $j \in \{1, \dots, n\}$ such that $B_{r/3}^{(i)} \cap B_{r/3}^{(k_j)} \neq \emptyset$, then $B_{r/3}^{(i)} \subseteq B_r^{(k_j)}$. If $\boldsymbol{B}^* = \{B_{r/3}^{(k_1)}, \dots, B_{r/3}^{(k_p)}\}$, then $B_{r/3}^{(i)} \cap B_{r/3}^{(k_j)} \neq \emptyset$ for every $i \geqslant k_n + 1$ and for some $j = 1, \dots, n$. On the other hand, if $k_j \leqslant i < k_{j+1}$, we can argue as before, and we reach the desired conclusion, in any case.

To finish we have to show the «M-intersection» property. Let x be in $B_{2r}^{(k_j)}$ for every $k_j \in I$, being $I \subseteq N$. Now $\bigcup_{k_j \in I} B_{2r}^{(k_j)} \subseteq B_d(x, 4r) \subseteq B_{6r}^{(k_i)}$ for every $k_i \in I$. Therefore for every $k_i \in I$

$$\sum_{k_j \in I} |B_{r/3}^{(k_j)}| = \left| \bigcup_{k_j \in I} B_{r/3}^{(k_j)} \right| \leqslant |B_d(x, 4r)| \leqslant |B_{6r}^{(k_i)}|.$$

Then (card I) $\min_{k_j \in I} |B_{r/3}^{(k_j)}| \leqslant |B_{6r}^{(k_i)}|$, for every $k_i \in I$. On the other hand, from (2.2) there is a positive constant $D = D(r)$ such that $|B_{6r}^{(k_j)}| \leqslant D|B_{r/3}^{(k_j)}|$ for every $k_j \in I$. This prove that card $I \leqslant D$ and completes the proof.

Some remarks about hypothesis (H_r) are now in order.

REMARK 2.1. If $(R^N, d, |\ |)$ is a homogeneous space in the sense of Coifman and Weiss (see [3]), that is there is a positive constant c such that

$$(2.3)\qquad |B_d(x, 2r)| \leqslant c|B_d(x, r)| \text{ for every } x \in R^N \text{ and for every } r > 0,$$

then evidently (2.2) holds for every $r > 0$. On the other hand (2.2) and (2.3) are not equivalent. In fact, let X_1, X_2 be the vector fields on R^2 defined by $X_1 = \partial_{x_1}$, $X_2 = \alpha(x_1)\partial_{x_2}$, where $\alpha(0) = 0$ and $\alpha(x_1) = \exp(-x_1^{-2})$ for $x_1 \neq 0$. Since the distance d is invariant with respect to the translation parallel to the x_2-axis (because such are X_1, X_2), and d is equivalent to Euclidean distance in $\{(x_1, x_2)/|x_1| \geqslant 1\}$, it is not difficult to shown that (2.2) holds.

Moreover we can prove that there exists $c > 0$ such that

$$|B_d(0, r)|/|B_d(0, 2r)| \leqslant c\exp(-1/2r) \text{ for every } r > 0.$$

From this inequality it follows immediately that (2.3) does not hold.

REMARK 2.2. If d is not continuous the compact embedding of $\overset{\circ}{W}{}^p_X(\Omega)$ in $L^p(\Omega)$ may fail, also in the case of bounded Ω. On the other hand, the continuity of the control distance d is not guaranteed in general by the only hypothesis of X-connection of R^N.

In fact, for example, if $\phi \in C^\infty(R^2, R)$, $\phi \geqslant 0$, supp $\phi \subseteq R^2 \setminus ([-1, 1] \times [-1, 1])$, then R^2 is connected with respect to the vector fields

$$X_1 = \partial_{x_1}, \qquad X_2 = \phi(x_1, x_2)\partial_{x_2}.$$

But, for all $h \neq 0$, $-1 < h < 1$, we can easily prove that $d((0,0),(0,h)) \geq 2$. So d is not continuous.

Moreover, if we consider the sequence

$$u_n(x_1, x_2) = n^{1/p} f(x_1 - 1/n) g(nx_2),$$

where $g \in C_0^\infty(R)$, supp $g \subseteq]-2, 2[$, $0 \leq g \leq 1$ and $g = 1$ near zero. $f \in C_0^\infty(R)$, supp$f \subseteq]-1, 1[$, $0 \leq f \leq 1$ and $f = 1$ near zero. It is easy to see that (u_n) is bounded in $\overset{\circ}{W}{}^p_X(\Omega)$, $\Omega = (-2, 2) \times (-2, 2)$. On the other hand, there are no subsequences of (u_n) which converge in $L^p(\Omega)$. Thus $\Omega \notin \mathcal{K}_p(X)$, for every $p \in [1, +\infty[$.

From Main Lemma we easily obtain the following:

Theorem 2.1. *Let Ω be an open subset of R^N. Suppose that, for a suitable fixed $r > 0$ the covering property* (C$_r$) *holds and the following hypotheses are satisfied:*

For every $\delta > 0$ there exists $\Omega_0 \in \mathcal{K}_p(X; \Omega)$, $\Omega_0 \subseteq \Omega$, such that

$$\text{(2.4)} \qquad \text{if } (\Omega \setminus \Omega_0) \cap B_d(x, r) \neq \emptyset \quad \text{then } \frac{|B_d(x,r) \cap \Omega|}{|B_d(x,r)|} \leq \delta;$$

For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{(2.5)} \qquad \int_{B_d(x,r)} |u(y)|^p dy \leq \varepsilon \int_{B_d(x,2r)} \left(|u(y)|^p + \sum_{j=1}^m |X_j u(y)|^p \right) dy;$$

For every $u \in C_0^1(\Omega)$ and $B_d(x,r)$ such that

$$\text{(2.6)} \qquad \frac{|B_d(x,r) \cap \operatorname{supp} u|}{|B_d(x,r)|} \leq \delta.$$

Then $\Omega \in \mathcal{K}_p(X)$.

Proof. We note that (2.4)-(2.6) imply condition (E$_r$) and, since convering property also holds, the theorem follows from Main Lemma.

The following corollaries as the condition (2.5)-(2.6) can be deduced from Poincaré-Sobolev type inequalities.

We use the notations

$$\fint_A f(x)\,dx = f_A = \frac{1}{|A|} \int_A f(x)\,dx.$$

Corollary 2.2. *Let Ω be an open subset of R^N. Suppose that the following Poincaré-Sobolev type inequality holds: for a suitable $r > 0$ there are a $k > 1$ and a constant $c = c(r) > 0$ such that*

$$\text{(2.7)} \qquad \left(\fint_{B_d(x,r)} |u(y)|^{kp} dy \right)^{1/kp} \leq c \left(\fint_{B_d(x,2r)} \sum_{j=1}^m |X_j u(y)|^p dy \right)^{1/p},$$

for every $x \in R^N$ *and for every* $u \in C^1(B_d(x, 2r))$ *such that*

$$|\{y \in B_d(x, r)/u(y) = 0\}| \geqslant |B_d(x, r)|/2 .$$

If (C_r) *and* (2.4) *hold then* $\Omega \in \mathcal{K}_p(X)$.

PROOF. We prove that, in this case, (2.5)-(2.6) holds with $\delta = \min\{1/2, \textit{pos. const.}\ \varepsilon^{k/(k-1)}\}$. Indeed, if E is the set of points of $B_d(x, r)$ where $u = 0$, then from Hölder inequality and (2.7), for every $u \in C_0^1(\Omega)$, we obtain

$$\left(\ \fint_{B_d(x,r)} |u(y)|^p\,dy\right)^{1/p} \leqslant \left(\frac{|B_d(x,r)\setminus E|}{|B_d(x,r)|}\right)^{(1-1/k)/p}\left(\ \fint_{B_d(x,r)} |u(y)|^{kp}\,dy\right)^{1/kp} \leqslant$$

$$\leqslant \left(\frac{|B_d(x,r)\cap \operatorname{supp} u|}{|B_d(x,r)|}\right)^{(1-1/k)/p}\left(\ \fint_{B_d(x,r)} |u(y)|^{kp}\,dy\right)^{1/kp} \leqslant$$

$$\leqslant c\delta^{(1-1/k)/p}\left(\ \fint_{B_d(x,2r)} \sum_{j=1}^{m} |X_j u(y)|^p\,dy\right)^{1/p} .$$

The conclusion of Corollary 2.2 is also true if it is satisfied a more familiar Poincaré-Sobolev inequality, precisely:

COROLLARY 2.3. *Let* Ω *be an open subset of* R^N. *Suppose that for a suitable* $r > 0$ *there are* $k > 1$ *and a constant* $c = c(r) > 0$ *such that*

$$\left(\ \fint_{B_d(x,r)} |u(y) - u_{B_d(x,r)}|^{kp}\,dy\right)^{1/kp} \leqslant c\left(\ \fint_{B_d(x,2r)} \sum_{j=1}^{m} |X_j u(y)|^p\,dy\right)^{1/p} \tag{2.8}$$

for every $x \in R^N$ *and* $u \in C^1(B_d(x, 2r))$. *If* (C_r) *and* (2.4) *hold then* $\Omega \in \mathcal{K}_p(X)$.

PROOF. We prove that (2.7) holds. Let $u \in C^1(B_d(x, 2r))$ such that $|E| = |\{y \in B_d(x, r)/u(y) = 0\}| \geqslant |B_d(x, r)|/2$. Thus by Hölder inequality

$$|u_{B_d(x,r)}| = \frac{1}{|E|}\int_E |u(y) - u_{B_d(x,r)}|\,dy \leqslant \frac{2}{|B_d(x,r)|}\int_{B_d(x,r)} |u(y) - u_{B_d(x,r)}|\,dy \leqslant$$

$$\leqslant \frac{2}{|B_d(x,r)|}\|u - u_{B_d(x,r)}\|_{L^{kp}(B_d(x,r))}\,|B_d(x,r)|^{1-1/kp} = 2\left(\ \fint_{B_d(x,r)} |u(y) - u_{B_d(x,r)}|^{kp}\,dy\right)^{1/kp} .$$

Then by (2.8)

$$\left(\ \fint_{B_d(x,r)} |u(y)|^{kp}\,dy\right)^{1/kp} \leqslant \left(\ \fint_{B_d(x,r)} |u(y) - u_{B_d(x,r)}|^{kp}\,dy\right)^{1/kp} + |u_{B_d(x,r)}| \leqslant$$

$$\leqslant 3\left(\ \fint_{B_d(x,r)} |u(y) - u_{B_d(x,r)}|^{kp}\,dy\right)^{1/kp} \leqslant 3c\left(\ \fint_{B_d(x,2r)} \sum_{j=1}^{m} |X_j u(y)|^p\,dy\right)^{1/p},$$

that is (2.7).

Finally we prove that condition (2.5)-(2.6) is satisfied if a Sobolev inequality holds for $\overset{\circ}{W}{}^p_X(B_d(x,r))$ and if there exist suitable cut-off functions.

COROLLARY 2.4. *Let Ω be an open subset of R^N. Suppose that for a suitable $r>0$ there exist $q>p$ and a constant $c=c(r)>0$ such that*

$$\left(\fint_{B_d(x,\,2r)} |u(y)|^q\,dy\right)^{1/q} \leq c(r)\left(\fint_{B_d(x,\,2r)} \sum_{j=1}^m |X_j u(y)|^p\,dy\right)^{1/p}, \tag{2.9}$$

for every $x\in R^N$ and $u\in C_0^1(B_d(x,2r))$.

Moreover we assume that for every $x\in R^N$ there exists $\psi\in C_0^1(B_d(x,2r))$, such that $\psi=1$ in $B_d(x,r)$, $0\leq\psi\leq 1$ and

$$\sum_{j=1}^m |X_j\psi(y)|^p \leq c(r) \qquad \textit{for every } y\in B_d(x,2r). \tag{2.10}$$

If (C_r) *and* (2.4) *hold then $\Omega\in\mathcal{K}_p(X)$.*

PROOF. We prove that in this case (2.5)-(2.6) holds with $\delta=$ *post. const.* $\varepsilon^{q/(q-p)}$, so that from Theorem 2.1, our corollary follows.

Let $u\in C_0^1(\Omega)$ and let ψ be as before. If $E=\{y\in B_d(x,r)/u(y)=0\}$, by Hölder inequality, (2.9) and (2.10) we have:

$$\begin{aligned}
\int_{B_d(x,r)} |u(y)|^p\,dy &\leq |B_d(x,r)\setminus E|^{1-p/q}\left(\int_{B_d(x,r)}|u(y)|^q\,dy\right)^{p/q}\leq\\
&\leq |B_d(x,r)\setminus E|^{1-p/q}\left(\fint_{B_d(x,\,2r)}|\psi(y)u(y)|^q\,dy\right)^{p/q}|B_d(x,2r)|^{p/q}\leq\\
&\leq c|B_d(x,r)\setminus E|^{1-p/q}\left(\fint_{B_d(x,\,2r)}\sum_{j=1}^m|X_j(\psi(y)u(y))|^p\,dy\right)|B_d(x,2r)|^{p/q}\leq\\
&\leq 2^p c\left(\frac{|B_d(x,r)\cap \operatorname{supp} u|}{|B_d(x,r)|}\right)^{1-p/q}\int_{B_d(x,\,2r)}\left(c\,|u(y)|^p+\sum_{j=1}^m|X_j u(y)|^p\right)dy\,.
\end{aligned}$$

Then if $\dfrac{|B_d(x,r)\cap \operatorname{supp} u|}{|B_d(x,r)|}\leq\delta$, we obtain

$$\int_{B_d(x,r)}|u(y)|^p\,dy\leq c'\delta^{1-p/q}\int_{B_d(x,\,2r)}\left(|u(y)|^p+\sum_{j=1}^m|X_j u(y)|^p\right)dy\,,$$

where c' is a positive constant depending only on p and c.

From this inequality, if $c'\delta^{1-p/q}=\varepsilon$, we immediately obtain (2.5)-(2.6).

3. Some examples

3a. *Invariant vector fields.*

Let $X_1, \dots, X_m$ be $C^\infty(R^N, R^N)$ vector fields satisfying Hörmander's condition that is: rank $Lie[X_1, \dots, X_m](x) = N$, at every point $x \in R^N$. Here $\mathcal{L} = Lie[X_1, \dots, X_m]$ denotes the Lie algebra generated by $X_1, \dots, X_m$. We suppose that $\mathcal{L}$ is nilpotent and has dimension N.

Then there exists a group $G = (R^N, \circ)$ such that the vector fields $X_1, \dots, X_m$ are invariant with respect to the left translation of G, (see Proposition 8.41 in [10]). Moreover the exponential map is merely the identity, so that the Lebesgue measure on R^N is a Haar measure on G, see Proposition 1.2 in [6].

In this case R^N is X-connected (see the Theorem of Chow-Hermann in [12, chapter 18]), and the control distance d is invariant with respect to the left translation of G. On the other hand the distance d is locally Hölder continuous, see [4].

Moreover, as a consequence of $|B_d(x, r)| = |B_d(0, r)|$ for every $x \in R^N$, we straightforwardly obtain (2.2), in (H_r), choosing $D(r) = |B_d(0, 2r)| / |B_d(0, r)|$, with $r \leq r_0$, r_0 such that $|B_d(0, 2r_0)| < +\infty$.

Then, from Covering lemma, (C_r) holds, for every $r \leq r_0$.

Now, for a fixed $r > 0$, choose a function $\phi \in C_0^\infty(B_d(0, 2r))$ such that $\phi = 1$ on $B_d(0, r)$, $0 \leq \phi \leq 1$. Therefore, if we define $\psi(y) = \phi(x^{-1} \circ y)$, ψ is a cut-off function as required by Corollary 2.4.

In addition Sobolev inequality (2.9) is true on the ball $B_d(x, r)$ with center $x = 0$, see [5], so again by translation, it holds with a constant independing on the center of the ball, as required by (2.9).

Finally, if $\tilde{\Omega}$ is an open bounded set then $\overset{\circ}{W}{}^p_X(\tilde{\Omega})$ is continuously embedded in a classical Sobolev space $\overset{\circ}{W}{}^{p,\varepsilon}(\tilde{\Omega})$, $0 < \varepsilon \leq 1$ suitable (Theorem 4.16 in [5]). Then, since $\overset{\circ}{W}{}^{p,\varepsilon}(\tilde{\Omega})$ is compactly embedded in $L^p(\tilde{\Omega})$, we get that $\overset{\circ}{W}{}^p_X(\tilde{\Omega})$ is compactly embedded in $L^p(\tilde{\Omega})$. As a consequence we easily obtain that $\Omega_0 \in \mathcal{K}_p(X; \Omega)$ for every bounded open set $\Omega_0 \subseteq \Omega$.

Then, from Corollary 2.4 we have:

Theorem 3.1. *Let Ω be an open subset of R^N and let $p \in [1, +\infty[$. If, for a suitable fixed r, $0 < r \leq r_0$ for every ε, there exists a bounded set K such that*

$$(3.1) \qquad (\Omega \setminus K) \cap B_d(x, r) \neq \emptyset \quad \textit{implies} \quad |B_d(x, r) \cap \Omega| \leq \varepsilon,$$

then $\Omega \in \mathcal{K}_p(X)$.

We note explicitely that, if $X_j = \partial_{x_j}$, $j = 1, \dots, N$ (then the space space $\overset{\circ}{W}{}^p_X(\Omega)$ is the classical Sobolev space and the control distance is the Euclidean distance), (3.1) is the Berger and Schechter's thinness condition for compact embedding theorem, see [2].

Moreover, if we consider the following vector fields on $R^n \times R^n \times R = R^N$:

$$X_j = \partial_{x_j} + 2y_j \partial_t, \qquad Y_j = \partial_{y_j} - 2x_j \partial_t, \qquad j = 1, \dots, n, \qquad (x, y, t) \in R^N,$$

then G is the Heisenberg group $\boldsymbol{H}_n$ and Theorem 3.1 give back a result proved by Garofalo and Lanconelli (Theorem 3.3 in [11]).

3b. Vector fields of Grushin type.

We consider the following vector fields on $R^n \times R^m = R^N$:

$$X_i = \partial_{x_i},\ Y_j = |x|^\alpha \partial_{y_j}, \quad i=1,\dots,n,\ j=1,\dots,m,\ \alpha>0,\ z=(x,y)\in R^n\times R^m.$$

Franchi and Lanconelli in [7] wrote explicitely the control distance d and proved that d is locally Hölder continuous, the d-balls are bounded and verify the doubling property. More precisely, there are $c_1, c_2 > 0$ such that, for every $z = (x, y) \in R^N$

$$c_1 \leqslant \frac{|B_d(z,r)|}{r^{n+m}(|x|+r)^{\alpha m}} \leqslant c_2\,, \tag{3.2}$$

(Theorem 2.7 in [7]). We remark that the constants c_1, c_2 are independing on z and r. Then the control distance d verifies hypothesis (H_r) for every $r > 0$.

Moreover, Theorem 2.6 in [8], assure that every bounded open subset of R^N is in $\mathcal{K}_p(X)$ while (2.7) is granted by Theorem 4.1 in [9].

Then, by Corollary 2.2 and (3.2) we have:

THEOREM 3.2. *Let Ω be an open subset of R^N and let $p \in [1, +\infty[$. If, for a suitable fixed $r > 0$,*

$$\frac{|\Omega \cap B_d(z,r)|}{(|x|+r)^{\alpha m}} \to 0 \quad \text{as } |z| \to \infty\,,$$

then $\Omega \in \mathcal{K}_p(X)$.

3c. We consider in R^2 the following vector fields:

$$X_1 = (1+x_1^2)\,\partial_{x_1}, \qquad X_2 = \partial_{x_2}, \qquad (x_1, x_2) \in R^2.$$

We prove the following

THEOREM 3.3. *Let Ω be an open subset of R^2 and let $p \in [1, +\infty[$. If, for every $L > 0$, we have:*

$$\lim_{|\lambda| \to +\infty} |\Omega \cap (]-L, L[\times]\lambda - \pi, \lambda + \pi[)| = 0 \tag{3.3}$$

then $\Omega \in \mathcal{K}_p(X)$.

First of all we remark that, in this case we can explicitely compute the control distance d:

$$d((x_1,x_2),(x_1',x_2')) \cong \max\{|\arctan x_1 - \arctan x_1'|,\ |x_2 - x_2'|\} = d_1((x_1,x_2),(x_1',x_2'))\,.$$

Then for big x_1 the d-balls of center (x_1, x_2) are not bounded and the d_1-ball with center (x_1, x_2) and radius $r = \pi$ is $S_{x_2} = R \times]x_2 - \pi, x_2 + \pi[$.

So that there are *unbounded sets with bounded diameter*, with respect to the control metric. Moreover, it easy to see that, with respect to the d_1-balls, the covering property (C_r) holds for every $r > 0$.

Now, we prove that the sets $\Omega_h = R \times]-h, h[$, $h > 0$, are in $\mathcal{K}_p(X)$, for every $p \in$

$\in [1, +\infty[$. We introduce the trasformation

$$T\colon R^2 \to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\times R$$

$$(x_1, x_2) \mapsto (\arctan x_1, x_2).$$

If we put $\tilde{\Omega}_b = T(\Omega_b)$ then compact embedding of $\overset{\circ}{W}{}^p_X(\Omega_b)$ in $L^p(\Omega_b)$is equivalent to the compact embedding of $\overset{\circ}{W}{}^p_\omega(\tilde{\Omega}_b)$ in $L^p_\omega(\tilde{\Omega}_b)$, where $\overset{\circ}{W}{}^p_\omega(\tilde{\Omega}_b)$, $L^p_\omega(\tilde{\Omega}_b)$ are the (classical) weighted Sobolev and L^p spaces, respectively, with weight $\omega(\xi, \eta) = 1 + \tan^2 \xi$. Now $\tilde{\Omega}_b$ is a bounded set of $]-\pi/2, \pi/2[\times R$, then from Theorem 1.9 in [1], $\overset{\circ}{W}{}^p_\omega(\tilde{\Omega}_b)$ is compactly embedded in $L^p_\omega(\tilde{\Omega}_b)$ that is $\Omega_b \in \mathcal{K}_p(X)$.

Now we are in position to prove Theorem 3.3.

Proof of the Theorem 3.3. Because condition (C_r) is satisfied for every $r > 0$, it is enough to prove condition (E_r) of Main Lemma for $r = \pi$. Now from the previous remark $S_\lambda \in \mathcal{K}_p(X)$, so that (by the translation invariant with respect to x_2) for every $\varepsilon > 0$ there exists $L > 0$ such that

$$\int_{S_\lambda \cap \{|x_1| \geq L\}} |u(y)|^p dy \leq \frac{\varepsilon}{2} \int_{S_\lambda} (|u(y)|^p + |D_X u(y)|^p) dy, \tag{3.4}$$

for every $u \in C^1(S_\lambda)$ and for every $\lambda \in R$. Here and in the following we use the notation $|D_X u|^p$ for $\sum_{j=1}^{2} |X_j u|^p$.

We choose a covering (S_{λ_i}) of R^2 by d_1-balls $S_{\lambda_i} = B_{d_i}((0, \lambda_i), \pi)$.

Let $\varepsilon > 0$ be fixed. We choose $L > 0$ as in (3.4) and $\delta > 0$ as we will fix later. From (3.3) there is a $\bar{\lambda} > 0$ such that

$$|\Omega \cap (]-L, L[\times]\lambda - \pi, \lambda + \pi[)| \leq \delta \quad \text{for every } \lambda \in R, \ |\lambda| \geq \bar{\lambda}. \tag{3.5}$$

We put $\Omega_0 = R \times]-\bar{\lambda} - \pi, \bar{\lambda} + \pi[$. Then $\Omega_0 \cap \Omega \in \mathcal{K}_p(X; \Omega)$, since Ω_0 is of Ω_b type. Therefore, if $(\Omega \setminus \Omega_0) \cap S_{\lambda_i} \neq \emptyset$ then $|\lambda_i| \geq \bar{\lambda}$. Now

$$\int_{(\Omega\setminus\Omega_0) \cap S_{\lambda_i}} |u(y)|^p dy =$$

$$= \int_{((\Omega\setminus\Omega_0) \cap S_{\lambda_i}) \cap \{|x_1| \geq L\}} |u(y)|^p dy + \int_{((\Omega\setminus\Omega_0) \cap S_{\lambda_i}) \cap \{|x_1| \leq L\}} |u(y)|^p dy = I_1 + I_2.$$

We estimate I_1 and I_2 separately.

From (3.4):

$$I_1 \leq \int_{S_{\lambda_i} \cap \{|x_1| \geq L\}} |u(y)|^p dy \leq \frac{\varepsilon}{2} \int_{S_{\lambda_i}} (|u(y)|^p + |D_X u(y)|^p) dy.$$

Moreover, from the classical Sobolev inequality, if we put $Q^i_L =]-L, L[\times]\lambda_i - \pi, \lambda_i + \pi[$, there exist $k > 1$ and a positive constant $c = c(L) = c(\varepsilon) > 0$ independing on i, such that

$$\left(\fint_{Q_L^i} |u(y)|^{kp}\,dy\right)^{1/kp} \leqslant c\left(\fint_{Q_L^i} (|u(y)|^p + |Du(y)|^p)\,dy\right)^{1/p},$$

where Du is the classical gradient. Then from Hölder inequality:

$$\left(\fint_{Q_L^i} |u(y)|^p\,dy\right)^{1/p} \leqslant \left(\frac{|Q_L^i \cap \operatorname{supp} u|}{|Q_L^i|}\right)^{(1-1/k)/p}\left(\fint_{Q_L^i} |u(y)|^{kp}\,dy\right)^{1/kp} \leqslant$$

$$\leqslant c(|Q_L^i \cap \operatorname{supp} u|)^{(1-1/k)/p}\left(\fint_{Q_L^i} (|u(y)|^p + |Du(y)|^p)\,dy\right)^{1/p}.$$

Since $|Q_L^i \cap \operatorname{supp} u| \leqslant |Q_L^i \cap \Omega|$, if we choose δ in (3.5) such that $c^p(\varepsilon)\,\delta^{1-1/k} \leqslant \varepsilon/2$, we thus obtain

$$I_2 \leqslant \frac{\varepsilon}{2}\int_{Q_L^i} (|u(y)|^p + |Du(y)|^p)\,dy \leqslant \frac{\varepsilon}{2}\int_{S_{\lambda_i}} (|u(y)|^p + |D_X u(y)|^p)\,dy\,.$$

Therefore we finally get

$$\int_{(\Omega\setminus\Omega_0)\cap S_{\lambda_i}} |u(y)|^p\,dy = I_1 + I_2 \leqslant \varepsilon\int_{S_{\lambda_i}} (|u(y)|^p + |D_X u(y)|^p)\,dy\,.$$

This proves (E_r) with $r = \pi$ and completes the proof of Theorem 3.3.

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