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Analisi matematica. — *Some perturbation results for non-linear problems.* Nota di
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ABSTRACT. — We discuss the existence of closed geodesic on a Riemannian manifold and the existence of periodic solution of second order Hamiltonian systems.

KEY WORDS: Critical point theory; Closed geodesics; Second order Hamiltonian systems.

RIASSUNTO. — *Alcuni risultati di perturbazione per problemi non-lineari.* Viene discussa l'esistenza di geodetiche chiuse su una varietà Riemanniana compatta e l'esistenza di moti periodici per sistemi Hamiltoniani del second'ordine.

1. INTRODUCTION

In this paper we prove two multiplicity results for two classical nonlinear problems: the existence of closed geodesics on a Riemannian manifold and the existence of periodic solutions of second-order Hamiltonian systems.

About the former we show:

THEOREM 1. *Let* $W \in C^2(\boldsymbol{R}^{n+1}; \mathcal{L}_s(\boldsymbol{R}^{n+1}; \boldsymbol{R}^{n+1}))$ *and consider the space* $\boldsymbol{R}^{n+1}$ *endowed with the scalar product*:

$$(v\,|\,w)_x = v \cdot w + \varepsilon W(x)\, v \cdot w\,. \tag{1}$$

Let $S^n = \{x \in \boldsymbol{R}^{n+1}\colon\ |x| = 1\}$ *and* $N_\varepsilon = (S^n, (\cdot\,|\,\cdot))$ *unit sphere endowed with the Riemannian structure inherited from the scalar product* (1). *There exist* $n+1$ *geometrically distinct closed geodesics on the Riemannian manifold* N_ε, *provided that* $\varepsilon > 0$ *is small enough*.

As far as we know, this theorem is not contained in the classical results about closed geodesics, see for example [7].

The second theorem deals with brake orbits of prescribed energy for second-order systems

$$\ddot{q} + V'(q) = 0\,, \qquad q(t) \in \boldsymbol{R}^n\,. \tag{2}$$

A *brake orbit* for (2) is a solution of (2) such that, for some $T > 0$,

$$\dot{q}(0) = \dot{q}(T) = 0\,. \tag{3}$$

Let us remark that if q is a brake orbit then $\tilde{q}(t) := q(|t|)$ is a periodic solution of (2).

THEOREM 2. *Let* $V_\varepsilon(x) = (1/2)Ax \cdot x + \varepsilon R(x)$, *where* A *is a symmetric* $n \times n$ *matrix* (*that, with no loss of generality, can be supposed diagonal*) *such that* $\det A \neq 0$, *and* $R \in C^2(\boldsymbol{R}^n; \boldsymbol{R})$. *Let* $0 < \omega_1^2 < \ldots < \omega_m^2$ *be the positive eigenvalues of* A *and*

(*) Nella seduta del 24 aprile 1993.

$X_j = \ker(A - \omega_j^2 I)$, $j = 1, \dots, m$, *be the corresponding eigenspaces of dimension respectively* $\alpha_1, \dots, \alpha_m$.

Moreover let the following non-resonance condition hold:

$$(\omega_i / \omega_j) \notin \mathbf{Z}, \qquad \forall i \neq j. \tag{4}$$

For every $E > 0$ *the equation* $\ddot{q} + V_\varepsilon'(q) = 0$ *has at least* $\alpha = \sum \alpha_j$ *brake orbits of energy* $1/2\, |\dot{q}|^2 + V_\varepsilon(q) = E$, *provided that* $\varepsilon > 0$ *is small.*

The existence of multiple periodic solutions of Hamiltonian systems (even more general than (2)) has been studied in [6, 10], see also [3]. For instance the result of Ekeland and Lasry [6], in the case of second-order systems like (2), claims the following. Let $H(p, q) = (1/2)\,|p|^2 + V(q)$ and $\Sigma_E := \{(p,q) \in \boldsymbol{R}^{2n} : H(p,q) = E\}$ and assume:

i) $\Sigma_E = \partial\Omega$, and Ω is a convex open set containing 0, and $H'(x)\cdot x > 0$, $\forall x \in \Sigma_E$.

ii) $R^2 < 2r^2$, where $r^2 := \min\{|p|^2 + |q|^2 : (p, q) \in \partial\Omega\}$ and $R^2 := \max\{|p|^2 + |q|^2 : (p, q) \in \partial\Omega\}$.

Then (2) has at least n distinct periodic solutions of energy $E > 0$.

One the one hand this result is more general than ours because the potential is not necessarily of the type $V(x) = (1/2) Ax\cdot x + \varepsilon R(x)$.

On the other hand:

1) If we perturb a quadratic potential $Q(x) = (1/2) Ax\cdot x$, where A is a matrix with some negative eigenvalue, the energy surface Σ_E of the perturbed problem will not, in general, be the boundary of a convex open set (in fact, it will not even be bounded).

2) If $V(x) = (1/2) Ax\cdot x$ where A is a positive defined matrix, the condition $R^2 < 2r^2$ implies that $r^2 \leqslant \omega_1^2 \leqslant \omega_n^2 \leqslant R^2 < 2r^2$.

This condition is obviously stronger than (4).

Essentially the same remarks can be made comparing Theorem 2 to the result about brake orbits obtained by Szulkin in [10].

The common feature of the two problems we will discuss is that they are perturbation problems variational in nature, namely their solution can be found looking for the critical points of some suitable functionals. As a matter of fact, they are both obtained exploiting an abstract critical point theory theorem contained in [4].

2. Preliminaries and notations

Let X be a topological space. Let us denote by $\operatorname{cat}(X)$ the Ljusternik-Schnirelman category of X with respect to itself, namely the least integer k such that $X \subset C_1 \cup \dots \cup C_k$, where every C_i is a closed subset of X and is contractible to a point in X. (For an exposition of the Ljusternik-Schnirelman theory see for example [9]). We recall the following classical result:

THEOREM 3. *Let Z be a compact manifold and $f \in C^1(Z; \mathbf{R})$. Then f has at least* cat(Z) *critical points on* Z.

The following definition and theorem are taken from [4].

DEFINITION 4. *Let M be a Hilbert manifold, $Z \subset M$ a compact connected submanifold of class C^2 and let $f \in C^2(M, \mathbf{R})$.*

Z is a non-degenerate critical manifold for f if

1) *All the points of Z are critical points of f.*
2) $\ker f''(z) = T_z Z$. *(Where $T_z Z$ denotes the tangent space to Z in z).*
3) *$f''(z)$ is a Fredholm operator of index zero.*

Let $f \in C^2(\mathbf{R} \times M, \mathbf{R})$ and let $f_\varepsilon(x) := f(\varepsilon, x)$. Moreover, let $f(x) = f(0, x)$.

THEOREM 5. *Let Z be a non-degenerate critical manifold for $f = f(0, \cdot)$, then there exists some $\bar{\varepsilon} > 0$ such that for every ε: $|\varepsilon| < \bar{\varepsilon}$ f_ε has* cat(Z) *critical points near Z.*

REMARKS. 1) Actually [4] deals with the case where $M = H$ is a Hilbert space. It is easy to check that the results hold true in the above more general formulation as well.

2) Let be given on M the action of a compact group G and let $f(\varepsilon, \cdot)$ be G-invariant. Then Z is invariant as well, and we can define the G-category of Z as the minimum number of closed G-contractible (1) sets covering Z. In this case, using the same techniques, we obtain that f_ε has G-cat(Z) critical points which are distinct modulo G. Remark that G-cat$(Z) \geq$ cat(Z/G).

Similarly, if instead of the G-category we consider i_G an index for G we obtain that f_ε has i_G critical points which are distinct modulo G. For the index theory we again refer to [9, p. 91 and ff].

3. CLOSED GEODESICS ON A SPHERE WITH A PERTURBED METRIC

Let $\mathcal{S}$ be a Riemannian manifold endowed with the scalar product $(\,|\,)$. A *closed geodesic* is a critical point of the functional

$$\int_0^{2\pi} (\dot{\gamma} | \dot{\gamma})^2 \, dt \quad \text{on} \quad H^1(S^1, \mathcal{S}).$$

Let $H := \{\gamma \in H^1([0, 2\pi]; \mathbf{R}^{n+1}) \colon \gamma(0) = \gamma(2\pi)\}$, and $M = \{\gamma \in H \colon \gamma(t) \in S^n \ \forall t \in [0, 2\pi]\}$. Consider $f_\varepsilon \in C^2(H; \mathbf{R})$ defined by:

$$\gamma \mapsto \int_0^{2\pi} |\dot{\gamma}|^2 \, dt + \varepsilon \int_0^{2\pi} (W(\gamma)\, \dot{\gamma} \cdot \dot{\gamma}) \, dt \, .$$

(1) A set C is G-contractible in Z if there exists an homotopy $h \in C([0, 1] \times C; Z)$ such that $h(t, \cdot)$ is G-equivariant and $h(1, C)$ is the G-orbit of one single point.

Theorem 1 will be proved by an application of Theorem 5. First we study the critical points of the unperturbed functional f.

LEMMA 6. *For all* $m \in \mathbf{Z}$ *the sets* Z_m *defined by*

$$Z_m = \{z \in M \,|\, z(t) = x\cos mt + y\sin mt\colon x, y \in \boldsymbol{R}^{n+1}\colon\ |x| = |y| = 1,\ x\cdot y = 0\}$$

are non-degenerate critical manifolds for f.

PROOF. It is easy to see that Z_m is a smooth compact manifold and $f'_{|M}(z) = 0$, $\forall z \in Z_m$.

If z is a geodesic, then

$$f''_{|M}(z)[h, k] = \int_0^{2\pi} [\dot{h}\cdot\dot{k} - |\dot{z}|^2 h\cdot k]\, dt\,, \qquad \forall h, k \in T_z M\,.$$

Moreover, the fact that $f''(z)$ is a Fredholm operator of index zero can be easily verified.

Now we claim that: $T_z Z_m = \ker f''_{|M}$.

Let us recall that $h \in \ker f''_{|M}(z)$ if and only if

$$\int_0^{2\pi} \dot{h}\cdot\dot{k}\, dt = \int_0^{2\pi} |\dot{z}|^2 h\cdot k\, dt\,, \qquad \forall k \in T_z M\,. \tag{5}$$

Let $S = \langle z(0); \dot{z}(0)\rangle$ be the span of the two vectors $z(0)$, $\dot{z}(0)$ and let $\{e_2, \ldots, e_n\}$ be an orthonormal base for $S^\perp$. For every $z \in Z_m$ let us take a base $\{e_i(t)\}_{i=1,\ldots,n}$ of the tangent space to S^n in $z(t)$ defined by:

$$e_i(t) = \begin{cases} \dot{z}(t)/m\,, & i = 1\,,\\ e_i\,, & i \neq 1\,.\end{cases}$$

Then, if $h, k \in T_z M$, one has

$$h(t) = \sum_{i=1}^n h_i(t)\, e_i(t) \qquad k(t) = \sum_{j=1}^n k_j(t)\, e_j(t)$$

where $h_i(t) = (h(t)\cdot e_i(t))$ and $k_j(t) = (k(t)\cdot e_j(t))$. From (5) it follows that

$$\int_0^{2\pi}\left\{\left[\sum_{i=1}^n (\dot{h}_i e_i + h_i \dot{e}_i)\right]\left[\sum_{j=1}^n (\dot{k}_j e_j + k_j \dot{e}_j)\right] - m^2\left[\sum_{i=1}^n h_i e_i\right]\left[\sum_{j=1}^n k_j e_j\right]\right\} dt = 0\,.$$

Therefore

$$\int_0^{2\pi}\left\{\sum_{i=1}^n \dot{h}_i\dot{k}_i + m^2 h_1 k_1 - m^2\sum_{i=1}^n h_i k_i\right\} dt = \int_0^{2\pi}\left\{\sum_{i=1}^n \dot{h}_i\dot{k}_i - m^2\sum_{i=2}^n h_i k_i\right\} dt = 0\,.$$

By regularity $h \in C^2([0, 2\pi]; \boldsymbol{R}^n)$ and is a solution of

$$\begin{cases} \ddot{h}_1 = 0\,,\\ \ddot{h}_j + m^2 h_j = 0\,, \qquad j = 2, \ldots, n\,.\end{cases}$$

Then, since in addition $h(0) = h(2\pi)$, one finds

$$\begin{cases} h_1(t) = \lambda_1\,, \\ h_j(t) = \lambda_j \cos mt + \mu_j \sin mt\,, \qquad j = 2, \dots, n\,. \end{cases}$$

And thus,

$$h(t) = \frac{\lambda_1}{m}\,\dot z(t) + \sum_{j=2}^{n} (\lambda_j \cos mt + \mu_j \sin mt)\, e_j\,.$$

As $h(t) \cdot z(t) = 0\ \forall t \in [0, 2\pi]$, it is immediate to see that $h \in T_z Z_m$. Since obviously $T_z Z_m \subset \ker f''_{|M}$, the lemma follows.

PROOF OF THEOREM 1. Let us point out that the functional $f(\varepsilon, \cdot)$ is invariant under the $O(2)$-action:

$$\{\pm 1\} \times S^1 \times H \to H\,, \qquad (\pm 1, \theta, \gamma) \mapsto \gamma(\pm t + \theta)\,.$$

And hence M and Z_m are invariant as well. Lemma 6 allows us to apply Theorem 5 and for all $m \in \boldsymbol{N}$ there are at least $O(2)$-cat(Z_m) critical points (which are distinct modulo $O(2)$) near the critical manifold Z_m. They correspond to geometrically distinct closed geodesics. It is known (see [8]) that

$$O(2)\text{-cat}(Z_m) \geqslant \operatorname{cat}(Z_m / O(2)) \geqslant \operatorname{cuplength}(Z_m / O(2)) + 1\,.$$

Moreover, Z_m is diffeomorphic to the unit tangent bundle

$$T_1 S^n := \{(x, y) \in \boldsymbol{R}^{n+1} \times \boldsymbol{R}^{n+1} \colon |x|^2 = |y|^2 = 1\,, (x \cdot y) = 0\}\,,$$

and hence, using a result of [1, p. 151], cuplength $(Z_m / O(2)) =$ cuplength $(T_1 S^n / O(2)) \geqslant n$. Hence $O(2)$-cat$(Z_m) \geqslant n + 1$. This ends the proof of Theorem 1.

4. EXISTENCE OF BRAKE ORBITS

Given $E > 0$, let $H = H^1([0, 1]; \boldsymbol{R}^n)$ and consider the functional $f_\varepsilon \in C^2(H, \boldsymbol{R})$,

$$f_\varepsilon(u) = \frac{1}{2}\left(\int_0^1 |\dot u|^2 dt\right)\left(\int_0^1 [E - V_\varepsilon(u)]\, dt\right).$$

It is well known (see for instance [5]) that, if $u \in H$ is a critical point for the functional f_ε and $f_\varepsilon(u) > 0$ then

$$\text{(6)} \qquad \begin{cases} \ddot u + T^2 V_\varepsilon(u) = 0\,, \\ h(1) \cdot \dot u(1) - h(0) \cdot \dot u(1) = 0\,, \end{cases}$$

where

$$T^2 = \left(\frac{1}{2} \int_0^1 |\dot u|^2\, dt\right) \Big/ \left(\int_0^1 [E - V_\varepsilon(u)]\, dt\right),$$

and $q(t) = u(t/T)$ is a brake orbit of (2).

The original problem is again reduced to the study of the critical points of the functional f_ε. Here the unperturbed functional is

$$f(u) = \frac{1}{2}\left(\int_0^1 |\dot{u}|^2\,dt\right)\left(\int_0^1 \left[E - \frac{1}{2}Au\cdot u\right]dt\right).$$

Let us consider the following critical manifolds

$$Z_j = \{z \in H \,|\, z = \xi(e^{i\pi t} + e^{-i\pi t})/2 \colon \xi \in X_j,\ |\xi|^2 = 2E/\omega_j^2\}.$$

We claim that

Lemma 7. *Z_j are non-degenerate critical manifolds if and only if $\omega_i/\omega_j \notin \mathbf{Z}$ $\forall i \neq j$.*

Proof. There results

$$(7)\qquad f''(z)[h,k] = \left(\int_0^1 \dot{h}\cdot\dot{k}\,dt\right)\left(\int_0^1 \left[E - \frac{1}{2}Az\cdot z\right]dt\right) - \left(\int_0^1 \dot{z}\cdot\dot{h}\,dt\right)\left(\int_0^1 Az\cdot k\,dt\right) -$$

$$-\left(\int_0^1 \dot{z}\cdot\dot{h}\,dt\right)\left(\int_0^1 Az\cdot h\,dt\right) - \left(\int_0^1 |\dot{z}|^2\,dt\right)\left(\int_0^1 Ak\cdot h\,dt\right).$$

If $z \in Z_j$ then $Az = \omega_j^2 z$, $\ddot{z} = -\pi^2 z$, and therefore

$$\left(\int_0^1 \dot{z}\cdot\dot{k}\,dt\right)\left(\int_0^1 Az\cdot h\,dt\right) = \pi^2\omega_j^2\left(\int_0^1 z\cdot k\,dt\right)\left(\int_0^1 z\cdot h\,dt\right),$$

$$\left(\int_0^1 \dot{z}\cdot\dot{h}\,dt\right)\left(\int_0^1 Az\cdot k\,dt\right) = \pi^2\omega_j^2\left(\int_0^1 z\cdot k\,dt\right)\left(\int_0^1 z\cdot h\,dt\right).$$

Moreover

$$\int_0^1 \left[E - \frac{1}{2}Az\cdot z\right]dt = \frac{E}{2}, \qquad \frac{1}{2}\int_0^1 |\dot{z}|^2\,dt = \frac{\pi^2 E}{2\omega_j^2}.$$

Substituting in (7) we obtain that, if $h \in \ker f''(z)$, then

$$(8)\quad f''(z)[h,k] = \frac{E}{2}\int_0^1 \dot{h}\cdot\dot{k}\,dt - 2\pi^2\omega_j^2\left(\int_0^1 z\cdot k\,dt\right)\left(\int_0^1 z\cdot h\,dt\right) -$$

$$-\pi^2\frac{E}{2}\int_0^1 Ah\cdot k = 0, \qquad \forall k \in H.$$

This implies that h satisfies weakly (and, by regularity, strongly) the equation

$$(9)\qquad \frac{E}{2}\ddot{h} + 2\pi^2\left(\int_0^1 z\cdot h\,dt\right)z + \frac{\pi^2 E}{2\omega_j^2}Ah = 0.$$

Moreover, integrating by parts (8) and taking in account (9), we find that

$\dot{h}(1)\cdot k(1) - \dot{h}(0)\cdot k(0) = 0$, $\forall k \in H$, and hence

$$\dot{h}(0) = \dot{h}(1) = 0 . \tag{10}$$

Let e_r be a vector of the canonical base of $\boldsymbol{R}^n$ such that $e_r \notin X_j$. Then the component $h_r := h \cdot e_r$ of h satisfies

$$\begin{cases} \ddot{h}_r + \pi^2 \omega_j^{-2} A h_r = 0 , \\ \dot{h}_r(0) = \dot{h}_r(1) = 0 . \end{cases}$$

If e_r is an eigenvector corresponding to a negative eigenvalue, it is obvious that the system will not admit nontrivial solutions. If e_r is an eigenvector corresponding to a positive eigenvalue ω_r^2, then we find $h_r(t) = a \cos(\pi \omega_r t / \omega_j) + b \sin(\pi \omega_r t / \omega_j)$, and (10) implies $a = 0$ e $b = 0 \Leftrightarrow \omega_r / \omega_j \notin \mathbf{Z}$. Therefore $\omega_r / \omega_j \notin \mathbf{Z}$ $\forall j \neq r$ is a necessary condition for Z_j to be non-degenerate. Let us prove that this condition is sufficient as well. A simple calculation shows that, extending z and h to even functions defined on $[-1, 1]$, they still satisfy to equation (9) on the interval $[-1, 1]$.

Let us set:

$$h(t) = \sum_{-\infty}^{+\infty} c_n e^{in\pi t} , \qquad c_n = \bar{c}_{-n} (^2) .$$

Since the support of h is contained in X_j, then $c_n = a_n + i b_n$ where $a_n, b_n \in X_j$, $\forall n \in \mathbf{Z}$. Therefore, substituting in (9),

$$-\frac{E}{2} \sum_{-\infty}^{+\infty} c_n \pi^2 n^2 e^{i\pi n t} + \pi^2 \xi \cdot (c_1 + \bar{c}_1) \xi \frac{e^{i\pi t} + e^{-\pi t}}{2} + \pi^2 \frac{E}{2} \sum_{-\infty}^{+\infty} c_n e^{i\pi n t} = 0 .$$

Whence

$$\frac{E}{2} \sum_{-\infty}^{+\infty} (1 - n^2) c_n e^{i\pi n t} + \xi \cdot (c_1 + \bar{c}_1) \xi \frac{e^{i\pi t} + e^{-i\pi t}}{2} = 0 ,$$

and then $c_n = 0$ $\forall n \neq \pm 1$, $c_1 = a + ib$, $a, b \in X_j$, $a \cdot \xi = 0$.

Moreover (10) implies that $b = 0$, and finally one has $h(t) = a(e^{i\pi t} + e^{-i\pi t})/2$, $a \in X_j$, $a \cdot \xi = 0$.

It is now evident that $h \in T_z Z_j$.

Proof of Theorem 2. Let us consider

$$\sigma: H \to H, \qquad u(t) \to u(1-t) .$$

It is clear that σ is an isometry and $\sigma^2 = \mathrm{id}$. Now, let $G = \{\mathrm{id}, \sigma\}$ be the group generated by σ. Let us set: $\mathrm{Fix}\, G = \{u \in H: u(t) = u(1-t)\ \forall t \in [0, 1]\}$; $\mathcal{A} = \{A \subset H,$ A closed, $\sigma(A) = A\}$; $\Gamma = \{h \in C(H; H): h \circ \sigma = \sigma \circ h\}$.

Define the index $i_G: \mathcal{A} \to \mathbf{N} \cup \{+\infty\}$ by setting: $i_G(A) = \min_n \{\exists \phi: A \to \boldsymbol{R}^n \setminus \{0\}, \phi \circ \sigma = -\phi\}$, and $i_G = +\infty$ if such an n does not exist.

(2) $\bar{c}$ denotes the complex coniugate of c.

It is easy to check that i_G is an index (see [10]). Moreover since $\sigma z = -z\ \forall z \in Z_j$, one has that i_G coincides with the Krasnoselskii genus γ on Z_j (see [9] for a definition). Therefore $i_G(Z_j) = \gamma(Z_j)$. From the fact that Z_j is diffeomorphic to the unit sphere in $\boldsymbol{R}^{\alpha_j}$, it follows that $\gamma(Z_j) = \alpha_j$ (see [2, p. 19]).

Since f_ε is G-invariant $\forall\varepsilon$, applying Theorem 5 we obtain that for all fixed $E > 0$, provided ε is small, there exist $\alpha = \sum \alpha_j$ geometrically distinct critical points that give rise to α distinct brake orbits for the potential V_ε.

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