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Remarks on balance laws in electromagnetism

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Fisica matematica. — *Remarks on balance laws in electromagnetism.* Nota di ANGELO MORRO, presentata (*) dal Corrisp. C. Cercignani.

ABSTRACT. — Two aspects of balance equations in electromagnetism are considered. First, concerning Faraday's law and Ampère's law it is emphasized that appropriate treatment of the time derivative of an integral over a time-dependent surface eliminates ambiguities occurring in various presentations of the subject. Second, standard forms of energy balance are shown not to be equivalent in the case of materials with memory.

KEY WORDS: Faraday's law; Time-dependent surface; Energy balance.

RIASSUNTO. — *Osservazioni sulle leggi di bilancio in elettromagnetismo.* Si considerano due aspetti delle equazioni di bilancio in elettromagnetismo. Primo, in connessione con le leggi di Faraday e di Ampère si sottolinea come un uso appropriato della derivata temporale di un integrale su una superficie dipendente dal tempo eviti le ambiguità di molte trattazioni dell'argomento. Secondo, si mostra la non equivalenza di forme del bilancio dell'energia quando sono presenti effetti di memoria.

1. INTRODUCTION

It is a customary procedure, in continuum physics, to derive local balance equations through global balance equations or to establish a precise correspondence between global and local balance equations. Roughly, in continuum mechanics the usual content of a balance law is expressed by the time derivative of a volume integral, over a three-dimensional region, being given by a volume integral and a surface integral. In electromagnetism, balance laws occur where the time derivative is associated with surface integrals. In particular such is the case for Faraday's law of induction and Ampère's law.

Time dependent volumes, surfaces and curves are extensively treated in continuum mechanics of electromagnetic solids (cf. [1, ch. 3]). Yet, according to the presentation of electromagnetism given in many textbooks, it looks as though appropriate use of time derivatives of surface integrals were deliberately ignored. The result is that very often presentations of Faraday's law and Ampère's law are given which, if not incorrect, are obscure and turn out to be difficult to understand properly.

The purpose of this *Note* is twofold. The first fold is to show that use of the appropriate mathematical setting makes the formulation of Faraday's and Ampère's laws precise and simple. Mathematically, the problem consists in the evaluation of the time-derivative of a flux through a time-dependent surface. In this regard the next section exhibits a short proof of Nanson's formula for the surface element and then, as an application, the derivation of the time derivative of an integral over a time-dependent surface. The second fold is to examine the seeming discrepancy between two standard forms of balance of energy.

2. TIME-DERIVATIVE OF THE FLUX THROUGH A TIME-DEPENDENT SURFACE

Let $\boldsymbol{X} = \boldsymbol{X}(\alpha, \beta)$, which maps an open region $\mathcal{A} \subset \mathbf{R}^2$ into the three-dimensional Euclidean space $\mathcal{E}$, be the representation of an open surface S_0. Let $\boldsymbol{\chi}: \mathcal{E} \times \mathbf{R} \to \mathcal{E}$ and

(*) Nella seduta del 13 marzo 1993.

assume that $\chi(\cdot,t)$ is bijective for each $t \in \mathbf{R}$. Both $\mathbf{X}(\alpha,\beta)$ and $\chi(\mathbf{X},t)$ are of class C^1 in the pertinent domains. Consider the map

$$\boldsymbol{x} = \chi(\mathbf{X}(\alpha,\beta),t).$$

As $(\alpha,\beta) \in \mathcal{A}$, we can view this map as defining a surface S, image of S_0 under χ, parameterized by the time t. We denote by $\boldsymbol{v}$ the time-derivative of χ, namely the velocity of the points of S.

Describe the position vector $\boldsymbol{x}$ by a triple of coordinates x^1,x^2,x^3 and $\mathbf{X}$ by X^1,X^2,X^3. We define F as the Jacobian matrix, $F^h_H = \chi^h_{;H}$, a semi-colon denoting covariant differentiation. Moreover we let $J = \det F$; since χ is bijective we have $J \neq 0$ and we let $J > 0$. Under $\chi(\cdot,t)$, a surface element $\mathbf{N}\,dA$ of S_0 is mapped into a surface element $\boldsymbol{n}\,da$ of S. Nanson's formula relates $\mathbf{N}\,dA$ to $\boldsymbol{n}\,da$; the following proof simplifies that given in [2, §20].

PROPOSITION. The surface elements $\mathbf{N}\,dA$ and $\boldsymbol{n}\,da$ are related by

$$n_k\,da = J(F^{-1})^H_k N_H\,dA.$$

PROOF. We represent the surface element of S_0 and S as

$$\mathbf{N}\,dA = \mathbf{X}_{,\alpha} \times \mathbf{X}_{,\beta}\,d\alpha\,d\beta, \qquad \boldsymbol{n}\,da = \boldsymbol{x}_{,\alpha} \times \boldsymbol{x}_{,\beta}\,d\alpha\,d\beta,$$

a comma denoting partial differentiation. In coordinate form,

$$(\boldsymbol{x}_{,\alpha} \times \boldsymbol{x}_{,\beta})_k = \varepsilon_{kpq} F^p_H F^q_K X^H_{,\alpha} X^K_{,\beta}\,d\alpha\,d\beta.$$

Because

$$\varepsilon_{kpq} F^p_H F^q_K F^k_L = \varepsilon_{HKL}\,J$$

then

$$\varepsilon_{kpq} F^p_H F^q_K = \varepsilon_{HKL}\,(F^{-1})^L_k\,J.$$

Substitution yields the result. □

Let L be the velocity gradient matrix which is related to F by the identity $L^h_k = \dot F^h_H (F^{-1})^H_k$, a superposed dot denoting time differentiation with $\mathbf{X}$ fixed.

PROPOSITION. For any C^1 vector function $\boldsymbol{w}(\boldsymbol{x},t)$

$$\frac{d}{dt}\int_S \boldsymbol{w}\cdot\boldsymbol{n}\,da = \int_S \{\dot w^k n_k + w^k [v^h_{;h}\delta^j_k - L^j_k]\,n_j\}\,da. \tag{1}$$

PROOF. Observe that by the definition of J and the relation $(F^{-1})^H_k F^k_K = \delta^H_K$ we have

$$\dot J = \frac{\partial J}{\partial F^m_M}\,\dot F^m_M = J(F^{-1})^M_m\,\dot F^m_M = J v^m_{;m},$$

$$\frac{d}{dt}\,(F^{-1})^H_h = -(F^{-1})^H_k\,L^k_h.$$

Now, by the change of variables $x^1,x^2,x^3 \to X^1,X^2,X^3$ and Nanson's formula we have

$$\frac{d}{dt}\int_S w^k n_k\,da = \frac{d}{dt}\int_{S_0} w^k J(F^{-1})^H_k N_H\,dA = \int_{S_0} \frac{d[w^k J(F^{-1})^H_k]}{dt}\,N_H\,dA.$$

Substitution, rearrangement, and the reverse change $X^1, X^2, X^3 \to x^1, x^2, x^3$ yields the desired result. □

For both formal simplicity and immediate comparison with the standard form of Maxwell equations, henceforth we use Cartesian coordinates and represent vectors and tensors in the compact notation. Accordingly the result (1) takes the form

$$\frac{d}{dt}\int_S \boldsymbol{w}\cdot\boldsymbol{n}\,da = \int_S \overset{\diamond}{\boldsymbol{w}}\cdot\boldsymbol{n}\,da$$

where $\overset{\diamond}{\boldsymbol{w}} = \dot{\boldsymbol{w}} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{v} + \boldsymbol{w}(\nabla\cdot\boldsymbol{v})$ is the convective derivative. In view of the identity $\nabla\times(\boldsymbol{w}\times\boldsymbol{v}) = \boldsymbol{w}(\nabla\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\nabla)\boldsymbol{w} - \boldsymbol{v}(\nabla\cdot\boldsymbol{w}) - (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}$, application of Stokes' theorem yields

$$\frac{d}{dt}\int_S \boldsymbol{w}\cdot\boldsymbol{n}\,da = \int_S \left[\frac{\partial\boldsymbol{w}}{\partial t} + \boldsymbol{v}(\nabla\cdot\boldsymbol{w})\right]\cdot\boldsymbol{n}\,da + \int_{\partial S}(\boldsymbol{w}\times\boldsymbol{v})\cdot\boldsymbol{t}\,dl \tag{2}$$

$\boldsymbol{t}$ being the unit tangent vector of ∂S.

As we show in a moment, eq. (2) is the basic tool for a precise statement of global laws in electromagnetism.

3. Faraday's law

The electric field $\boldsymbol{E}$, the magnetic induction $\boldsymbol{B}$, as well as the velocity $\boldsymbol{v}$ are C^1 functions on $S\times\mathbf{R}$. Start from the Maxwell equation

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}\,. \tag{3}$$

Irrespective of the dependence of S on the time t, integration of (3) on S and Stokes' theorem give

$$\int_{\partial S}\boldsymbol{E}\cdot\boldsymbol{t}\,dl = -\int_S \frac{\partial\boldsymbol{B}}{\partial t}\cdot\boldsymbol{n}\,da\,. \tag{4}$$

Apply (2) to $\boldsymbol{w} = \boldsymbol{B}$. Since $\nabla\cdot\boldsymbol{B} = 0$, by use of Stokes' theorem we have

$$\int_{\partial S}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\boldsymbol{t}\,dl = -\frac{d}{dt}\int_S \boldsymbol{B}\cdot\boldsymbol{n}\,da\,, \tag{5}$$

whereby the line integral of $\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$ along ∂S is equal and opposite to the time derivative of the flux of $\boldsymbol{B}$ through S. The field $\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}$ is the effective electric field, at the local co-moving frame of the element of ∂S [3, §6.1]. Equivalently, one may start from (5) as the axiom for Faraday's law and then obtain (3) by means of (2).

It is of interest to give some examples of how Faraday's law is introduced in known books. First, review the presentation in [4, p. 8]. Integration of (3) over S yields (4). Then, if the contour is *fixed*, the operator $\partial/\partial t$ may be brought out from under the sign of integration and then

$$\int_{\partial S}\boldsymbol{E}\cdot\boldsymbol{t}\,dl = -\frac{\partial}{\partial t}\int_S \boldsymbol{B}\cdot\boldsymbol{n}\,da\,. \tag{6}$$

The experiments of Faraday indicated that the relation (6) holds whatever the cause of the flux (of $\boldsymbol{B}$) variation. To take this into account the Faraday law is written generally in the form

$$\int_{\partial S} \boldsymbol{E}\cdot\boldsymbol{t}\,dl = -\frac{d}{dt}\int_S \boldsymbol{B}\cdot\boldsymbol{n}\,da\,. \tag{7}$$

The same procedure is followed in [5, § 1.1].

Sommerfeld [6, p. 13] starts from (7) as an *axiom*. Then, on the assumption that the surface is *fixed*, the local form (3) is derived. This presentation is common to a great many books; cf., *e.g.* [7, p. 22; 8, p. 266; 9, p. 87; 10, p. 230].

Feynman *et al.* [11, p. 1.5] start from the «flux rule» whereby the circulation of $\boldsymbol{E}$ around a contour C is equal to the rate of change of the flux of $\boldsymbol{B}$ through any surface whose edge is C. Then [11, § 17.1], they take (3) as Faraday's law, integrate on an open, *fixed* in space, surface S and, via Stokes' theorem, obtain the flux rule in the form (6). They say also that if the contour ∂S moves then, by regarding $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$ as the force per unit charge, we get the flux rule once again. Interestingly, they say that «the two possibilities – *circuit moves* or *field changes* – are not distinguished in the statement of the flux rule. Yet ... we have used two completely distinct laws for the two cases – $\boldsymbol{v} \times \boldsymbol{B}$ for *circuit moves* and $\nabla \times \boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$ for *field changes*. We know of no other place in physics where such a simple principle requires ... two different phenomena».

Halliday and Resnick [12, ch. 35] state Faraday's law as the flux rule

$$\oint \boldsymbol{E}\cdot\boldsymbol{t}\,dl = -\frac{d\phi_B}{dt}\,.$$

Then ϕ_B through S is written as

$$\phi_B = \int_S \boldsymbol{B}\cdot\boldsymbol{n}\,da\,.$$

Next, in the Appendix II, proof is given that, for a *fixed* surface, the flux rule leads to (3).

In all these formulations it is not clear at all that the field $\boldsymbol{E}$ in (3) and (4) is the electric field in a single frame (laboratory) while $\boldsymbol{E}$ in (7), cf. (6), should be in fact $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$, namely the electric field in the local co-moving frame. Furthermore, apart from the apparent inconsistency of (6) and (7), a natural question arises as to which other equation should follow, in place of (3), from the flux rule if the surface S is allowed to depend on time. Restricting attention to fixed surfaces is somewhat contradictory in that almost every book on electromagnetism considers the application of Faraday's law to a loop, in a constant induction field, that has a fixed U-shaped part and a movable crossbar that can slide along the two legs of the U.

4. Ampère's law

On way of stating Amphère's law, in the general form improved by Maxwell, is to say that the line integral of the magnetic field $\boldsymbol{H}$ along ∂S is equal to the current I_S through S plus the flux of the displacement current $\partial\boldsymbol{D}/\partial t$, viz.

$$\int_{\partial S} \boldsymbol{H}\cdot\boldsymbol{t}\,dl = I_S + \int_S \frac{\partial\boldsymbol{D}}{\partial t}\cdot\boldsymbol{n}\,da\,. \tag{8}$$

For simplicity assume that the current I_S is expressed as the flux of a current density vector $\boldsymbol{j}$,

$$I_S = \int_S \boldsymbol{j} \cdot \boldsymbol{n} \, da .$$

Now apply (2) to $\boldsymbol{w} = \boldsymbol{D}$. Because of the Gauss law $\nabla \cdot \boldsymbol{D} = \rho$, ρ being the free charge density, we obtain

$$\int_{\partial S} (\boldsymbol{H} + \boldsymbol{D} \times \boldsymbol{v}) \cdot \boldsymbol{t} \, dl = \int_S (\boldsymbol{j} - \rho \boldsymbol{v}) \cdot \boldsymbol{n} \, da + \frac{d}{dt} \int_S \boldsymbol{D} \cdot \boldsymbol{n} \, da . \tag{9}$$

In view of (9), the line integral of $\boldsymbol{H} + \boldsymbol{D} \times \boldsymbol{v}$ along the contour ∂S is equal to the flux of $\boldsymbol{j} - \rho\boldsymbol{v}$ plus the time derivative of the flux of $\boldsymbol{D}$. This shows that the effective current density is $\boldsymbol{j} - \rho\boldsymbol{v}$, namely the current density relative to the local co-moving frame. Perhaps to avoid the use of the relation (2), common books on electromagnetism exhibit (8) as the global form of Ampère's law.

5. Balance of energy

Also in the simple case of electromagnetism in undeformable matter, there are various forms of the balance of energy which generally are not equivalent. Here we contrast two of them and clarify the discrepancy.

One form is based on the observation that the Lorentz force $\boldsymbol{F}$, on a charge q with a velocity $\boldsymbol{v}$, is $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ and then the power is $\Pi = q\boldsymbol{v} \cdot \boldsymbol{E}$. The result is extended to a continuous distribution of charge by saying that the power on a current density is $\boldsymbol{j} \cdot \boldsymbol{E}$.

For simplicity, consider a body at rest. If, in addition to electric conduction, heat conduction occurs then the observation that $\boldsymbol{j} \cdot \boldsymbol{E}$ is the power on $\boldsymbol{j}$ results in the balance of energy

$$\frac{\partial e}{\partial t} = -\nabla \cdot \boldsymbol{q} + r + \boldsymbol{j} \cdot \boldsymbol{E} \tag{10}$$

where e is the internal energy (per unit volume), $\boldsymbol{q}$ is the heat flux and r is the heat supply.

Another form is based on Poynting's theorem whereby

$$\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\boldsymbol{j} \cdot \boldsymbol{E} . \tag{11}$$

If $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$, for constant ε and μ, then, letting $u = (\varepsilon E^2 + \mu H^2)/2$ we can write (11) in the form $\partial u / \partial t = -\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \boldsymbol{j} \cdot \boldsymbol{E}$, which can be phrased by saying that the energy density u of the electromagnetic field changes by the net input of the Poynting vector $\boldsymbol{E} \times \boldsymbol{H}$ and the power expended on the charges that provide the current density $\boldsymbol{j}$. If we consider the physical system of matter and electromagnetic field then the overall effect of $\boldsymbol{j} \cdot \boldsymbol{E}$ is zero. Then, still for matter at rest, letting w be the energy density of the system we write $\partial w / \partial t = -\nabla \cdot (\boldsymbol{q} + \boldsymbol{E} \times \boldsymbol{H}) + r$ or, by (11),

$$\frac{\partial w}{\partial t} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{j} \cdot \boldsymbol{E} - \nabla \cdot \boldsymbol{q} + r . \tag{12}$$

The balance (12) is equal, or equivalent, to that given in [13].

The expressions (12) and (10) are not equivalent whenever $\boldsymbol{E} \cdot \partial \boldsymbol{D} / \partial t + \boldsymbol{H} \cdot \partial \boldsymbol{B} / \partial t$ cannot be written as the time derivative of an energy function u. A significant case when

u does not exist occurs when $\boldsymbol{D}$ and $\boldsymbol{B}$ are given by memory functionals of $\boldsymbol{E}$ and $\boldsymbol{H}$. Memory effects are physically well grounded and the model

$$\boldsymbol{D}(t) = f_0 \boldsymbol{E}(t) + \int_0^\infty f(\xi)\, \boldsymbol{E}(t-\xi)\, d\xi \tag{13}$$

for dielectrics is customarily used – cf. [14, §58]. Indeed, by (12), letting $w = e + u$, where u is the electromagnetic energy now undetermined, we have

$$\frac{\partial e}{\partial t} = \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \frac{\partial u}{\partial t} + \boldsymbol{j} \cdot \boldsymbol{E} - \nabla \cdot \boldsymbol{q} + r .$$

This suggests that we regard (10) as incomplete in that it does not contain the «dissipative» power $\boldsymbol{E} \cdot \partial \boldsymbol{D}/\partial t + \boldsymbol{H} \cdot \partial \boldsymbol{B}/\partial t - \partial u/\partial t$ of electromagnetic field in matter. For instance, with reference to (13), if we let $\boldsymbol{E}(t) = \boldsymbol{E}_0 \sin \omega t$, $\omega > 0$, then the integral of $\partial u/\partial t$ over the period $[0, T]$, $T = 2\pi/\omega$, vanishes while

$$\int_0^T \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}\, dt = \pi f_s(\omega)\, \boldsymbol{E}_0^2$$

where

$$f_s(\omega) = \int_0^\infty f(\xi) \sin \omega\xi\, d\xi .$$

If memory effects do not occur then the laws (12) and (10) are equivalent (cf. [15, §9.5]).

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