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Maximum principle for viscosity sub solutions and viscosity sub solutions of the Laplacian


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Abstract. — The aim of this paper is to characterize the u.s.c. (resp. l.s.c.) viscosity sub (resp. super) solutions of the Laplacian which do not take the value $+\infty$ (resp. $-\infty$) as precisely the sub (resp. super) harmonic functions.

Key words: Viscosity solutions; Harmonic functions; Maximum principle.

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1. Basic Definitions

Let $\Omega$ be a nonempty open set in $\mathbb{R}^n$ and let $M_n$ be the space of all symmetric $n \times n$ matrices. Let $F$ be a mapping from $\Omega \times M_n$ to $\mathbb{R}^1$.

Definition 1. $F$ is said uniformly elliptic if there exists a $\lambda > 0$ such that for all $A, B \in M_n$, $B$ positive definite and for all $x \in \Omega$, one has $F(x, A + B) - F(x, A) \geq \lambda \|B\|$ where $\| \cdot \|$ is any norm on $M_n$.

Any second order linear elliptic differential operator with second order terms only is an example of $F$ satisfying Definition 1.

Definition 2. An extended real-valued function $u$ defined on $\Omega$ is said to be a viscosity sub (resp. super) solution of $F$ if for all $\phi \in C^2(\Omega)$ with $u - \phi$ having a local maximum at a point $x_0 \in \Omega$ implies that $F(x_0, D^2 \phi(x_0)) \geq 0$, where $D^2 \phi(x_0)$ stands for the Hessian of $\phi$ at $x_0$, that is the matrix $((\partial^2 \phi/\partial x_i \partial x_j(x_0)))$ (resp. $u - \phi$ having a local minimum at a point $x_0 \in \Omega$ implies that $F(x_0, D^2 \phi(x_0)) \leq 0$).

Definition 3. A real-valued function $u$ is said to be a viscosity solution of $F = 0$ if it is both a viscosity sub and super solution of $F$.

2. A Maximum Principle

Proposition 1. Let $F$ be uniformly elliptic and let us assume that $F(x, 0) = 0$ for all $x \in \Omega$. Then, we have $F(x, A) < 0$ for all matrix $A = (a_{ij})$ which is negative-definite in the sense that

$$\sum_i \sum_j a_{ij} \alpha_i \alpha_j < 0, \quad \forall \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \neq 0.$$

Proof. \(0 = F(x, 0) = F(x, A + (-A))\). Therefore,
\[
0 - F(x, A) = F(x, A + (-A)) - F(x, A) \geq \lambda \|(-A)\|
\]
as \(-A\) is positive definite and \(F\) is uniformly elliptic. Therefore,
\[
F(x, A) \leq -\lambda \|(-A)\| < 0.
\]

**Theorem 1.** Let \(\Omega\) be bounded. Let \(F\) be uniformly elliptic with \(F(x, 0) = 0\), \(\forall x \in \Omega\). Let \(u\) be a viscosity sub solution of \(F\) such that \(u(x) < \infty\), \(\forall x \in \Omega\). If \(u\) is upper semi continuous on \(\overline{\Omega}\), then
\[
\sup_{x \in \partial \Omega} u(x) = \sup_{x \in \overline{\Omega}} u(x).
\]

**Proof.** It is obvious that
\[
\sup_{x \in \partial \Omega} u(x) \leq \sup_{x \in \overline{\Omega}} u(x).
\]

Suppose that
\[
\sup_{x \in \partial \Omega} u(x) < \sup_{x \in \overline{\Omega}} u(x).
\]
\(u\) being upper semi continuous on \(\overline{\Omega}\) and \(\overline{\Omega}\) being compact, \(\sup_{x \in \overline{\Omega}} u(x)\) is attained at some point \(x_0 \in \overline{\Omega}\). Equation (1) implies that \(x_0 \notin \partial \Omega\). Hence \(x_0 \in \Omega\). Thus \(u\) has a local maximum at \(x_0\).

**Claim.** The function \(u_\varepsilon = u + \varepsilon |x - x_0|^2\) also has a local maximum in \(\Omega\) for small values of \(\varepsilon > 0\).

**Proof of the claim.** Suppose for some \(\varepsilon > 0\), \(u_\varepsilon\) attains its maximum only on \(\partial \Omega\). Let \(X \in \partial \Omega\) be such that \(u_\varepsilon(X) \geq u_\varepsilon(x)\), \(\forall x \in \overline{\Omega}\).

In particular, \(u_\varepsilon(X) \geq u_\varepsilon(x_0) = u(x_0)\). That is \(u(X) + \varepsilon |X - x_0|^2 \geq u(x_0)\).

\[
\Rightarrow \varepsilon |X - x_0|^2 \geq u(x_0) - u(X) \geq u(x_0) - \sup_{x \in \partial \Omega} u(x)
\]

\[
\Rightarrow \varepsilon \geq \frac{u(x_0) - \sup_{x \in \partial \Omega} u(x)}{|X - x_0|^2} \geq \frac{u(x_0) - \sup_{x \in \partial \Omega} u(x)}{\sup_{y \in \partial \Omega} |y - x_0|^2}.
\]

Let us observe that \(u(x_0) - \sup_{x \in \partial \Omega} u(x) > 0\). Hence if \(\varepsilon > 0\) is

\[
< \frac{u(x_0) - \sup_{x \in \partial \Omega} u(x)}{\sup_{y \in \partial \Omega} |y - x_0|^2},
\]

\(u_\varepsilon\) has a local maximum in \(\Omega\) and thus the claim is proved.
Fix one such $\varepsilon > 0$. Let $u_\varepsilon$ have a local maximum at a point $y \in \Omega$. As $u$ is a viscosity subsolution, applying the definition taking $\phi$ to be $-\varepsilon |x - x_0|^2$, we see that $F(y, -2\varepsilon I_n) \geq 0$ where $I_n$ is the $n \times n$ identity matrix. As $\varepsilon > 0$, $-2\varepsilon I_n$ is negative-definite and hence, Proposition 1 is contradicted. □

3. Viscosity sub (resp. super) solutions of the Laplacian $\Delta$: A characterization

**Theorem 2.** Let $u$ be an upper semi continuous (resp. lower semi continuous) function $u$ such that $u < \infty$ (resp. $u > -\infty$). Then $u$ is a viscosity sub (resp. super) solution for $\Delta$, if and only if $u$ is subharmonic (resp. superharmonic).

**Proof.** Sufficient to prove the characterization in the subharmonic case. Let us recall the definition of a subharmonic function.

**Definition 4.** An extended real-valued function defined on an open set $\Omega \neq \emptyset$ is said to be **subharmonic** if

i) $u$ is upper semi continuous,

ii) $u(x) < \infty \ \forall x \in \Omega$ and

iii) $\forall x_0 \in \Omega, \ \exists r_0 > 0$ such that

$$u(x_0) \leq \int_{\partial B(x_0; r)} u(x) d\sigma(x), \quad \forall r \leq r_0,$$

where $d\sigma$ is the unit surface measure on $\partial B(x_0; r)$, the boundary of $B(x_0; r)$.

**Proof of Theorem 2.** (i) *If part:* Let us assume that $u$ is subharmonic. Let $\phi \in C^2(\Omega)$ be such that $u - \phi$ has a local maximum at a point $x_0 \in \Omega$. Let us assume that $u(x_0) - \phi(x_0)$ is a maximum of $u - \phi$ in a ball $B(x_0; \varepsilon)$ for some $\varepsilon > 0$.

If $\Delta \phi(x_0) < 0$, then $\Delta \phi(x) < 0 \ \forall x$ in some neighbourhood of $x_0$, say for example in $B(x_0; \eta)$ for some $\eta \in (0, \varepsilon)$. Therefore, $u - \phi$ is subharmonic in $B(x_0; \eta)$, as $u$ is subharmonic and $\phi$ is super harmonic in $B(x_0; \eta)$. Therefore, by the classical maximum principle for subharmonic functions, $u - \phi$ must be equal to $u(x_0) - \phi(x_0)$ in $B(x_0; \eta)$.

That is $\phi = u - u(x_0) + \phi(x_0)$ in $B(x_0; \eta)$. Therefore $\phi$ is subharmonic in $B(x_0; \eta)$ if and only if $\Delta \phi \geq 0$ in $B(x_0; \eta)$.

In particular, $\Delta \phi(x_0) \geq 0$.

This contradicts that $\Delta \phi(x_0) < 0$ proving that $u$ is a viscosity subsolution.

(ii) *Only if part:* Before we start the proof, let us make the following remark, which is an easy consequence of the definitions.

**Remark.** If $u$ is a viscosity subsolution for $\Delta$, and if $h$ is any harmonic function, then $u + h$ is also a viscosity subsolution.

Let $u$ be upper semi continuous and let $u(x) < \infty \ \forall x \in \Omega$. Let $u$ be a viscosity subsolution for $\Delta$. Let $x_0 \in \Omega$. Let $R > 0$ be less than $d(x_0, \partial \Omega)$ so that $B(x_0; R) \subset \Omega$. Let $r \leq R$. 


Since $u$ is upper semi-continuous, there exists a decreasing sequence \( \{f_m\}_{m=1}^{\infty} \) of real-valued continuous functions on \( \partial B(x_0; r) \) such that \( f_m(x) \downarrow u(x) \), \( \forall x \in \partial B(x_0; r) \).

Consider the Poisson integral,

\[
I_{f_m}^r(x) = r^{n-2} \int_{\partial B(x_0; r)} f_m(X) \frac{r^2 - |x - x_0|^2}{|x - X|^n} \, d\sigma_r(X)
\]

in \( B(x_0; r) \).

Then, it is well known that \( I_{f_m}^r \) is a harmonic function in \( B(x_0; r) \) and that

\[
\forall X \in \partial B(x_0; r), \quad I_{f_m}^r(x) \to f_m(X), \quad \text{as } x \to X, \ x \in B(x_0; r).
\]

Consider \( u - I_{f_m}^r \) in \( B(x_0; r) \). As \( u \) is a viscosity subsolution of \( \Delta \) in \( B(x_0; r) \) and \( I_{f_m}^r \) is harmonic, by the remark above, \( u - I_{f_m}^r \) is also a viscosity subsolution of \( \Delta \) in \( B(x_0; r) \).

Define \( v \) in \( B(x_0; r) \) as

\[
v(x) = \limsup_{y \to x} \{u(y) - I_{f_m}^r(y)\}.
\]

Then \( v \) is upper semi-continuous in \( \overline{\Omega} = \partial B(x_0; r) \) and \( \forall X \in \partial B(x_0; r) \)

\[
v(X) = \limsup_{y \to X} \{u(y) - f_m(X)\}.
\]

\[
\leq u(X) - f_m(X) \quad \text{as} \quad u \text{ is upper semicontinuous}.
\]

\[
\forall X \in \partial B(x_0; r), \quad u(X) - f_m(X) \leq 0.
\]

Therefore, \( v(X) \leq 0, \forall X \in \partial B(x_0; r) \). By the maximum principle proved in Theorem 2,

\[
\sup_{X \in \partial B(x_0; r)} v(X) = \sup_{X \in \partial B(x_0; r)} v(x).
\]

The L.H.S. is \( \leq 0 \). Hence \( v(x) \leq 0 \ \forall x \in B(x_0; r) \). In particular, \( v(x_0) \leq 0 \).

\[
v(x_0) = u(x_0) - \int_{\partial B(x_0; r)} f_m(X) \, d\sigma_r(X).
\]

Therefore,

\[
\forall m \in \mathbb{N}, \quad u(x_0) \leq \int_{\partial B(x_0; r)} f_m(X) \, d\sigma_r(X).
\]

Hence

\[
\forall m \in \mathbb{N}, \quad u(x_0) \leq \int_{\partial B(x_0; r)} u(X) \, d\sigma_r(X),
\]

proving that \( u \) is subharmonic.

Corollary. A real-valued continuous function on \( \Omega \) is a viscosity solution for \( \Delta \) if and only if it is a harmonic function.
4. Concluding Remarks

The definition of uniformly elliptic second order non-linear differential operators given here is taken from L. A. Caffarelli [1] and the definitions of viscosity sub and super solutions are taken from Ishii and Lions [2]. The definitions of viscosity sub and super solutions given in [1] are apparently not the same as given in [2]. The equivalence of the definitions in [1] and [2] are proved in [3], for some class of uniformly elliptic operators.

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References


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