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Boundaries of prescribed mean curvature

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Calcolo delle variazioni. — *Boundaries of prescribed mean curvature.* Nota di
EDUARDO H. A. GONZALES, UMBERTO MASSARI e ITALO TAMANINI, presentata (*) dal Corrisp. M. Miranda.

ABSTRACT. — The existence of a singular curve in $\boldsymbol{R}^2$ is proven, whose curvature can be extended to an L^2 function. The curve is the boundary of a two dimensional set, minimizing the length plus the integral over the set of the extension of the curvature. The existence of such a curve was conjectured by E. De Giorgi, during a conference held in Trento in July 1992.

KEY WORDS: Calculus of variations; Geometric measure theory; Mean curvature; Singular boundaries of finite measure.

RIASSUNTO. — *Superfici di curvatura media assegnata*. È dimostrata l'esistenza di una curva singolare nello spazio euclideo a due dimensioni, la cui curvatura può essere estesa ad una funzione di quadrato integrabile. La curva è la frontiera di un insieme a due dimensioni, ed è minimizzante un funzionale ottenuto sommando alla lunghezza della curva, l'integrale sull'insieme di cui essa è frontiera della funzione curvatura. L'esistenza di una tale curva era stata congetturata da E. De Giorgi, durante un Convegno a Trento nel luglio del 1992.

0. INTRODUCTION

The study of the functional

$$\mathcal{F}_H(X) = |\partial X|(\Omega) + \int_\Omega \phi_X(x)\, H(x)\, dx \tag{0.1}$$

began in 1974 with the work of U. Massari (see [6]). Here Ω is an open subset of $\boldsymbol{R}^n$ $(n \geq 2)$, $H \in L^1(\Omega)$, ϕ_X is the characteristic function of the measurable set $X \subset \boldsymbol{R}^n$ and $|\partial X|(\Omega)$ is the perimeter of X in Ω, *i.e.*

$$|\partial X|(\Omega) = \sup\left\{\int \phi_X(x)\operatorname{div} G(x)\, dx\colon G \in C_0^1(\Omega; \boldsymbol{R}^n), \|G\|_\infty \leq 1\right\}. \tag{0.2}$$

If E is a local minimizer of $\mathcal{F}_H$ (*i.e.* if $\mathcal{F}_H(E) \leq \mathcal{F}_H(X)$ for all X such that $(E - X) \cup (X - E) \subset\subset \Omega$), then we will say that «E has (generalized) mean curvature H», or simply that «H is a curvature for E». The reason is that in this case, assuming moreover the continuity of function H at the point $x \in \Omega \cap \partial E$ together with the regularity of ∂E itself in a neighbourhood of x, the (classical) mean curvature of ∂E at x is given by $-H(x)/(n-1)$, as one readily verifies.

In [6, 7] U. Massari proved that if E minimizes $\mathcal{F}_H$ with $H \in L^p(\Omega)$, $p > n$, then one can find an open subset Ω_1 of Ω such that $\Omega_1 \cap \partial E$ is a hypersurface of class $C^{1,\alpha}$, $\alpha = (p-n)/4p$ and $H_S((\Omega - \Omega_1) \cap \partial E) = 0$ $\forall s > n - 8$, where H_S denotes the s-dimensional Hausdorff measure in $\boldsymbol{R}^n$.

In 1987 E. Barozzi, E. H. A. Gonzales and I. Tamanini [2] proved that any set of fi-

(*) Nella seduta del 13 marzo 1993.

nite perimeter has generalized mean curvature in $L^1(\Omega)$. Indeed, a suitable H (with countable rank), which is the L^1-limit of approximating curvatures of finite rank, can be constructed (see also [1, 3, 11]). It is then apparent that if E has curvature $H \in L^p(\Omega)$ with $1 \leq p < n$, then ∂E can contain many singular points, *e.g.* cusps, points of density 1 (or 0), and so on (see *e.g.* [3, 4]).

In this paper we consider the case $p = n$, *i.e.* sets of curvature $H \in L^n(\Omega)$. In the first section we recall some general properties (density, blow-up) that are more or less present in the existing literature. The main result of the paper is contained in the second section where we study a 2-dimensional set with singular boundary. The definition of this set was given by E. De Giorgi during a Conference held in Trento last July 1992. The singular boundary is the union of two spiral curves converging to the same point. De Giorgi conjectured that the curvature of this line could be seen as the restriction of an L^2 function. Here we present the proof of De Giorgi's conjecture.

1. General results

Let $\Omega \subset \mathbf{R}^n$ be open, let $E \subset \Omega$ be a set of curvature $H(x) \in L^n(\Omega)$, $n \geq 2$, and let us assume that $0 \in \Omega \cap \partial E$. Denote by $B_r = B_r(0)$ the open ball centered at 0 with radius r, by $\overline{B}_r = \overline{B}_r(0)$ the corresponding closed ball, by $\|H\|_{n,r}$ and, respectively, by $\|H\|_{n,r,E}$ the L^n-norm of H in B_r (respectively in $B_r \cap E$), *i.e.*

$$\|H\|_{n,r} = \sqrt[n]{\int_{B_r} |H(x)|^n dx}\,, \qquad \|H\|_{n,r,E} = \sqrt[n]{\int_{B_r \cap E} |H(x)|^n dx}\,.$$

For $X \subset \mathbf{R}^n$ denote by $|X|$ its n-dimensional Lebesgue measure and set $\omega_n = |B_1|$. It is well known (see *e.g.* [5, Prop. 3.1]) that, by possibly redefining E on a set of measure zero, we can assume that

$$0 \in \partial E \Leftrightarrow 0 < |E \cap B_r| < \omega_n r^n \qquad \forall r > 0\,.$$

Throughout the paper, the boundary of any set of (locally) finite perimeter has to be intended in this stronger sense.

Lemma 1.1. If $E \subset \Omega$ has mean curvature $H \in L^n(\Omega)$ and $0 \in \Omega \cap \partial E$ then

$$\liminf_{r \to 0} \frac{|E \cap B_r|}{\omega_n r^n} \geq \frac{1}{2^n}\,. \tag{1.1}$$

Proof (see also [4]). By assumption, E is a local minimizer of $\mathcal{F}_H$, hence by comparing E with $E - B_r$ we get

$$|\partial E|(B_r) + \int_{B_r \cap E} H(x)\,dx \leq \int_{\partial B_r} \phi_E \, dH_{n-1} \tag{1.2}$$

and Hölder's inequality then gives

$$|\partial E|(B_r) - \|H\|_{n,r,E}\, |B_r \cap E|^{1-1/n} \leq \int_{\partial B_r} \phi_E \, dH_{n-1}\,. \tag{1.3}$$

The isoperimetric inequality states that

$$c(n)\,|B_r \cap E|^{1-1/n} \leq |\partial(E \cap B_r)|(\boldsymbol{R}^n), \tag{1.4}$$

where $c(n) = n\sqrt[n]{\omega_n}$. Since

$$|\partial(E \cap B_r)|(\boldsymbol{R}^n) = |\partial E|(B_r) + \int_{\partial B_r} \phi_E\, dH_{n-1},$$

from (1.3) and (1.4) we get

$$[c(n) - \|H\|_{n,r,E}]\,|B_r \cap E|^{1-1/n} \leq 2\int_{\partial B_r} \phi_E\, dH_{n-1}. \tag{1.5}$$

Now, for every $\varepsilon \in (0,1)$ we can find $r_\varepsilon > 0$ such that

$$\|H\|_{n,r,E} \leq \varepsilon c(n) \qquad \forall r\colon 0 \leq r \leq r_\varepsilon \tag{1.6}$$

so that

$$(1-\varepsilon)\,c(n)\,|B_r \cap E|^{1-1/n} \leq 2\int_{\partial B_r} \phi_E\, dH_{n-1}. \tag{1.7}$$

Set $g(r) = |B_r \cap E|$; function $g(r)$ is non-descreasing and for almost all r we have

$$g'(r) = \int_{\partial B_r} \phi_E\, dH_{n-1}.$$

We can than rewrite (1.7) as follows

$$(1-\varepsilon)\,c(n)\,g(r)^{1-1/n} \leq 2g'(r) \tag{1.8}$$

i.e. $(1-\varepsilon)\,c(n)/2n \leq [g(r)^{1/n}]'$ which holds for almost all $r \in (0, r_\varepsilon)$; by integrating between 0 and ρ we get $\rho(1-\varepsilon)\,c(n)/2n \leq g(\rho)^{1/n}$ *i.e.* $\rho^n(1-\varepsilon)^n\,\omega_n/2^n \leq g(\rho) = |B_\rho \cap E|$ $\forall \rho \in [0, r_\varepsilon]$ and (1.1) follows at once. ♦

LEMMA 1.2. *If* $E \subset \Omega$ *has mean curvature* $H \in L^n(\Omega)$ *and if* $B_r \subset \Omega$, *then*

$$|\partial E|(B_r) \leq c_1(n)(1 + \|H\|_{n,r})\,r^{n-1}. \tag{1.9}$$

PROOF. By comparison of E with $E - B_r$ we get

$$|\partial E|(B_r) + \int_{B_r \cap E} H(x)\,dx \leq \int_{\partial B_r} \phi_E\, dH_{n-1}. \tag{1.10}$$

Similarly, by comparison of E with $E \cup B_r$ we get

$$|\partial E|(B_r) + \int_{B_r \cap E} H(x)\,dx \leq \int_{\partial B_r} (1 - \phi_E)\,dH_{n-1} + \int_{B_r} H(x)\,dx. \tag{1.11}$$

By addition, (1.10) and (1.11) yield

$$2\left(|\partial E|(B_r) + \int_{B_r \cap E} H(x)\,dx\right) \leq n\,\omega_n r^{n-1} + \int_{B_r} H(x)\,dx$$

and from that, using Hölder's inequality, we obtain

$$|\partial E|(B_r) \leqslant \frac{n\,\omega_n r^{n-1}}{2} + \|H\|_{n,r}\,\frac{(\omega_n r^n)^{1-1/n}}{2} = \left[\frac{n\,\omega_n}{2} + \|H\|_{n,r}\,\frac{\omega_n^{1-1/n}}{2}\right] r^{n-1}. \qquad \blacklozenge$$

For $\lambda > 0$ we now consider the sets $\lambda E = \{\lambda x : x \in E\}$, $\lambda\Omega = \{\lambda x : x \in \Omega\}$.

With a change of variable $x = y/\lambda$ we get

$$\mathcal{F}_H(E) = |\partial E|(\Omega) + \int_\Omega \phi_E(x) H(x)\,dx = \lambda^{1-n}\,|\partial(\lambda E)|(\lambda\Omega) +$$

$$+\lambda^{-n}\int_{\lambda\Omega} \phi_{\lambda E}(y) H(y/\lambda)\,dy = \lambda^{1-n}\left(|\partial(\lambda E)|(\lambda\Omega) + \lambda^{-1}\int_{\lambda\Omega} \phi_{\lambda E}(y) H(y/\lambda)\,dy\right)$$

which shows that if E has mean curvature H in Ω, then λE has mean curvature $H_\lambda(y) = = \lambda^{-1} H(y/\lambda)$ in $\lambda\Omega$.

THEOREM 1.1. If E has mean curvature $H \in L^n(\Omega)$, $0 \in \Omega \cap \partial E$, and $\lambda(k)$ is a sequence of positive numbers with $\lambda(k) \underset{k}{\to} +\infty$, then there exist a subsequence $\mu(k)$ of $\lambda(k)$ and a set E_∞ with zero mean curvature (*i.e.*, a local minimizer of $\mathcal{F}_H$ in $\boldsymbol{R}^n$, with $H \equiv 0$) such that

$$\mu(k) E \underset{k}{\to} E_\infty$$

in $L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. When $n \leqslant 7$, we have specifically that E_∞ is a half-space; in this case moreover

$$\lim_{r\to 0} \frac{|E \cap B_r|}{\omega_n r^n} = \frac{1}{2} \qquad \text{and} \qquad \lim_{r\to 0} \frac{|\partial E|(B_r)}{\omega_{n-1} r^{n-1}} = 1.$$

PROOF (see also [9]). Lemma 1.2 gives, for all $R > 0$ and $\lambda > 0$:

$$|\partial(\lambda E)|(B_R) = \lambda^{n-1}\,|\partial E|(B_{R/\lambda}) \leqslant c_1(n)(1 + \|H\|_{n,R/\lambda}) R^{n-1}$$

whence the family $\{\lambda(k) E : k \in \boldsymbol{N}\}$ is relatively compact in $L^1_{\mathrm{loc}}(\boldsymbol{R}^n)$. Let then $\mu(k)$ (subsequence of $\lambda(k)$) and $E_\infty \subset \boldsymbol{R}^n$ be such that

$$\mu(k) E \underset{k}{\to} E_\infty \qquad \text{in } L^1_{\mathrm{loc}}(\boldsymbol{R}^n).$$

We have $0 \in \partial E_\infty$: indeed, from (1.1) it follows that

$$|E_\infty \cap B_R| = \lim_k |\mu(k) E \cap B_R| = \lim_k \mu(k)^n\,|E \cap B_{R/\mu(k)}| \geqslant \frac{\omega_n R^n}{2^n}.$$

Similarly, $|B_R - E_\infty| \geqslant \omega_n R^n / 2^n$ (notice that E has mean curvature H if and only if $\Omega - E$ has mean curvature $-H$).

To check the minimality of E_∞, choose $\rho \in (0, R)$ so that

$$\lim_k \int_{\partial B_\rho} |\phi_{\mu(k)E} - \phi_{E_\infty}|\,dH_{n-1} = 0. \tag{1.12}$$

For F such that $(F - E_\infty) \cup (E_\infty - F) \subset\subset B_\rho$, set

$$F_k = (F \cap B_\rho) \cup (\mu(k) E - B_\rho).$$

Since $\mu(k)E$ has mean curvature $H_k(x) = \mu(k)^{-1} H(x/\mu(k))$, we have

$$|\partial(\mu(k)E)|(B_R) + \int_{B_R} \phi_{\mu(k)E}(x) H_k(x)\,dx \leq |\partial F_k|(B_R) + \int_{B_R} \phi_{F_k}(x) H_k(x)\,dx =$$

$$= |\partial F|(B_\rho) + \int_{\partial B_\rho} |\phi_F - \phi_{\mu(k)E}|\,dH_{n-1} + |\partial(\mu(k)E)|(B_R - \overline{B}_\rho) + \int_{B_R} \phi_{F_k}(x) H_k(x)\,dx$$

hence

$$|\partial(\mu(k)E)|(B_\rho) \leq |\partial F|(B_\rho) + \int_{\partial B_\rho} |\phi_{E_\infty} - \phi_{\mu(k)E}|\,dH_{n-1} + \int_{B_R} |H_k|\,dx. \tag{1.13}$$

Now

$$\int_{B_R} |H_k|\,dx = \int_{B_R} \mu(k)^{-1} |H(x/\mu(k))|\,dx = \mu(k)^{-1} \int_{B_{R/\mu(k)}} |H(y)|\,\mu(k)^n\,dy \leq \tag{1.14}$$

$$\leq \mu(k)^{n-1} \|H\|_{n,R/\mu(k)} \, |B_{R/\mu(k)}|^{1-1/n} = \|H\|_{n,R/\mu(k)}\, \omega_n^{1-1/n} R^{n-1} \underset{k}{\to} 0.$$

From (1.13), thanks to (1.12), (1.14) and the semicontinuity of the perimeter, we get $|\partial E_\infty|(B_R) \leq |\partial F|(B_R)$ which holds for all $R > 0$, thus showing that E_∞ has zero mean curvature in $\boldsymbol{R}^n$. Moreover

$$\lim_k |\partial(\mu(k)E)|(B_R) = |\partial E_\infty|(B_R). \tag{1.15}$$

It is well known that when $n \leq 7$ the set E_∞ is a half-space with $0 \in \partial E_\infty$, so that

$$|E_\infty \cap B_R| = \frac{\omega_n R^n}{2} = \lim_k |\mu(k)E \cap B_R| = \lim_k \mu(k)^n \, |E \cap B_{R/\mu(k)}| =$$

$$= \omega_n R^n \lim_k \frac{|E \cap B_{R/\mu(k)}|}{\omega_n (R/\mu(k))^n}$$

i.e.

$$\lim_k \frac{|E \cap B_{R/\mu(k)}|}{\omega_n (R/\mu(k))^n} = \frac{1}{2}. \tag{1.16}$$

Similarly, by (1.15)

$$|\partial E_\infty|(B_R) = \omega_{n-1} R^{n-1} = \lim_k |\partial(\mu(k)E)|(B_R) =$$

$$= \lim_k \mu(k)^{n-1} |\partial E|(B_{R/\mu(k)}) = \omega_{n-1} R^{n-1} \lim_k \frac{|\partial E|(B_{R/\mu(k)})}{\omega_{n-1}(R/\mu(k))^{n-1}}$$

i.e.

$$\lim_k \frac{|\partial E|(B_{R/\mu(k)})}{\omega_{n-1}(R/\mu(k))^{n-1}} = 1. \tag{1.17}$$

The proof is easily concluded. ♦

We now show that any set E with mean curvature $H \in L^n(\Omega)$ can be redefined on a set of Lebesgue measure zero, so as to produce a new set E^* which has the same curvature H and some kind of «topological regularity». For simplicity in the exposition, we only consider the case when $\Omega = \boldsymbol{R}^n$. In the following proof, we denote by $E(\alpha)$, $0 \leqslant \alpha \leqslant 1$, the set of points where E has density α, *i.e.*

$$E(\alpha) = \left\{ x \in \boldsymbol{R}^n : \lim_{r \to 0} \frac{|E \cap B_{x,r}|}{\omega_n r^n} = \alpha \right\}.$$

THEOREM 1.2. Let $E \subset \boldsymbol{R}^n$ be a set of mean curvature $H \in L^n(\boldsymbol{R}^n)$, and set $E^* = E(1) \cup \partial E$. Then E^* differs from E by a set of Lebesgue measure zero, H is a curvature for E^*, E^* coincides with the closure of its interior, and $|\partial E^*|(A) = H_{n-1}(A \cap \partial E^*)$ for all open $A \subset \boldsymbol{R}^n$.

PROOF. If $x \notin \partial E$, then (recall the convention at the beginning of this section) we can find $r > 0$ such that either $|E \cap B_{x,r}| = 0$ or $|E \cap B_{x,r}| = \omega_n r^n$. Moreover, if $x \in E(0)$, then (by Lemma 1.1) $x \notin \partial E$ so that $|E \cap B_{x,r}| = 0$ for a suitable $r > 0$. This shows that $E(0)$, $E(1)$ and ∂E form a partition of $\boldsymbol{R}^n$, and that $E(0)$ is open. Similarly, $E(1)$ is open, and ∂E is closed. Clearly, $\partial E \subset \partial E(1) \cap \partial E(0)$, since $E(1)$ is equivalent to E and $E(0)$ is equivalent to $\boldsymbol{R}^n - E$. On the other hand, the closure of $E(0)$ is a subset of $\boldsymbol{R}^n - E(1)$, since $E(1)$ is open, so that $\partial E(0) \subset \boldsymbol{R}^n - [E(0) \cup E(1)] = \partial E$.

By similar considerations, we get $\partial E(0) = \partial E(1) = \partial E = \boldsymbol{R}^n - [E(0) \cup E(1)]$.

Now, it is well known that any set $F \subset \boldsymbol{R}^n$ of locally finite perimeter satisfies:

$$F(1/2) \subset \partial F,$$

$$|\partial F|(A) = H_{n-1}(F(1/2) \cap A) < +\infty \quad \text{for all open } A \subset\subset \boldsymbol{R}^n,$$

$$H_{n-1}(\boldsymbol{R}^n - [F(0) \cup F(1) \cup F(1/2)]) = 0,$$

so that $H_{n-1}(\partial E - E(1/2)) = 0$, hence $H_n(\partial E) = 0$. In conclusion $E^* = E(1) \cup \partial E = \boldsymbol{R}^n - E(0)$ differs from E by a set of Lebesgue measure zero (hence H is a curvature for E^* too), coincides with the closure of its interior $E(1)$, and since $\partial E^* = \partial E$ we have in addition $|\partial E^*|(A) = H_{n-1}(A \cap \partial E^*)$ for all open $A \subset \boldsymbol{R}^n$, as was to be proven. ♦

REMARK 1.1. Define $\Psi(r) = |\partial E|(\bar{B}_r) - \inf\{|\partial F|(\bar{B}_r) : F - \bar{B}_r = E - \bar{B}_r\}$.

If E has mean curvature $H \in L^p(\Omega)$ and $B_r \subset \Omega$, then clearly

$$\Psi(r) \leqslant \int_{B_r} |H(x)|\, dx$$

and by Hölder's inequality we obtain

$$\Psi(r) \leqslant \omega_n^{1-1/p} \|H\|_{p,r} r^{n-n/p} . \tag{1.18}$$

When $p > n$ we thus get $\Psi(r) \leqslant \text{const} \cdot r^{n-1+\varepsilon}$, $\varepsilon = 1 - n/p > 0$. As shown by Massari in [7], this is enough to start the procedure leading to the (partial) regularity of ∂E quoted in the introduction. Actually, it was shown in [10] that the following estimate is sufficient for the regularity result to hold:

$$\Psi(r) \leq \alpha(r)\, r^{n-1}, \qquad \text{with} \quad \int_0^\delta \frac{\sqrt{\alpha(r)}}{r}\, dr < +\infty, \qquad \delta > 0. \tag{1.19}$$

This condition may fail when $p \leq n$. Indeed, examples of singular sets having mean curvature in L^p, with $p < n$, are well known (see *e.g.* [3]). In the next section we settle the question of what happens when $p = n$ – an open problem since 1975 – by the explicit construction of a set $E \subset \boldsymbol{R}^2$ which is singular at $0 \in \partial E$ and mean curvature $H \in L^2(\boldsymbol{R}^2)$.

A general method yielding a mean curvature of an arbitrary set $E \subset \boldsymbol{R}^n$ with $|\partial E|(\boldsymbol{R}^n) < +\infty$ has been presented in [2]. However, when E has a simple geometry, it may be convenient to apply, the following.

LEMMA 1.3. Suppose $E \subset \Omega$ is a set with a smooth boundary in Ω, and assume that the exterior normal unit vector $\nu(x)$ at $x \in \Omega \cap \partial E$ can be extended to a vector field $V: \Omega \to \boldsymbol{R}^n$ with $V \in W^{1,1}_{\text{loc}}(\Omega) \cap C(\Omega)$, $\|V\|_\infty \leq 1$; then

$$H(x) = -\operatorname{div} V(x) \tag{1.20}$$

is a curvature for E.

The proof is a straightforward application of the divergence theorem, see *e.g.* [3, 11]. We remark explicitly that the method works also when isolated singularities are present: *e.g.*, if $0 \in \Omega \cap \partial E$ is the only singular point of E, then we can apply Lemma 1.3 to $\Omega - B_r$ and pass to the limit as $r \to 0$.

2. A SINGULAR SET IN $\boldsymbol{R}^2$ WITH CURVATURE IN L^2

Let us consider the following «antipodal spirals» in the (x,y)-plane:

$$\Sigma(t) = x(t) + iy(t) = t\,[\cos\theta(t) + i \sin\theta(t)], \tag{2.1}$$

$$\Sigma^*(t) = x^*(t) + iy^*(t) = t\,[\cos(\theta(t) + \pi) + i\sin(\theta(t) + \pi)], \tag{2.2}$$

where $t \in (0,1)$ and the «angular function» $\theta: (0,1) \to (0, +\infty)$ is required to be convex, strictly decreasing and of class C^2 on the interval $(0,1)$, with

$$\lim_{t\to 0} \theta(t) = +\infty, \qquad \lim_{t \to 1} \theta(t) = 0. \tag{2.3}$$

We have clearly

$$\Sigma(t) = -\Sigma^*(t), \qquad \|\Sigma(t)\| = t \qquad \forall t \in (0,1), \tag{2.4}$$

$$\lim_{t\to 1} \Sigma(t) = (1,0), \qquad \lim_{t\to 0}\Sigma(t) = (0,0). \tag{2.5}$$

Call

$$E = \{t[\cos(\theta(t) + \alpha) + i\sin(\theta(t) + \alpha)]: 0 < t < 1,\ \pi < \alpha < 2\pi\} \tag{2.6}$$

the subset of the unit ball $B_1 \subset \boldsymbol{R}^2$ lying «between» the two spirals Σ, Σ^*. Clearly $B_1 \cap \cap\, \partial E = \Sigma \cup \Sigma^* \cup \{(0,0)\}$, the origin being the only singular point of $B_1 \cap \partial E$.

Now, for every $\rho \in (0,1)$, the circle $x^2 + y^2 = \rho^2$ has exactly two points in common with ∂E, namely $\Sigma(\rho)$ and $\Sigma^*(\rho) = -\Sigma(\rho)$. Moreover, the exterior unit normals to E at these two points coincide; we can then extend the exterior normals $\nu\colon \Sigma \cup \Sigma^* \to S^1$ to a vector field $V\colon B_1 - \{(0,0)\} \to S^1$ by the obvious requirement

$$V(x,y) = \nu(x(\rho), y(\rho)), \qquad x^2 + y^2 = \rho^2 . \tag{2.7}$$

On the account of Lemma 1.3 and the remark following it, we try to find conditions on $\theta(t)$ assuring that

$$\operatorname{div} V \in L^2(B_1). \tag{2.8}$$

Now, on Σ we have

$$\begin{aligned}\nu(x(t), y(t)) &= (-y'(t), x'(t))\,(x'^2(t) + y'^2(t))^{-1/2} = \\ &= (-\sin\theta(t) - t\theta'(t)\cos\theta(t),\ \cos\theta(t) - t\theta'(t)\sin\theta(t))\,(1 + t^2\theta'^2(t))^{-1/2}\end{aligned} \tag{2.9}$$

hence by (2.7)

$$V(x,y) = (\varphi_1(\rho), \varphi_2(\rho)) \tag{2.10}$$

where for $\rho \in (0,1)$

$$\varphi_1(\rho) = -\sin\theta(\rho) - \rho\,\theta'(\rho)\cos\theta(\rho)\,(1 + \rho^2\theta'^2(\rho))^{-1/2}, \tag{2.11}$$

$$\varphi_2(\rho) = \cos\theta(\rho) - \rho\,\theta'(\rho)\sin\theta(\rho)\,(1 + \rho^2\theta'^2(\rho))^{-1/2}. \tag{2.12}$$

If follows that

$$\operatorname{div} V = \varphi_1'(\rho)\cdot\cos\alpha + \varphi_2'(\rho)\cdot\sin\alpha \tag{2.13}$$

where of course $x = \rho\cos\alpha$, $y = \rho\sin\alpha$.

We check immediately that

$$\varphi_1'(\rho) = (2\theta' + \rho^2\theta'^3 + \rho\theta'')(-\cos\theta + \rho\theta'\sin\theta)\,(1 + \rho^2\theta'^2)^{-3/2}, \tag{2.14}$$

$$\varphi_2'(\rho) = (2\theta' + \rho^2\theta'^3 + \rho\theta'')(-\sin\theta - \rho\theta'\cos\theta)\,(1 + \rho^2\theta'^2)^{-3/2}. \tag{2.15}$$

Therefore, from (2.13), (2.14) and (2.15) we obtain

$$|\operatorname{div} V(x,y) \leqslant |2\theta' + \rho^2\theta'^3 + \rho\theta''|/(1 + \rho^2\theta'^2). \tag{2.16}$$

Thus, condition (2.8) is certainly satisfied if

$$\int_0^1 \frac{\rho(2\theta' + \rho^2\theta'^3 + \rho\theta'')^2}{(1 + \rho^2\theta'^2)^2}\, d\rho < +\infty . \tag{2.17}$$

By expanding the numerator in the preceding integral we find

$$4\theta'^2 + \rho^4\theta'^6 + \rho^2\theta''^2 + 4\rho^2\theta'^4 + 4\rho\,\theta'\,\theta'' + 2\rho^3\theta'^3\theta'' \leqslant 4\theta'^2(1 + \rho^2\theta'^2) + \rho^4\theta'^6 + \rho^2\theta''^2$$

since by assumption $\theta' < 0$ and $\theta'' \geqslant 0$.

The integrability condition (2.17) then holds, provided both functions

$$\rho\theta'^2(\rho) \quad \text{and} \quad \rho^3\theta''^2(\rho) \tag{2.18}$$

are integrable in $(0,1)$.

Among the simplest choices of functions $\theta(t)$, satisfying the convexity and monotonicity conditions stated above, (2.3) and (2.18), we have the following one:

$$\theta(t) = \log(1 - \log t), \qquad 0 < t < 1 . \tag{2.19}$$

We have thus proved

THEOREM 2.1. The set $E \subset \boldsymbol{R}^2$ defined by (2.6), with $\theta(t)$ given by (2.19), has mean curvature in $L^2(B_1)$. The origin is a singular point of ∂E.

Additional properties of E are given in the following remarks.

REMARK 2.1. Thanks to the symmetry of the construction, the set E given in Theorem 2.1 satisfies $|E \cap B_r| = |B_r|/2 \ \forall r \in (0,1)$ (compare with Theorem 1.1).

REMARK 2.2. Let r_α be the half-line issuing from 0 with direction (cos α, sen α), let $t_k(\alpha)$ be the value of the parameter t corresponding to the k-th intersection of $\Sigma(t)$ with r_α. One has $t_k(\alpha) = \exp(1 - \exp(\alpha + 2k\pi))$.

By choosing $\lambda(k) = (t_k(\alpha))^{-1}$, we see that $\lambda(k)E$ converges in $L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$ to the half-space through 0 with exterior normal vector $(-\operatorname{sen}\alpha, \cos\alpha)$ (compare with Theorem 1.1). Therefore in this case every half-space through 0 is the limit of a suitable sequence of dilations of E.

REMARK 2.3. We find easily that

$$\Psi(r) = |\partial E|(B_r) - 2r = 2\left[\int_0^r \sqrt{1 + \frac{1}{(1-\log t)^2}}\, dt - r\right]$$

so that

$$\lim_{r \to 0^+} \frac{\Psi(r)(1-\log r)^2}{r} = 1, \qquad i.e. \qquad \Psi(r) \approx \frac{r}{(1-\log r)^2}.$$

Compare with Remark 1.1: this shows in a sense that (1.19) is an «optimal condition» for regularity.

REMARK 2.4. One verifies easily that the mapping $f: B_1 \subset \boldsymbol{R}^2_{u,v} \to B_1 \subset \boldsymbol{R}^2_{x,y}$ given (in polar coordinates $u = t\cos\beta$, $v = t\sin\beta$ and, respectively, $x = \rho\cos\alpha$, $y = \rho\sin\alpha$) by $f(t,\beta) = (t, \theta(t) + \beta)$ for $0 < t < 1$, $0 \leq \beta < 2\pi$, $f(0,0) = (0,0)$ with $\theta(t)$ as in (2.19), is a bilipschitzian trasformation of B_1 in itself, which maps the lower hemidisk $B_1 \cap \{v < 0\}$ onto E and the diameter $B_1 \cap \{v = 0\}$ onto $B_1 \cap \partial E$.

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