Andrew Granville

Solution to a problem of Bombieri


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Abstract. — We solve a problem of Bombieri, stated in connection with the «prime number theorem» for function fields.

Key words: Distribution of primes; Elementary proofs; Prime number theorem.

Riassunto. — Soluzione di un problema di Bombieri. Questa Nota risolve in senso affermativo un problema posto da Bombieri in una recente Nota Lincea, dimostrando che la formula del tipo di Selberg considerata da Bombieri ammette solamente due soluzioni asintotiche.

In [1], Bombieri states that if \( a_1, a_2, \ldots \) is a sequence of non-negative real numbers satisfying the Selberg-type formula

\[
ma_m + \sum_{i=1}^{m-1} a_i a_{m-i} = 2m + O(1)
\]

for each \( m \geq 1 \) then \( a_m = 1 + o(1) \); however there is an error in the proof as may be seen by the counterexample \( a_m = 1 - (-1)^m \). In [2], Bombieri shows that this original result may be recovered by also having the analogous formula to (1) for the sequence \( a_2, a_4, \ldots \); and, in [4], Zhang improves the error term in this result to \( a_m = 1 + O(1/m) \).

Herein we return to the original question and solve (slightly more than) a problem stated by Bombieri in [2]:

**Theorem 1.** If \( a_1, a_2, \ldots \) is a sequence of non-negative real numbers satisfying

\[
ma_m + \sum_{i=1}^{m-1} a_i a_{m-i} = 2m + o(m)
\]

for each \( m \geq 1 \) then either (i) \( a_m = 1 + o(1) \); or (ii) \( a_m = 1 - (-1)^m + o(1) \).

In [3] (Theorem 2'), Erdös showed that for any sequence of non-negative real numbers satisfying (2) we have

\[
\sum_{i=1}^{m} a_i = m + o(m).
\]

We note also that as each \( a_i \geq 0 \), thus \( ma_m \leq 2m + o(m) \) by (2), and so

\[
0 \leq a_m \leq 2 + o(1).
\]

Therefore, by taking \( b_j = 1 - a_j \) for each \( j \), we see that Theorem 1 follows immediately from

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THEOREM 2. If \( b_1, b_2, \ldots \) is a sequence of real numbers satisfying
\[
|b_m| \leq 1 + o(1)
\]
and
\[
mb_m = \sum_{i=1}^{m-1} b_i b_{m-i} + o(m)
\]
for each \( m \geq 1 \) then one of the following cases holds:

(i) \( b_m = o(1) \);  
(ii) \( b_m = (-1)^m + o(1) \);  
(iii) \( b_m = 1 + o(1) \).

Proof. We start by showing that either (i) holds or
\[
B := \limsup_{m \to \infty} |b_m| = 1.
\]
First note that \( B \geq 0 \) by definition, and \( B \leq 1 \) by (5). Now \( m|b_m| \leq mB^2 + o(m) \) by (6) and so, choosing \( m \) with \( b_m = B + o(1) \), we have \( B \leq B^2 \). Therefore either \( B = 0 \) (in which case (i) holds) or \( B \geq 1 \), so that \( B = 1 \).

Next we show that if (7) holds then
\[
\max_{2m \leq n \leq 3m} |b_n| = 1 + o(1)
\]
as \( m \to \infty \). Suppose that (8) is false so that there exists \( \delta > 0 \) such that, for certain arbitrarily large \( m \), we have \( |b_n| < 1 - 10\delta \) for all \( n \) in the range \( 2m \leq n \leq 3m \). From this we can deduce (by induction on \( n \)) that \( |b_n| < 1 - \delta \) for all \( n > 3m \), if \( m \) is sufficiently large, which contradicts (7). The induction proceeds in a straightforward way, by using (6) in the form
\[
n|b_n| \leq \sum_{i=1}^{n-1} |b_i| |b_{n-i}| + o(n),
\]
and
\[
|b_j| \leq \begin{cases} 
O(1) & \text{for } j \leq j_0; \\
1 + \delta/10 & \text{for } j_0 < j < 2m; \\
1 - 10\delta & \text{for } 2m \leq j \leq 3m; \\
1 - \delta & \text{for } 3m < j \leq n - 1,
\end{cases}
\]
where \( j_0 \) is chosen so that \( |b_j| < 1 + \delta/10 \) for all \( j \geq j_0 \) (which is possible, by (5)).

We now prove two Lemmas.

LEMMA 1. If \( |b_m| = 1 + o(1) \) then \( |b_i| = 1 + o(1) \) and \( b_i b_{m-i} = b_m + o(1) \) for all but \( o(m) \) values of \( i \leq m \).

Proof. Let \( c_i = b_i b_{m-i}/b_m \) so that \( |c_i| \leq 1 + o(1) \) by (5) (if \( m - i \to \infty \)) and \( \sum_{i=1}^{m} c_i = m + o(m) \) by (6). Therefore \( c_i = 1 + o(1) \) for all but \( o(m) \) values of \( i \leq m \), which is equivalent to the second assertion of the Lemma. Moreover \( c_i = 1 + o(1) \) implies that \( |b_i| |b_{m-i}| = 1 + o(1) \) for such \( i \), whereas both \( |b_i| \) and \( |b_{m-i}| \) are \( \leq 1 + o(1) \) by (5). Thus both \( |b_i| \) and \( |b_{m-i}| \) equal \( 1 + o(1) \).
LEMMA 2. Fix \( \varepsilon > 0 \). For any sufficiently large \( m \) and for any integers \( k \) and \( n \) in the range \( 1 \leq k \leq em, \ m \leq n \leq 2m \), for which \( |b_n|, \ |b_{n+k}| = 1 + o(1) \), we have the estimate

\[
|b_{m+k}| = b_m b_{n+k}/b_n + O(\varepsilon),
\]

where the constant implied by \( O(\varepsilon) \) is absolute.

PROOF. Let \( \sigma = 1 \) if \( b_n \) and \( b_{n+k} \) have the same sign, and let \( \sigma = -1 \) otherwise. Now, as \( |b_n| \) and \( |b_{n+k}| \) both equal \( 1 + o(1) \), we see that \( |b_i| = 1 + o(1) \), \( b_i b_{n-i} = b_n + o(1) \) and \( b_i b_{n+k-i} = b_{n+k} + o(1) \) for all but \( o(m) \) values of \( i \leq n \), by Lemma 1. Therefore, by taking \( j = n - i \), we see that \( b_j = \sigma b_{j+k} + o(1) \) for all but \( o(m) \) values of \( j \leq m \). Substituting this into (6) gives

\[
mb_m - \sigma \left( (m+k)b_{m+k} - \sum_{i=1}^{k} b_i b_{m+k-i} \right) = \sum_{j=1}^{m-1} (b_j - \sigma b_{j+k}) b_{m-j} + o(m) = o(m),
\]

and the result follows from (5) as

\[
k|b_{m+k}| + \sum_{i=1}^{k} |b_i b_{m+k-i}| = O(k).
\]

Completion of the proof of Theorem 2. Fix \( \varepsilon > 0 \) and suppose that \( m \) is sufficiently large. By (8) there exists \( n \) in the range \( 2m \leq n \leq 3m \) with \( |b_n| = 1 + o(1) \), and so \( |b_i| = 1 + o(1) \) for all but \( o(m) \) values of \( i \leq 2m \), by Lemma 1. Therefore there exists an integer \( k \) in the range \( 1 \leq k \leq em \) such that both \( |b_{m+k}| \) and \( |b_{m+2k}| = 1 + o(1) \). Taking \( n = m + k \) in Lemma 2, we see that \( b_m = b_{m+2k} + O(\varepsilon) \); and letting \( \varepsilon \to 0 \), we then get

\[
|m| = 1 + o(1)
\]

as \( m \to \infty \).

Now take \( k = 1 \) and \( m = m + 1 \) in Lemma 2. By (9) this implies that \( b_m b_{m+2} = b_{m+1} + o(1) = 1 + o(1) \), so that \( b_m \) and \( b_{m+2} \) have the same sign if \( m \) is sufficiently large. Therefore there exist constants \( \nu \) and \( \gamma \), equal to \( -1 \) or \( 1 \), such that \( b_{2m} = \nu + o(1) \); and \( b_{2m+1} = \gamma + o(1) \). Substituting this into (6) for even \( m \) we obtain \( \nu = (\nu^2 + \gamma^2)/2 = 1 \); therefore we get (ii) if \( \gamma = -1 \), and (iii) if \( \gamma = 1 \).

REFERENCES


Department of Mathematics - University of Georgia
ATHENS, Ga. 30602 (U.S.A.)