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The Julia-Carathéodory theorem for distance-decreasing maps on infinite dimensional hyperbolic spaces

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Matematica. — *The Julia-Carathéodory theorem for distance-decreasing maps on infinite dimensional hyperbolic spaces.* Nota di Kazimierz Włodarczyk, presentata (*) dal Socio E. Vesentini.

Abstract. — A classical Julia-Carathéodory theorem concerning radial limits of holomorphic maps in one dimension is extended to hyperbolic contractions of bounded symmetric domains in J^*-algebras.

Key words: Infinite dimensional bounded symmetric homogeneous domains; Hyperbolic metrics; Distance-decreasing maps; Julia-Carathéodory theorem; J^*-algebras.

Riassunto. — *Il teorema di Julia-Carathéodory per applicazioni contrattive in spazi iperbolici di dimensioni infinite.* Un classico risultato di Julia e Carathéodory, relativo a limiti radiali di funzioni olomorfe di una variabile, viene esteso alle contrazioni iperboliche in domini limitati simmetrici in algebra J^*.

1. Introduction

Let H and K be Hilbert spaces over $\boldsymbol{C}$, let $\mathcal{L}(H,K)$ denote the Banach space of all bounded linear operators X from H to K with the operator norm, and let $\mathcal{A} \subset \mathcal{L}(H,K)$ be a J^*-algebra, *i.e.* a closed complex linear subspace of $\mathcal{L}(H,K)$ such that $XX^*X \in \mathcal{A}$ whenever $X \in \mathcal{A}$.

J^*-algebras, being natural generalizations of C^*-algebras, B^*-algebras, JC^*-algebras, ternary algebras, complex Hilbert spaces and others, are infinite dimensional complex Banach spaces whose open unit balls are bounded symmetric homogeneous domains. In particular, all four types of the classical Cartan bounded symmetric homogeneous domains in $\boldsymbol{C}^n$[5] and their infinite dimensional analogues [14, 16] are the open unit balls in some J^*-algebras.

There is, of course, a large literature concerning holomorphic maps of $\Delta = \{u \in \boldsymbol{C}: |u| < 1\}$ into itself. The kind of the results we are interested in started with the works of Denjoy, Wolff, Schwarz, Pick, Julia, Carathéodory, Koebe and Landau. For the generalizations of some of them to higher dimensional spaces, see [2, 9-12, 6, 18, 19, 1, 3, 21] and others. The following classical Julia-Carathéodory theorem (see *e.g.* [4, p. 57]) concerning the existence of radial limits of holomorphic maps is well known.

Let $f\colon \Delta \to \Delta$ be a holomorphic map and let there exist a sequence $\{x_n\}$ in Δ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_n) = 1$ and

$$\alpha = \lim_{n\to\infty} \frac{1 - |f(x_n)|}{1 - |x_n|} < +\infty .$$

Then $|1 - f(x)|^2/(1 - |f(x)|^2) \le \alpha \cdot |1 - x|^2/(1 - |x|^2)$, for all $x \in \Delta$, and the following limit equations hold, where x tends to its limit through arbitrary real values and α is

(*) Nella seduta del 13 febbraio 1993.

a real positive constant,

$$\alpha = \lim_{x \to 1} \frac{1 - |f(x)|}{1 - x} = \lim_{x \to 1} \frac{|1 - f(x)|}{1 - x} = \lim_{x \to 1} \frac{1 - f(x)}{1 - x}.$$

The main results of this paper provide the versions of the above theorem for distance-decreasing maps (hyperbolic contractions) on infinite dimensional hyperbolic spaces of operators. These hyperbolic spaces are bounded symmetric homogeneous domains in J^*-algebras endowed with hyperbolic metrics. The special case when these domains are the open Hilbert unit balls is considered.

2. Definitions and notations

Let H, K, M and N be complex Hilbert spaces, $\mathcal{A} \subset \mathcal{L}(H,K)$ and $\mathcal{B} \subset \mathcal{L}(M,N)$ – J^*-algebras, let I_H, I_K, I_M and I_N stand for the identity operators on H, K, M and N, respectively, and let

$$\mathcal{A}_0 = \{X \in \mathcal{A}: \|X\| < 1\}, \qquad \mathcal{B}_0 = \{X \in \mathcal{B}: \|X\| < 1\}.$$

Further, set

$$A_Z = I_H - Z^*Z \quad \text{and} \quad B_Z = I_K - ZZ^* \quad \text{for } Z \in \mathcal{A}_0, \tag{2.1}$$

$$A_Z = I_M - Z^*Z \quad \text{and} \quad B_Z = I_N - ZZ^* \quad \text{for } Z \in \mathcal{B}_0. \tag{2.2}$$

The hyperbolic metric ρ_1 on $\mathcal{A}_0$ is defined by the formula

$$\rho_1(X,Z) = \tanh^{-1}\|T_Z(X)\|, \qquad X,\ Z \in \mathcal{A}_0, \tag{2.3}$$

where T_Z, $Z \in \mathcal{A}_0$, is a biholomorphic map of $\mathcal{A}_0$ onto itself defined by (cf. [14, Theorem 2, p. 20])

$$T_Z(X) = B_Z^{-1/2}(X - Z)(I_H - Z^*X)^{-1}A_Z^{1/2}, \qquad X \in \mathcal{A}_0, \tag{2.4}$$

$(\mathcal{A}_0, \rho_1)$ is a complete metric space and, moreover, since any automorphism f of $\mathcal{A}_0$ with $f(W) = 0$ is given by $f = L \circ T_W$ where $L: \mathcal{A} \to \mathcal{A}$ is a surjective linear isometry (cf. [14, Theorem 3, p. 23]), the metric ρ_1 is also represented in the following way:

$$\rho_1(Z,W) = \inf\left\{\frac{1}{2}\log\frac{1+r}{1-r}: 0 < r < 1 \text{ and } r\mathcal{A}_0 \ni f(Z) \text{ for some } f \in \operatorname{Aut}(\mathcal{A}_0) \text{ with } f(W) = 0\right\},$$

where $\operatorname{Aut}(\mathcal{A}_0)$ denotes the group of automorphisms of $\mathcal{A}_0$ and $r\mathcal{A}_0$ denotes the set $\{rX: X \in \mathcal{A}_0\}$. Furthermore, we have

$$\{Z \in \mathcal{A}_0: \rho_1(0,Z) < s\} = r\mathcal{A}_0 \qquad \text{where} \qquad r = (e^{2s} - 1)/(e^{2s} + 1).$$

Let us denote

$$\rho_2(X,Z) = \tanh^{-1}\|T_Z(X)\|, \qquad X,\ Z \in \mathcal{B}_0, \tag{2.5}$$

where

$$T_Z(X) = B_Z^{-1/2}(X - Z)(I_M - Z^*X)^{-1}A_Z^{1/2}, \qquad X \in \mathcal{B}_0. \tag{2.6}$$

The ρ_2 and T_Z for $\mathcal{B}_0$ defined by (2.5) and (2.6) play the corresponding role as ρ_1 and T_Z for $\mathcal{A}_0$ defined by (2.3) and (2.4), respectively.

The map $F\colon \mathcal{A}_0 \to \mathcal{B}_0$ is a hyperbolic contraction if $\rho_2[f(X), f(Z)] \le \rho_1(X,Z)$ for all $X, Z \in \mathcal{A}_0$, and a hyperbolic isometry if $\rho_2[f(X), f(Z)] = \rho_1(X,Z)$ for all $X, Z \in \mathcal{A}_0$. It is known that each holomorphic map $f\colon \mathcal{A}_0 \to \mathcal{B}_0$ is a hyperbolic contraction and, in particular, each biholomorphic map $f\colon \mathcal{A}_0 \to \mathcal{B}_0$ is a hyperbolic isometry.

For further details concerning distances and pseudodistances in infinite dimensional complex spaces, see *e.g.* [7, 8, 13, 15, 17, 20].

3. Statement of results

For a map $f\colon \mathcal{A}_0 \to \mathcal{B}_0$ and for $X, Y \in \mathcal{B}_0$, let $D(X,Y) \in \mathcal{L}(M,M)$ be the operator defined as follows

$$D(X,Y) = A_{f(0)}^{1/2}[I_M - Y^* f(0)]^{-1}(I_M - Y^* X)[I_M - f(0)^* X]^{-1} A_{f(0)}^{1/2}\,.$$

Using these notations, we give the following conditions which guarantee the existence of the radial limits for hyperbolic contractions on bounded symmetric homogeneous domains in J^*-algebras endowed with hyperbolic metrics.

Theorem 3.1. *Let $f\colon \mathcal{A}_0 \to \mathcal{B}_0$ be a hyperbolic contraction, $U \in \partial\mathcal{A}_0$ and $W \in \partial B_0$. Let*

$$\alpha = \liminf_{X \to U,\, X \in \mathcal{A}_0} \frac{\|D(f(X), f(X))\|}{1 - \|X\|^2} < +\infty \tag{3.1}$$

and

$$\alpha = \lim_{n \to \infty} \frac{\|D(f(X_n), f(X_n))\|}{1 - \|X_n\|^2} \tag{3.2}$$

for some sequence $\{X_n\}$ in $\mathcal{A}_0$ such that $X_n \to U$ and $f(X_n) \to W$. Then $\alpha \ge 1$ and

$$\|(D[f(X), W] D[f(X), f(X)]^{-1} D[W, f(X)])\| \le \alpha \cdot \|(I_H - U^* X) A_X^{-1} (I_H - X^* U)\| \tag{3.3}$$

for all $X \in \mathcal{A}_0$. Moreover, if U is an isometry in $\mathcal{A}$ and

$$\alpha = \liminf_{X \to U,\, X \in \mathcal{A}_0} \frac{1 - \|D(f(X), f(X))^{-1}\|^{-1}}{1 - \|X\|^2}, \tag{3.4}$$

then

$$\begin{aligned} \alpha &= \lim_{\lambda \to 1} \frac{1 - \|D(f(\lambda U), f(\lambda U))^{-1}\|^{-1}}{1 - \lambda^2}, \\ &= \lim_{\lambda \to 1} \frac{1 - [1 - \|D(f(\lambda U), f(\lambda U))^{-1}\|^{-1}]^{1/2}}{1 - \lambda}, \end{aligned} \tag{3.5}$$

and if, additionally, J^*-algebra $\mathcal{B}$ is such that

$$\|B_Z\| = 1 - \|Z\|^2 \tag{3.6}$$

for all $Z \in \mathcal{B}_0$, *then*

$$(3.7) \qquad \alpha = \lim_{\lambda \to 1} \frac{\|D(f(\lambda U), W)\|}{1-\lambda}.$$

In (3.5) *and* (3.7) λ *tends to its limit through arbitrary real values.*

Let $H_0 = \{x \in H: \|x\| < 1\}$, $K_0 = \{x \in K: \|x\| < 1\}$ and, for a map $f: H_0 \to K_0$ and for x, $y \in K_0$, let $d(x,y) = (1 - \langle x,y\rangle)(1 - \|f(0)\|^2)/[(1 - \langle f(0),y\rangle)(1 - \langle x, f(0)\rangle)]$. Let E_z, $z \in H_0$, denote the linear projection of H onto the subspace $\{uz: u \in \mathbf{C}\}$. By identifying H with the J^*-algebra $\mathcal{L}(\mathbf{C}, H)$, the biholomorphic map T_z, $z \in H_0$, of H_0 onto itself is defined by the formula

$$T_z(x) = \frac{[E_z + (1 - \|z\|^2)^{1/2}(I_H - E_z)](x-z)}{1 - \langle x,z\rangle}.$$

A special case of Theorem 3.1 is

THEOREM 3.2. *Let* $f: H_0 \to K_0$ *be a hyperbolic contraction,* $u \in \partial H_0$, $w \in \partial K_0$,

$$(3.8) \qquad \alpha = \lim_{x \to u} \inf_{x \in H_0} \frac{1 - |d(f(x), f(x))|}{1 - \|x\|^2} < +\infty$$

and $f(x_n) \to w$ *for some sequence* $\{x_n\}$ *in* H_0 *such that* $x_n \to u$ *and*

$$(3.9) \qquad \alpha = \lim_{n \to \infty} \frac{1 - |d(f(x_n), f(x_n))|}{1 - \|x_n\|^2}.$$

Then $\alpha \geq 1$,

$$(3.10) \qquad \frac{|d(f(x), w)|^2}{|d(f(x), f(x))|} \leq \alpha \frac{|1 - \langle x, u\rangle|^2}{1 - \|x\|^2}$$

for all $x \in H_0$ *and*

$$(3.11) \qquad \alpha = \lim_{\lambda \to 1} \frac{1 - |d(f(\lambda u), f(\lambda u))|}{1 - \lambda^2} = \lim_{\lambda \to 1} \frac{1 - [1 - |d(f(\lambda u), f(\lambda u))|]^{1/2}}{1 - \lambda} = $$
$$= \lim_{\lambda \to 1} \frac{|d(f(\lambda u), w)|}{1 - \lambda} = \lim_{\lambda \to 1} \frac{d(f(\lambda u), w)}{1 - \lambda}$$

where λ *tends to its limit through arbitrary real values.*

REMARK 3.1. In general, the proof of the classical Julia-Carathéodory theorem on the existence of radial limits of holomorphic maps $f: \Delta \to \Delta$, $\Delta = \{u \in \mathbf{C}: |u| < 1\}$, based on the geometric properties of the oricycles in Δ, differs from those given in our paper. For details, see [4, pp. 53-57].

Also, we show the following characterization of ρ_1.

THEOREM 3.3. *If* $V \in \partial\mathfrak{A}_0$ *and* $0 \leq \tau < \gamma < \mu < 1$, *then* $\rho_1(\tau V, \mu V) = \rho_1(\tau V, \gamma V) + \rho_1(\gamma V, \mu V)$.

4. Proof of Theorem 3.1

First, let us observe that if $X,\ Y,\ Z \in \mathcal{A}_0$, then

$$\|T_{f(Y)}[f(X)]\| \le \frac{\|T_{f(Y)}[f(Z)]\| + \|T_Z(X)\|}{1 + \|T_{f(Y)}[f(Z)]\| \cdot \|T_Z(X)\|} \tag{4.1}$$

and

$$\|f(X)\| \le \frac{\|f(0)\| + \|X\|}{1 + \|f(0)\| \cdot \|X\|}, \qquad \frac{1 - \|f(0)\|}{1 + \|f(0)\|} \le \frac{1 - \|f(X)\|}{1 - \|X\|}. \tag{4.2}$$

Indeed, the triangle inequality implies

$$\rho_2[f(X), f(Y)] \le \rho_2[f(Z), f(Y)] + \rho_2[f(X), f(Z)], \qquad X,\ Y,\ Z \in \mathcal{A}_0 .$$

Moreover, by our assumption, $\rho_2[f(X), f(Z)] \le \rho_1(X,Z)$, $X,\ Z \in \mathcal{A}_0$. Hence we infer, using (2.1)-(2.6), that (4.1) follows. Analogously, using $\rho_2[f(X),0] \le \rho_2[f(0),0] + \rho_2[f(X), f(0)]$, we prove (4.2).

Next, let us observe that, using [22, equalities (7)], we obtain

$$(I_M - T_{f(0)}(Y)^* T_{f(0)}(X)) = D(X,Y), \qquad X,\ Y \in \mathcal{B}_0, \tag{4.3}$$

and, from [23, equality (18)] we get

$$\|D(X,X)^{-1}\| = (1 - \|T_{f(0)}(X)\|^2)^{-1}, \qquad X \in \mathcal{B}_0. \tag{4.4}$$

Let now $F = T_{f(0)} \circ f$. Since $T_{f(0)}$ acts transitively on $\mathcal{B}_0$, the map $F\colon \mathcal{A}_0 \to \mathcal{B}_0$ is also a hyperbolic contraction, *i.e.*, in particular,

$$\rho_2[F(X), F(X_m)] \le \rho_1(X, X_m) \tag{4.5}$$

for all $X \in \mathcal{A}_0$ and $m \in \mathbf{N}$. By (2.1)-(2.6), from (4.5) we have

$$\|A_{F(X_m)}^{-1/2}[I_M - F(X_m)^* F(X)] A_{F(X)}^{-1} [I_M - F(X)^* F(X_m)] A_{F(X_m)}^{-1/2}\| \le \tag{4.6}$$
$$\le \|A_{X_m}^{-1/2}(I_H - X_m^* X) A_X^{-1} (I_H - X^* X_m) A_{X_m}^{-1/2}\|.$$

But $\|A_X^{-1}\| = (1 - \|X\|^2)^{-1}$ for $\|X\| < 1$. Thus (4.6) implies

$$\|[I_M - F(X_m)^* F(X)] A_{F(X)}^{-1} [I_M - F(X)^* F(X_m)]\| \le \tag{4.7}$$
$$\le \|A_{F(X_m)}\| (1 - \|X_m\|^2)^{-1} \|(I_H - X_m^* X) A_X^{-1} (I_H - X^* X_m)\|.$$

Inequality (4.7), by (4.3) and (3.2), implies (3.3).

Since $F(0) = 0$ thus, in virtue of (4.2), we have $\|F(X_m)\| \le \|X_m\|$ and, consequently,

$$1 \le (1 - \|F(X_m)\|^2)/(1 - \|X_m\|^2) \le \|A_{F(X_m)}\|/(1 - \|X_m\|^2). \tag{4.8}$$

A consequence of (4.8), by (4.3), (3.2) and (2.2), is

$$\alpha \ge 1. \tag{4.9}$$

Assume that

$$0 < 1 - \lambda < 1/\alpha \tag{4.10}$$

and

$$1 - \lambda = 2c, \qquad 0 < c < 1. \tag{4.11}$$

By (4.9), we have $c \leq \alpha c$. Thus $\alpha c < 1/2$ by (4.10) and (4.11). Consequently,

$$\alpha c/(1-c) \leq \alpha c/(1-\alpha c). \tag{4.12}$$

Let U be an isometry and let (3.4) hold. For $X = \lambda U$, from (4.7), by (4.8), (3.4) and (4.4), we get

$$\|[I_M - T_{f(0)}(W)^* F(\lambda U)] A_{F(\lambda U)}^{-1} [I_M - F(\lambda U)^* T_{f(0)}(W)]\| \leq \alpha \cdot (1-\lambda)/(1+\lambda). \tag{4.13}$$

Now, from (4.13), using (4.12) and (4.11), we obtain

$$\|A_{F(\lambda U)}^{-1}\| \leq \alpha c (1-\alpha c)^{-1} \|[I_M - T_{f(0)}(W)^* F(\lambda U)]^{-1}\|^2,$$

which yields

$$[1 - \|F(\lambda U)\|]/[1 + \|F(\lambda u)\|] \leq \alpha c/(1-\alpha c) \tag{4.14}$$

since

$$\|A_{F(\lambda U)}^{-1}\| = (1 - \|F(\lambda U)\|^2)^{-1}$$

and

$$\|[I_M - T_{f(0)}(W)^* F(\lambda U)]^{-1}\| \leq (1 - \|F(\lambda U)\|)^{-1}.$$

From (4.14) we obtain $\|F(\lambda U)\| \geq 1 - 2\alpha c$, which, by (4.11), is equivalent to the inequality

$$1 - \|F(\lambda U)\| \leq \alpha(1-\lambda). \tag{4.15}$$

Moreover, by (4.2), $\|F(\lambda U)\| \leq \lambda$. Consequently, $1 + \|F(\lambda U)\| \leq 1 + \lambda$. From this and (4.15) we obtain

$$\frac{1 - \|F(\lambda U)\|^2}{1-\lambda^2} \leq \frac{1 - \|F(\lambda U)\|}{1-\lambda} \leq \alpha. \tag{4.16}$$

Thus (4.16), (4.4) and (3.4) imply (3.5).

On the other hand, from (4.13) we get

$$\frac{\|I_M - T_{f(0)}(W)^* F(\lambda U)\|^2}{(1-\lambda)^2} \leq \alpha \cdot \frac{\|A_{F(\lambda U)}\|}{1-\lambda^2}. \tag{4.17}$$

If (3.6) holds, then (4.17) and (4.16) give

$$\frac{\|I_M - T_{f(0)}(W)^* F(\lambda U)\|}{1-\lambda} \leq \alpha. \tag{4.18}$$

But

$$\frac{1 - \|F(\lambda U)\|}{1-\lambda} \leq \frac{1 - \|T_{f(0)}(W)^* F(\lambda U)\|}{1-\lambda} \leq \frac{\|I_M - T_{f(0)}(W)^* F(\lambda U)\|}{1-\lambda}. \tag{4.19}$$

Thus, from (3.4), (4.16), (4.18), (4.19) and (3.5), by (4.3), we get (3.7).

5. Proof of Theorem 3.2

Since in an arbitrary complex Hilbert space $H = \mathcal{L}(C, H)$ the formulae $x^* x = \|x\|^2 I_C$ and $x^* y = \langle y, x \rangle I_C$ hold for $x, y \in H$, any vector with norm one is an isometry and, in this case, conditions (3.1) and (3.4) are identical and (3.6) holds. From (3.5) and (3.7) we obtain

$$(5.1) \qquad \alpha = \lim_{\lambda\to 1} \frac{1-\|F(\lambda u)\|^2}{1-\lambda^2} = \lim_{\lambda\to 1} \frac{1-\|F(\lambda u)\|}{1-\lambda} = \\ = \lim_{\lambda\to 1} \frac{1-|\langle F(\lambda u), T_{f(0)}(w)\rangle|}{1-\lambda} = \lim_{\lambda\to 1} \frac{|1-\langle F(\lambda u), T_{f(0)}(w)\rangle|}{1-\lambda}.$$

We shall now prove the equality

$$(5.2) \qquad \alpha = \lim_{\lambda\to 1} \frac{1-\langle F(\lambda u), T_{f(0)}(w)\rangle}{1-\lambda}.$$

From (5.1) and (4.9) we have

$$(5.3) \qquad \lim_{\lambda\to 1} \frac{|1-\langle F(\lambda u), T_{f(0)}(w)\rangle|}{1-|\langle F(\lambda u), T_{f(0)}(w)\rangle|} = 1.$$

Let us denote $1-\langle F(\lambda u), T_{f(0)}(w)\rangle = r\cdot\exp(it)$, $r>0$, $t\in \boldsymbol{R}$. Then

$$\frac{|1-\langle F(\lambda u), T_{f(0)}(w)\rangle|}{1-|\langle F(\lambda u), T_{f(0)}(w)\rangle|} = \frac{1+(1-2r\cdot\cos t + r^2)^{1/2}}{2\cos t - r}.$$

If λ tends to 1, the r tends to 0 and, therefore, by (5.3), $\cos(t)$ must tend to 1 (thus also $\exp(it)$ tends to 1). But

$$\frac{1-\langle F(\lambda u), T_{f(0)}(w)\rangle}{1-|\langle F(\lambda u), T_{f(0)}(w)\rangle|} = \frac{\exp(it)\cdot[1+(1-2r\cdot\cos t + r^2)^{1/2}]}{2\cos t - r}.$$

This also implies

$$(5.4) \qquad \lim_{\lambda\to 1} \frac{1-\langle F(\lambda u), T_{f(0)}(w)\rangle}{1-|\langle F(\lambda u), T_{f(0)}(w)\rangle|} = 1.$$

By (5.1) and (5.4), we have (5.2). So, finally, from (5.1) and (5.2) we immediately obtain (3.11).

6. Proof of Theorem 3.3

First, we prove that

$$(6.1) \qquad \rho_1(\tau V, \gamma V) = \rho_1(0, \tau V) - \rho_1(0,\gamma V), \qquad 0<\tau<\gamma<1.$$

Indeed, then the map $g: \Delta \to \mathfrak{C}_0$ defined by $g(u) = uV$, $u\in\Delta$, is holomorphic, and thus, a hyperbolic contraction, *i.e.* $\rho_1[g(u), g(v)] \le \rho_C(u,v)$, $u,v\in\Delta$, where $\rho_C(u,v) = \tanh^{-1}|(u-v)/(1-\bar{v}u)|$. Consequently, in particular, we have $\rho_1(\tau V,\gamma V) \le \rho_C(\tau,\gamma) = \rho_C(0,\gamma) - \rho_C(0,\tau) = \rho_1(\gamma V, 0) - \rho_1(\tau V, 0)$. Simultaneously, by the triangle inequality, $\rho_1(\tau V, \gamma V) \ge \rho_1(0,\gamma V) - \rho_1(\tau V, 0)$.

Thus (6.1) holds.

Now, using (6.1), we have, for $0\le\tau<\gamma<\mu<1$,

$$\rho_1(\tau V, \mu V) = \rho_1(0,\tau V) - \rho_1(0,\mu V) = \\ = \{\rho(0,\tau V) - \rho_1(0,\gamma V)\} + \{\rho_1(0,\gamma V) - \rho_1(0,\mu V)\} = \rho_1(\tau V,\gamma V) + \rho_1(\gamma V,\mu V).$$

7. Special cases

(*i*) If $f(0) = 0$ in Theorem 3.1, then, by (2.1) and (2.2), equalities (3.1)-(3.5) and (3.7) are of the forms, respectively,

$$\alpha = \lim_{X \to U,\ X \in \mathfrak{a}_0} \inf \frac{\|I_M - f(X)^* f(X)\|}{1 - \|X\|^2} < +\infty, \tag{3.1'}$$

$$\alpha = \lim_{n \to \infty} \frac{\|I_M - f(X_n)^* f(X_n)\|}{1 - \|X_n\|^2}, \tag{3.2'}$$

$$\|[I_M - W^* f(x)][I_M - f(X)^* f(X)]^{-1}[I_M - f(X)^* W]\| \leq \\ \leq \alpha \|[I_H - U^* X][I_H - X^* X]^{-1}[I_H - X^* U]\|, \tag{3.3'}$$

$$\alpha = \lim_{X \to U, X \in \mathfrak{a}_0} \inf \frac{1 - \|f(X)\|^2}{1 - \|X\|^2}, \tag{3.4'}$$

$$\alpha = \lim_{\lambda \to 1} \frac{1 - \|f(\lambda U)\|^2}{1 - \lambda^2} = \lim_{\lambda \to 1} \frac{1 - \|f(\lambda U)\|}{1 - \lambda} \tag{3.5'}$$

and

$$\alpha = \lim_{\lambda \to 1} \frac{\|I_M - W^* f(\lambda U)\|}{1 - \lambda}. \tag{3.7'}$$

(*ii*) If $f(0)=0$ in Theorem 3.2, then (3.8)-(3.11) are of the forms, respectively,

$$\alpha = \lim_{x \to u, x \in H_0} \inf \frac{1 - \|f(x)\|^2}{1 - \|x\|^2} < +\infty, \tag{3.8'}$$

$$\alpha = \lim_{n \to \infty} \frac{1 - \|f(x_n)\|^2}{1 - \|x_n\|^2}, \tag{3.9'}$$

$$\frac{|1 - \langle f(x), w \rangle|^2}{1 - \|f(x)\|^2} \leq \alpha \frac{|1 - \langle x, u \rangle|^2}{1 - \|x\|^2}, \tag{3.10'}$$

$$\alpha = \lim_{\lambda \to 1} \frac{1 - \|f(\lambda u)\|^2}{1 - \lambda^2} = \lim_{\lambda \to 1} \frac{1 - \|f(\lambda u)\|}{1 - \lambda} = \\ = \lim_{\lambda \to 1} \frac{|1 - \langle f(\lambda u), w \rangle|}{1 - \lambda} = \lim_{\lambda \to 1} \frac{1 - \langle f(\lambda u), w \rangle}{1 - \lambda}. \tag{3.11'}$$

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