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A footnote to a paper by Noma

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Geometria algebrica. — *A footnote to a paper by Noma.* Nota di ANTONIO LANTERI e FRANCESCO RUSSO, presentata (*) dal Socio E. Marchionna.

ABSTRACT. — Let E be a globally generated ample vector bundle of rank ≥ 2 on a complex projective smooth surface X. By extending a recent result by A. Noma, we classify pairs (X, E) as above satisfying $c_2(E) = 2$.

KEY WORDS: Algebraic surfaces (complex, projective); Vector bundles (ample); Classification.

RIASSUNTO. — *Un'osservazione su un lavoro di Noma.* Sia E un fibrato vettoriale ampio e globalmente generato di rango ≥ 2 su una superficie algebrica proiettiva complessa non singolare X. Una semplice osservazione consente di estendere un recentissimo risultato di A. Noma, classificando le coppie (X, E) come sopra nell'ipotesi che E abbia seconda classe di Chern $c_2(E) = 2$.

Recently Noma [4] obtained the classification of ample and globally generated rank-2 vector bundles with $c_2 = 2$ over a smooth projective algebraic surface. The same problem was previously considered in several papers (*e.g.* [1], where the final result was affected by a gap in a proof of a previous step [2, (4.6)]) The final result is the following.

THEOREM [4, Thm. 8.1]. Let E be an ample and spanned rank-2 vector bundle on a complex projective smooth surface X. Then $c_2(E) = 2$ if and only if (X, E) is one of the following: 1) $(\boldsymbol{P}^2, \mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{O}_{\boldsymbol{P}^2}(2))$; 2) $(\boldsymbol{Q}^2, \mathcal{O}_{\boldsymbol{Q}^2}(1)^{\oplus 2})$, where $\boldsymbol{Q}^2$ is a smooth quadric in $\boldsymbol{P}^3$; 3) $X = \boldsymbol{P}_B(\mathcal{F})$ is a $\boldsymbol{P}^1$-bundle over an elliptic curve B and $E = H(\mathcal{F}) \otimes \otimes \pi^* \mathcal{E}$, where $\pi: X \to B$ is the ruling projection, $\mathcal{E}$ and $\mathcal{F}$ are normalized rank-2 vector bundles of degree 1 on B and $H(\mathcal{F})$ is the tautological bundle of $\mathcal{F}$; 4) there exists a finite morphism $f: X \to \boldsymbol{P}^2$ of degree 2 and $E = f^* \mathcal{O}_{\boldsymbol{P}^2}(1)^{\oplus 2}$.

The aim of this short *Note* is to supplement Noma's result with the following observation removing the assumption on the rank of E.

COROLLARY. Let E be an ample and spanned vector bundle of rank $r \geq 2$ on a complex projective smooth surface X. Then $c_2(E) = 2$ if and only if $r = 2$ and (X, E) is one of the pairs in the theorem above.

PROOF. We show that if $c_2(E) = 2$ then $r = 2$. By contradiction let $r \geq 3$. Since E is spanned, we have the Serre exact sequence (*e.g.* see [5, p. 81]).

$$0 \to \mathcal{O}_X^{\oplus(r-2)} \to E \to F \to 0, \qquad (*)$$

where the rank-2 vector bundle F is ample and spanned, as a quotient of E. Moreover $c_2(F) = c_2(E) = 2$, so that the pair (X, F) is as in 1),...,4) in the Theorem above. Let l be a line in cases 1), 2), and a fibre of π in case 3). Since l is a smooth $\boldsymbol{P}^1$ inside X, from

(*) Nella seduta del 12 dicembre 1992.

$r \leqslant c_1(E_l) = c_1(F_l)$, we get a contradiction in cases 2), 3), while $r = 3$ in case 1). In case 4) we can assume that the branch locus of the double cover has degree $\geqslant 4$; otherwise (X, F) would be as in case 2). Let C be the inverse image of a general line via f. Since C is a smooth curve of genus $\geqslant 1$ we have by [3, (0.2)]

$$r + 1 \leqslant c_1(E_C) = c_1(F_C) = 4,$$

so that $r = 3$. Finally look at $(*)$ in cases 1) and 4), with $r = 3$. Let F^* denote the dual of F. The extension class of $(*)$ lives in $H^1(X, F^*)$, which is immediately seen to be zero in both cases. Therefore $E = \mathcal{O}_X \oplus F$, contradicting the ampleness. This concludes the proof.

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