A stable method for the inversion of the Fourier transform in $\mathbb{R}^N$


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Abstract. — A general method is given for recovering a function $f: \mathbb{R}^N \rightarrow \mathbb{C}$, $N \geq 1$, knowing only an approximation of its Fourier transform.

Key words: Fourier transform; Inversion; Well posed.

Riassunto. — Un metodo stabile per l'inversione della trasformata di Fourier in $\mathbb{R}^N$. È dato un metodo generale per ricostruire una funzione $f: \mathbb{R}^N \rightarrow \mathbb{C}$, $N \geq 1$ conoscoendo solo un'approssimazione della sua trasformata di Fourier.

1. — In some previous papers [1-3,5] stable inversion methods for multiple Fourier series were suggested and analyzed; moreover the effectiveness of the methods was discussed in [6,7,4]. In this paper we deal with the non compact case. The basic ideas are the same, but some technical difficulties arise from the lack of inclusion relations between the various $L^p$ spaces. Moreover minor formal problems are due to the not satisfactory representation of Fourier transform for $L^p$ functions with $p > 2$.

The motivations of the method are the same as in the compact case and can be found in some details in [8].

Notations and some preliminaries are contained in §2; the basic theorems are stated in §3, and as applications in §4 are given some a priori estimates for suitable classes of functions.

2. — Let us first introduce some notations.

If $N \geq 1$ let

$$B_p = \begin{cases} L^p(\mathbb{R}^N) & \text{if } 1 \leq p \leq 2 \\ L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) & \text{if } 2 < p < +\infty \\ C_0(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) & \text{if } p = +\infty \end{cases}$$

with the usual $L^p$ norm.

If $f \in B_p$ ($1 \leq p \leq +\infty$), we denote with $\tilde{f}$ its usual Fourier transform,

$$\tilde{f}(\mathbf{x}) = \int_{\mathbb{R}^N} f(t) e^{-2\pi i \mathbf{x} \cdot \mathbf{t}} \, d\mathbf{t},$$

(\text{where } x \cdot t \text{ is usual inner product}).

Through the paper $G$ will be a real function in $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ with $\tilde{G}(0) = 1$.

For every $\sigma > 0$ let us set $G_\sigma(x) = \sigma^{-N} G(x/\sigma)$ and for every $\tau > 0$

$$R_{\sigma,\tau}(\lambda) \sim (\lambda \cdot \tilde{G}_\sigma)^{\top}(x_\tau)$$

where $\lambda \in L^q, 2 \leq q \leq +\infty$, $(\cdot)^\vee$ is the inverse Fourier transform and $\chi_\tau$ is the characteristic function of the interval $[-\tau/2, \tau/2]^N$.

Finally let

$$R_\tau(\lambda) \sim (\lambda \cdot \widehat{G_\tau})^\vee.$$ 

Now we relate $\lambda$ and $\widehat{G_\tau}$ in such a way that the formal definitions of $R_{\tau, \varepsilon}$ and $R_\tau$ give us correctly defined functions.

**Proposition 1.** If $\lambda \in L^2(\mathbb{R}^N)$, then for every $\tau > 0$ $R_\tau(\lambda) \in L^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$ and for every $p$, $2 \leq p \leq +\infty$ we have

$$\|R_\tau \lambda\|_p \leq a_p \sigma^{-N[1 - 2/p]/2}\|\lambda\|_2$$

where

$$a_p = \|G\|_{1 - [1 - 2/p]} \cdot \|G\|_{1 - [2/p]}.$$ 

**Proof.** Indeed $(\lambda \cdot \widehat{G_\tau})^\vee \in C_0$ because $\lambda \cdot \widehat{G_\tau} \in L^1$. Moreover

$$G \in L^1 \Rightarrow \widehat{G_\tau} \in C_0 \Rightarrow (\lambda \cdot \widehat{G_\tau})^\vee \in L^2$$

then $(\lambda \cdot \widehat{G_\tau})^\vee \in L^p \forall p, 2 \leq p \leq +\infty$.

Since

$$\|R_\tau \lambda\|_2 = \|\lambda \cdot \widehat{G_\tau}\|_2 \leq \|\lambda\|_2 \cdot \|\widehat{G_\tau}\|_\infty \leq \|\lambda\|_2 \cdot \|G\|_1$$

and $\|R_\tau \lambda\|_\infty \leq \|\lambda \cdot \widehat{G_\tau}\|_1 \leq \|\lambda\|_2 \cdot \|G\|_2 = \sigma^{-N/2} \|\lambda\|_2 \cdot \|G\|_2$ by interpolation (2.1) follows.

**Proposition 2.** Let $1 \leq p < 2$ and $\widehat{G} \in L^p(\mathbb{R}^N)$. If $\lambda \in L^q(\mathbb{R}^n)$, $(1/p + 1/q = 1)$, then for every $\tau > 0$, $R_\tau \lambda \in C_0(\mathbb{R}^N)$ and for every $\tau > 0$ we have

$$\|R_{\tau, \varepsilon} \lambda\|_p \leq a_p (\tau/\varepsilon)^{N[1 - 2/p]/2}\|\lambda\|_q$$

where $a_p$ is given by (2.2).

**Proof.** Since $\lambda \cdot \widehat{G_\tau} \in L^1$, then $(\lambda \cdot \widehat{G_\tau})^\vee \in C_0$ and $R_{\tau, \varepsilon} \lambda \in L^p$ for every $p \geq 1$.

From $\|R_{\tau, \varepsilon} \lambda\|_1 \leq \|(\lambda \cdot \widehat{G_\tau})^\vee\|_2 \cdot \|\chi_\tau\|_2 \leq \|\lambda\|_\infty \cdot \|\widehat{G_\tau}\|_2 \cdot \tau^{N/2} = (\tau/\varepsilon)^{N/2} \|\lambda\|_\infty \cdot \|G\|_2 \|\tau\|_\infty$ and (2.3), by interpolation we have (2.4).

3. **Theorem 1.** Let $f \in B_p, p \geq 2, \varepsilon = \varepsilon(\delta) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ such that as } \delta \rightarrow 0$

$$\sigma(\varepsilon) \rightarrow 0, \text{ and } \delta \sigma(\varepsilon)^{-N[1 - 2/p]/2} \rightarrow 0.$$ 

Then for every $\varepsilon > 0$ there exists $\delta_0 = \delta_0(\varepsilon, f)$ such that if $\delta \leq \delta_0$, for every $\lambda \in L^2(\mathbb{R}^N)$ with $\|\lambda - \tilde{f}\|_2 < \delta$ we have $\|f - R_{\sigma(\varepsilon)} \lambda\|_p < \varepsilon$.

**Proof.** By Prop. 1, for every $\varepsilon > 0$ and $\delta < \delta_1(\varepsilon)$

$$\|\lambda - \tilde{f}\|_2 < \delta \Rightarrow \|R_{\sigma(\varepsilon)} (\lambda - \tilde{f})\|_p < \varepsilon/2.$$ 

On the other hand, since $\tilde{f}$ and $\widehat{G_\tau}$ are in $L^2$, then

$$\|f - R_\tau \tilde{f}\|_p = \|f - f \ast G_\tau\|_p.$$
Since \( \{G_\tau\} \) is an approximate unit, if \( \delta < \delta_2(\epsilon, f) \) we have 
\[
\|f - f * G_\tau\|_p < \epsilon/2.
\]
Then if \( \delta_0 = \min(\delta_1, \delta_2) \) the theorem follows.

The situation in the case \( 1 \leq p < 2 \) is different. We have not to restrict ourselves to some subclass of \( L^p \) but as usual we have to introduce some cut functions \( \{\chi_\tau\}_{\tau > 0} \) in order to have an approximation of \( f \) in \( L^p \) for every \( \lambda \) in \( L^q \). Obviously the choice of the family \( \{\chi_\tau\} \) is quite free. Practical problems may suggest suitable families. In this paper for sake of simplicity we use characteristic functions of intervals \([-\tau/2, \tau/2]^N\).

**Theorem 2.** Let \( \tilde{G} \in L^p(\mathbb{R}^N), 1 \leq p < 2 \) and \( \sigma = \sigma(\delta) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), \( \tau = \tau(\delta) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that as \( \delta \rightarrow 0 \)
\[
\sigma(\delta) \rightarrow 0, \quad \tau(\delta) \rightarrow +\infty, \quad \delta \left( \frac{\tau(\delta)}{\sigma(\delta)} \right)^{N(1 - 2/p)/2} \rightarrow 0.
\]

Then, if \( f \in L^p(\mathbb{R}^N) \), for every \( \epsilon > 0 \) there exists \( \delta_0 = \delta_0(\epsilon, f) \) such that if \( \delta < \delta_0 \) and \( 1/p + 1/q = 1 \) for every \( \lambda \in L^q(\mathbb{R}^N) \) and \( \|\lambda - \tilde{f}\|_q < \delta \) we have \( \|f - R_{\sigma(\delta), \tau(\delta)} \lambda\|_p < \epsilon \).

**Proof.** By Prop. 2 for every \( \epsilon > 0 \) and \( \delta < \delta_1(\epsilon) \)
\[
\|\lambda - \tilde{f}\|_q < \epsilon \Rightarrow \|R_{\sigma(\delta), \tau(\delta)}(\lambda - \tilde{f})\|_p < \epsilon/2.
\]
Moreover
\[
\|f - R_{\sigma(\delta), \tau(\delta)} \tilde{f}\|_p = \|f - (f * G_{\tau(\delta)}) \chi_{\tau(\delta)}\|_p = \left\|f - f \chi_{\tau(\delta)}\right\|_p + \|f - f * G_{\tau(\delta)}) \chi_{\tau(\delta)}\|_p.
\]
If \( \delta < \delta_2(\epsilon, f) \) we have \( \|f - f \chi_{\tau(\delta)}\|_p < \epsilon/4 \).
Since \( \{G_\tau\}_{\tau > 0} \) is an approximate unit, if \( \delta < \delta_3(\epsilon, f) \) \( \|(f - f * G_{\tau(\delta)} \chi_{\tau(\delta)})\|_p \leq \epsilon/4 \).
If \( \delta_0 = \min(\delta_1, \delta_2, \delta_3) \) we have the result.

For pointwise convergence, we don’t need to distinguish between \( p < 2 \) and \( p \geq 2 \). Nevertheless, as in the compact case we need a little bit more regularity of \( G \).

Let
\[
M(x) = \sup_{|y| \geq |x|} |G(y)|,
\]
and for every \( p, 1 \leq p \leq +\infty \) let \( p_0 = \min(p, 2) \), \( 1/q_0 + 1/p_0 = 1 \).

**Theorem 3.** Let \( M \in L^1(\mathbb{R}^N), \tilde{G} \in L^{p_0}(\mathbb{R}^N) \) and \( \sigma = \sigma(\delta) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that as \( \delta \rightarrow 0 \)
\[
(3.2) \quad \sigma(\delta) \rightarrow 0, \quad \delta \sigma(\delta)^{-N/p_0} \rightarrow 0.
\]
Then if \( f \in B_p, 1 \leq p \leq +\infty \) and \( x \) is a Lebesgue point of \( f \), for every \( \epsilon > 0 \) there exists \( \delta_0 = \delta_0(\epsilon, f, x) \) such that if \( \delta < \delta_0 \)
\[
\lambda \in L^q(\mathbb{R}^N) \quad \text{and} \quad \|\lambda - \tilde{f}\|_q < \delta \Rightarrow |f(x) - R_{\sigma(\delta)} \lambda(x)| < \epsilon.
\]
Of course, we can always suppose \( x = 0 \). We have (see e.g. [9], Th. 1.25 p. 13) for \( \delta \leq \delta_1 (\varepsilon, f, x) \)

\[
|f(0) - f \ast G_{\sigma(\delta)}(0)| < \varepsilon/2.
\]

Moreover

\[
|f \ast G(0) - R_{\sigma(\delta)}(0)| = \left| ((\tilde{f} - \lambda) \cdot \tilde{G}_{\sigma(\delta)})^\vee(0) \right| = \\
\left| \int_{\mathbb{R}^N} (\tilde{f} - \lambda)(x) \tilde{G}_\varepsilon(x) \, dx \right| \leq \|\tilde{f} - \lambda\|_{q_0} \cdot \sigma(\delta)^{-N/p} \|\tilde{G}\|_{p_0}.
\]

From (5.2), (5.3) and (5.1) we obtain the theorem.

4. As in the compact case, we give some *a priori* estimates of \( \|f - R_\sigma \lambda\|_p \) if \( p \geq 2 \) and \( \|f - R_{\sigma, \tau} \lambda\|_p \) if \( 1 \leq p < 2 \), for Lipschitz classes of functions \( K \text{ Lip}(\alpha, B_p) \), \( 0 < \alpha \leq 1 \). We recall that \( f \in K \text{ Lip}(\alpha, B_p) \) if for every \( y \in \mathbb{R}^N \) the function \( \Delta_y f(x) = f(x + y) - f(x) \) satisfies \( \|\Delta_y f\|_p \leq K \|y\|^\alpha \).

**Theorem 4.** If \( \int_{\mathbb{R}^N} |x|^\alpha \, |G(x)| \, dx = c_\alpha < + \infty \) then for every \( \sigma > 0 \), \( p \geq 2 \), if \( f \in K \text{ Lip}(\alpha, B_p) \) and \( \lambda \in L^2(\mathbb{R}^N) \) we have

\[
\|f - R_\sigma \lambda\|_p \leq Kc_\alpha \sigma^\alpha + a_p \sigma^{-N[1 - 2/p]/2} \|\lambda - \tilde{f}\|_2.
\]

The proof is obtained as in the following Theorem 5 assuming \( \chi_\tau \equiv 1 \).

For \( 1 \leq p < 2 \) it is easy to see that an analogous estimate for \( f - R_{\sigma, \tau} \lambda \) it is not available; some control of the decay at infinity of \( f \) is needed.

The simplest one is the following.

Let \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \) a decreasing function such that \( \lim_{x \to + \infty} \psi(x) = 0 \). Then we set \( H\Psi_p \)

the class of \( L^p(\mathbb{R}^N) \) functions such that \( \|f(1 - \chi_\tau)\|_p \leq H\Psi(\tau) \forall \tau > 0 \).

**Theorem 5.** If \( \tilde{G} \in L^p(\mathbb{R}^N) \) and \( \int_{\mathbb{R}^N} |x|^\alpha \, |G(x)| \, dx = c_\alpha < + \infty \), then for every \( \sigma > 0 \), \( \tau > 0 \) if \( f \in K \text{ Lip}(\alpha, B_p) \cap H\Psi_p \), \( 1 \leq p < 2 \) and \( \lambda \in L^q(\mathbb{R}^N) \) we have

\[
\|f - R_{\sigma, \tau} \lambda\|_p \leq Kc_\alpha \sigma^\alpha + H\Psi(\tau) + a_p (\tau/\sigma)^{N[1 - 2/p]/2} \|\tilde{f} - \lambda\|_q.
\]

**Proof.** We have

\[
\|f - R_{\sigma, \tau} \tilde{f}\|_p \leq \|f(1 - \chi_\tau)\|_p + \|(f - f \ast G_\sigma) \chi_\tau\|_p \leq \\
\leq H\Psi(\tau) + \left| \int_{\mathbb{R}^N} \Delta_y f(x) G_\varepsilon(-y) \chi_\tau(x) \, dy \right|_p \leq H\Psi(\tau) + \int_{\mathbb{R}^N} K \|y\|^\alpha |G_\varepsilon(y)| \, dy \leq H\Psi(\tau) + Kc_\alpha \sigma^\alpha.
\]

Moreover, Prop. 2 gives \( \|R_{\sigma, \tau}(\lambda - \tilde{f})\|_p \leq a_p (\tau/\sigma)^{N[1 - 2/p]/2} \|\lambda - \tilde{f}\|_q \) and the theorem holds.

Finally we give an *a priori* estimate of the pointwise approximation. In order to do this we recall the notion of \( K \text{ Leb}(\alpha, x) \) classes of functions. We say that a function
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Theorem 6. Let \( \tilde{G} \in L^p_0(\mathbb{R}^N) \) and \( \int |x|^\alpha M(x) \, dx = \gamma_\alpha < +\infty \). Then, if \( f \in K \text{Leb}(\alpha, x) \cap B_p \) and \( \lambda \in L^{q_0}(\mathbb{R}^N) \), for every \( \sigma > 0 \) we have

\[
|f(x) - \tilde{R}_\sigma \lambda(x)| \leq K \tilde{c}_\alpha \sigma^\alpha + \sigma^{-N/p_0} \|\tilde{G}\|_{p_0} \|\tilde{f} - \lambda\|_{q_0}
\]

where

\[
\tilde{c}_\alpha = \frac{N + \alpha}{2} \pi^{-N/2} \Gamma\left(\frac{N}{2}\right) \gamma_\alpha.
\]

Proof. We can always suppose \( x = 0 \). We have

\[
|f(0) - \tilde{R}_\sigma \lambda(0)| \leq |f(0) - \tilde{R}_\sigma \tilde{f}(0)| + |\tilde{R}_\sigma (\tilde{f} - \lambda)(0)|.
\]

From (3.4)

\[
|\tilde{R}_\sigma (\tilde{f} - \lambda)(0)| \leq \sigma^{-N/p_0} \|\tilde{G}\|_{p_0} \|\tilde{f} - \lambda\|_{q_0}.
\]

Moreover

\[
|f(0) - \tilde{R}_\sigma \tilde{f}(0)| = \left| \int (f(x) - f(0)) G_\sigma(x) \, dx \right| \leq \int |M_\sigma(x)| f(x) - f(0) \, dx.
\]

Let

\[
S(r) = \int_{|x| = r} |f(x) - f(0)| \, ds
\]

where \( ds \) is the surface area element of the sphere \( |x| = r \) and

\[
F(r) = \int_0^r S(y) \, dy.
\]

From (4.3) we obtain

\[
|f(0) - \tilde{R}_\sigma \tilde{f}(0)| \leq \int_0^{+\infty} M_\sigma(r) S(r) \, dr \leq F(r) M_\sigma(r) \left| \tilde{f}(r) \right| \leq F(r) dM_\sigma(r) \leq K r^{N+\alpha} M_\sigma(r) \left| \tilde{f}(r) \right|.
\]

\[
- K \int_0^{+\infty} r^{N+\alpha} dM_\sigma(r) \leq K (N + \alpha) \int_0^{+\infty} r^{N+\alpha-1} M_\sigma(r) \, dr = K (N + \alpha) \sigma^\alpha \tilde{c}_\alpha m(S_N)
\]

where \( m(S_N) \) is the surface measure of the unit ball of \( \mathbb{R}^N \). From the above inequality and (4.2) the result follows.

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