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Quaternionic-like structures on a manifold: Note I. 1-integrability and integrability conditions

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Geometria differenziale. — *Quaternionic-like structures on a manifold: Note I. 1-integrability and integrability conditions.* Nota (*) di Dmitri V. Alekseevsky e Stefano Marchiafava, presentata dal Socio E. Martinelli.

Abstract. — This *Note* will be followed by a *Note II* in these *Rendiconti* and successively by a wider and more detailed memoir to appear next. Here six quaternionic-like structures on a manifold M (almost quaternionic, hypercomplex, unimodular quaternionic, unimodular hypercomplex, Hermitian quaternionic, Hermitian hypercomplex) are defined and interrelations between them are studied in the framework of general theory of G-structures. Special connections are associated to these structures. 1-integrability and integrability conditions are derived. Decompositions of appropriate spaces of curvature tensors are given. In *Note II* the automorphism groups of these quaternionic-like structures will be considered.

Key words: G-structures; Quaternionic structures; Special connections; Integrability conditions; Curvature tensors.

Riassunto. — *Strutture di tipo quaternionale su una varietà: Nota I. Condizioni di 1-integrabilità e di integrabilità.* A questa *Nota* farà seguito una *Nota II* negli stessi *Rendiconti* e una successiva memoria più ampia e più dettagliata che apparirà prossimamente. Qui si definiscono su una varietà M sei strutture di tipo quaternionale (quasi quaternionale, ipercomplessa, unimodulare quaternionale, unimodulare ipercomplessa, Hermitiana quaternionale, Hermitiana ipercomplessa) e si studiano le loro interrelazioni nell'ambito della teoria generale delle G-strutture. Si associano a tali strutture connessioni speciali. Si determinano le condizioni di 1-integrabilità e di integrabilità. Si danno opportune decomposizioni degli spazi dei rispettivi tensori di curvatura. Nella *Nota II* si considereranno i gruppi degli automorfismi di tali strutture di tipo quaternionale.

1. Definition of q-like structures on a vector space

Let V be a real vector space of dimension $4n$. Now we define some quaternionic-like structures (shortly, *q-like structures*) on V.

Definitions. 1) A triple $H = (J_1, J_2, J_3)$ of anticommuting complex structures on V with $J_3 = J_1 J_2$ is called a *hypercomplex structure on* V.

2) The 3-dimensional subalgebra $Q \equiv \langle H \rangle = \boldsymbol{R}J_1 + \boldsymbol{R}J_2 + \boldsymbol{R}J_3 \approx sp_1$ of the Lie algebra of endomorphisms $\mathrm{End}\, V$ is called a *quaternionic structure* on V.

Note that two hypercomplex structures $H = (J_\alpha)$, $H' = (J'_\alpha)$ generate the same quaternionic structure $Q = \langle H \rangle = \langle H' \rangle$ iff they are related by a rotation, that is

$$J'_\alpha = \sum_\beta A_\alpha^\beta J_\beta \qquad (\alpha = 1, 2, 3)$$

with $A = (A_\alpha^\beta) \in SO_3$.

Definition. An Euclidean metric g in V is called *Hermitian* with respect to a hypercomplex structure $H = (J_\alpha)$ (respectively, the quaternionic structure $Q = \langle H \rangle$) iff

(*) Pervenuta all'Accademia il 3 agosto 1992.

for any $x, y \in V$

$$g(J_\alpha x, J_\alpha y) = g(x, y) \quad (\alpha = 1, 2, 3)$$

(respectively, $g(Jx, Jy) = g(x, y)$ for any complex structure $J \in Q$).

REMARK. Note that if a metric g is Hermitian with respect to a hypercomplex structure H then it is Hermitian with respect to the quaternionic structure $Q = \langle H \rangle$.

We recall that the group of automorphisms of V that preserve a given hypercomplex structure H (resp., quaternionic structure $Q = \langle H \rangle$) is isomorphic to $GL_n(\boldsymbol{H})$ (resp., $Sp_1 \cdot GL_n(\boldsymbol{H})$).

Let g be a metric which is Hermitian with respect to H (resp., Q): the group of automorphisms of V that preserve H and g (resp., Q and g) is isomorphic to Sp_n (resp., $Sp_1 \cdot Sp_n$).

Let vol be a given volume form on V: the group of automorphisms of V that preserve H and vol (resp., Q and vol) is isomorphic to $SL_n(\boldsymbol{H})$ (resp., $Sp_1 \cdot SL_n(\boldsymbol{H})$).

2. DEFINITIONS OF SIX (ALMOST) q-LIKE STRUCTURES ON A MANIFOLD

Let M be a $4n$-manifold, $n > 1$.

DEFINITIONS. 1) An *almost hypercomplex* (resp., *almost quaternionic*) structure on M is a field H (resp., Q) of hypercomplex (resp., quaternionic) structures on the tangent bundle.

2) An almost hypercomplex structure H together with a volume form vol (resp., an Hermitian metric g) is called a *almost unimodular hypercomplex* (resp., *almost Hermitian hypercomplex*) structure. Analogous definitions are given for *almost unimodular quaternionic* and *almost Hermitian quaternionic* structures. If there exists a torsionless connection ∇ that preserves a given structure of above type we say that the structure is *1-integrable* and to mean this we will omit the attribute «almost» in the definition. As an example, a *quaternionic structure* on M is an almost quaternionic structure Q which is preserved by a torsionless connection ∇.

Note that a manifold M with a quaternionic (resp., hypercomplex) Hermitian structure (Q, g) (resp., (H, g)) in our sense is usually called *quaternionic Kähler* (resp., *hyperKähler*).

3. G-STRUCTURE ASSOCIATED WITH AN (ALMOST) q-LIKE STRUCTURE

Let $\pi_M \colon CF(M) \to M$ be the principal $GL_n(\boldsymbol{R})$-bundle of coframes on a manifold M. Let $G \subset GL_n(\boldsymbol{R})$ be a matrix group.

DEFINITIONS. 1) A G-structure on M is a principal G-subbundle $\pi \colon P \to M$ of the bundle of coframes π_M.

2) Let π: $P \to M$ and π': $P' \to M$ be a G-structure and a G'-structure respectively. We say that π is subordinated to π' if $G \subset G'$ and $P \subset P'$.

Let π: $P \to M$ be a G-structure. For any $x \in M$ we shall denote by $G_x \subset GL_n(T_x M)$ the group of linear transformations of $T_x M$ that preserve the set of coframes $P_x = \pi^{-1}(x)$ and by $\mathfrak{g}_x$ its Lie algebra.

3) A G-structure is called *1-integrable* if it admits a torsionless connection.

This notion of 1-integrability agrees with 1-integrability condition of q-like structures (see n. 2).

We defined the six q-like structures on a manifold M. The generic one will be refered as $\mathcal{S}$: it may be considered as G-structure with appropriated group G. The corresponding groups G and the inclusion relations between them are indicated in diagram below.

$$\begin{array}{ccc} \dfrac{Sp_1\cdot GL_n(\boldsymbol{H})}{Q\text{: quaternionic}} & \hookleftarrow & \dfrac{GL_n(\boldsymbol{H})}{H=(J_\alpha)\text{: hypercom.}} \\ \uparrow & & \uparrow \\ \dfrac{Sp_1\cdot SL_n(\boldsymbol{H})}{(Q,\text{vol})\text{: unimodular quat.}} & \hookleftarrow & \dfrac{SL_n(\boldsymbol{H})}{(H,\text{vol})\text{: unimodular hypercom.}} \\ \uparrow & & \uparrow \\ \dfrac{Sp_1\cdot Sp_n}{(Q,g)\text{: Hermitian quat.}} & \hookleftarrow & \dfrac{Sp_n}{(H,g)\text{: Hermitian hypercom.}} \end{array}$$

REMARK. Here we intend that each inclusion refers to the appropriate choice of the structures. For example, the inclusion $GL_n(\boldsymbol{H}) \hookrightarrow Sp_1 \cdot GL_n(\boldsymbol{H})$ refers to the quaternionic structure $Q = \langle H \rangle$ generated by a hypercomplex structure H. Also, for inclusion $Sp_n \hookrightarrow SL_n(\boldsymbol{H})$ the volume form vol is the volume form vol^g defined by the metric g.

4. $\mathcal{D}$-CONNECTIONS OF A G-STRUCTURE

Let $G \subset GL(V)$ be a linear reductive Lie group with Lie algebra $\mathfrak{g} \subset gl(V) = V \otimes V^*$. We fix a G-invariant complement $\mathcal{D} = \mathcal{D}(\mathfrak{g})$ to the subspace $\partial(\mathfrak{g} \otimes V^*)$ into $V \otimes \Lambda^2 V^*$, where ∂: $\mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 V^*$ is the Spencer operator of alternation. We recall that $\mathfrak{g}^{(1)} = \operatorname{Ker} \partial = (\mathfrak{g} \otimes V^*) \cap (V \otimes S^2 V^*)$ is called the *first prolongation* of $\mathfrak{g}$.

DEFINITION. Let π: $P \to M$ be a G-structure and ∇ be a connection in π. Denote by t^∇: $P \to V \otimes \Lambda^2 V^* = \partial(\mathfrak{g} \otimes V^*) \oplus \mathcal{D}(\mathfrak{g})$ the torsion function of ∇, that associates to $p \in P$ the coordinates of the torsion tensor $\operatorname{Tor}(\nabla)$ with respect to the coframe p. The connection ∇ is called *$\mathcal{D}$-connection* if its torsion function takes values in $\mathcal{D}$.

THEOREM 1 ([1]). 1) *Any G-structure π: $P \to M$ admits a $\mathcal{D}$-connection ∇.*

2) *Any two $\mathcal{D}$-connections ∇, ∇' are related by $\nabla' = \nabla + S$ where S is a tensor field such that for any $x \in M$, S_x belongs to the first prolongation $\mathfrak{g}_x^{(1)}$ of the Lie algebra $\mathfrak{g}_x \subset gl(T_x M)$ (see n. 3).*

COROLLARY 1. *Assume that the first prolongation* $\mathfrak{g}^{(1)} = 0$. *Then* $\mathcal{D}$*-connection is unique.*

Denote by κ: $V \otimes \Lambda^2 V^* = \delta(\mathfrak{g} \otimes V^*) \oplus \mathcal{D}(\mathfrak{g}) \to \mathcal{D}(\mathfrak{g})$ the natural projection. For any connection ∇ in G-structure π: $P \to M$ the $\mathcal{D}(\mathfrak{g})$-component $\kappa \circ t^\nabla$: $P \to \mathcal{D}(\mathfrak{g})$ of the torsion function t^∇ is called the *structure function* of G-structure π: it is G-equivariant and does not depend on the choice of connection ∇. The associated tensor field on M is called the *structure tensor* of π.

5. CANONICAL CONNECTION OF A q-LIKE STRUCTURE DIFFERENT FROM ALMOST QUATERNIONIC ONE: 1-INTEGRABILITY CONDITION

To apply Theorem 1 for a q-like structure we need the following result (see [12, 9, 10]):

LEMMA. *The first prolongation of Lie algebra* $sp_1 + gl_n(\boldsymbol{H}) \subset gl(V)$ *is given by* $(sp_1 + gl_n(\boldsymbol{H}))^{(1)} = \{S^\xi, \xi \in V^*\}$ *where*

$$S^\xi = \xi \otimes \mathrm{Id} + \mathrm{Id} \otimes \xi - \sum_\alpha [(\xi \circ J_\alpha) \otimes J_\alpha + J_\alpha \otimes (\xi \circ J_\alpha)] \tag{5.1}$$

and (J_α), $\alpha = 1, 2, 3$, *is a hypercomplex structure that generates* sp_1.

COROLLARY 2. *The first prolongations of Lie algebras* $gl_n(\boldsymbol{H})$, $sl_n(\boldsymbol{H})$, sp_n, $sp_1 + sl_n(\boldsymbol{H})$, $sp_1 + sp_n$ *associated to all* q*-like structures different from a quaternionic one are zero.*

Applying Theorem 1 to the G-structure π: $P \to M$ associated with a q-like structure $\mathcal{S}$ different from almost quaternionic one we obtain the existence of a unique $\mathcal{D}$-connection $\nabla^{\mathcal{S}}$. It preserves $\mathcal{S}$. We shall call it the *canonical connection* of q-like structure $\mathcal{S}$. The torsion tensor $T^{\mathcal{S}}$ of the canonical connection $\nabla^{\mathcal{S}}$ is the structure tensor of $\mathcal{S}$. We have immediately

THEOREM 2. *Let* $\mathcal{S}$ *be a* q*-like structure different from almost quaternionic one.* $\mathcal{S}$ *is 1-integrable iff the canonical connection* $\nabla^{\mathcal{S}}$ *has no torsion.*

6. ALMOST QUATERNIONIC CONNECTIONS AND 1-INTEGRABILITY CONDITION FOR AN ALMOST QUATERNIONIC STRUCTURE

Let Q be an almost quaternionic structure on a manifold M and let π: $P \to M$ be the associated $Sp_1 \cdot GL_n(\boldsymbol{H})$-structure.

DEFINITION. A linear connection ∇ on M is called an *almost quaternionic connection* (with respect to Q) if it preserves Q, that is the parallel transport along a curve γ: $[0, 1] \to M$ transforms $Q_{\gamma(0)}$ into $Q_{\gamma(1)}$.

REMARK. Similarly as before, we will identify an almost quaternionic connection ∇ (with respect to an almost quaternionic structure Q) with a connection in $Sp_1 \cdot GL_n(\boldsymbol{H})$-structure π: $P \to M$ associated with Q.

PROPOSITION 1. *Let Q be an almost quaternionic structure on a manifold M and let ∇ be an almost quaternionic connection. Then any other almost quaternionic connection (with respect to Q) is given by*

$$\nabla' = \nabla + F \tag{6.1}$$

where F is a section of the vector bundle

$$\bigcup_x N(Q_x) \otimes T_x^* M \to M$$

and $N(Q_x) \cong sp_1 + gl_n(\boldsymbol{H})$ is the normalizer of Q_x into Lie algebra of endomorphisms $\mathrm{End}(T_x M)$.

In particular, the connections $\nabla, \nabla' = \nabla + F$ have the same torsion tensor iff $F = S^{\xi}$, for some 1-form $\xi \in \Lambda^1 M$, where S^{ξ} is given by (5.1) and $H = (J_\alpha)$ is any local almost hypercomplex structure that generates Q. (See also [8]).

LEMMA (Salamon [13]). *Let $G = Sp_1 \cdot GL_n(\boldsymbol{H})$ and $\mathcal{G} = sp_1 + gl_n(\boldsymbol{H})$. Then there exist the following decompositions of G-modules: $\mathcal{G} \otimes V^* = \mathcal{G}^{(1)} \oplus W$, $V \otimes \Lambda^2 V^* = \delta(\mathcal{G} \otimes V^*) \oplus \mathcal{D} = \delta W \oplus \mathcal{D}$ where $\mathcal{G}^{(1)}$ is the first prolongation of $\mathcal{G}$, W is an G-invariant complement of $\mathcal{G}^{(1)}$ into $\mathcal{G} \otimes V^*$ and $\mathcal{D} = \mathcal{D}(\mathcal{G})$ is unique irreducible G-submodule complement to $\delta W \cong W$.*

REMARK. Salamon proves that G-module $\mathcal{D}^{\boldsymbol{C}} = \mathcal{D} \otimes \boldsymbol{C} \cong (E^* \otimes \Lambda^2 E)_0 \otimes S^3 \boldsymbol{H}$, where $V^{\boldsymbol{C}} = E \otimes_{\boldsymbol{C}} \boldsymbol{H}$, $E = \boldsymbol{C}^{2n}$, $\boldsymbol{H} = \boldsymbol{C}^2$ and $(E^* \otimes \Lambda^2 E)_0$ denotes the subspace of all traceless tensors belonging to the $GL_n(\boldsymbol{H})$-module in the bracket. He proves that $\delta W \cong W$ doesn't contain such submodule.

Due to this Lemma the submodule $\mathcal{D}$ is uniquely defined and we may speak about almost quaternionic $\mathcal{D}$-connections without misleading.

Applying Theorem 1, we obtain

THEOREM 3. *An almost quaternionic structure Q on a manifold M is 1-integrable iff an almost quaternionic $\mathcal{D}$-connection has no torsion.*

7. EXPLICIT FORMULAS FOR ALMOST QUATERNIONIC CONNECTIONS OF AN ALMOST QUATERNIONIC AND AN ALMOST HYPERCOMPLEX STRUCTURE

For simplicity, in this Section we use the following notation: $\boldsymbol{G} = Sp_1 \cdot GL_n(\boldsymbol{H})$, $\boldsymbol{G}' = GL_n(\boldsymbol{H})$, $\boldsymbol{\mathcal{G}} = sp_1 + gl_n(\boldsymbol{H})$, $\boldsymbol{\mathcal{G}}' = gl_n(\boldsymbol{H})$. We have the following decompositions:

$$V \otimes \Lambda^2 V^* = \delta(\boldsymbol{\mathcal{G}} \otimes V^*) \oplus \mathcal{D}(\boldsymbol{\mathcal{G}}) =$$

$$= \delta(\boldsymbol{\mathcal{G}}' \otimes V^*) \oplus \delta(sp_1 \otimes V^*) \oplus \mathcal{D}(\boldsymbol{\mathcal{G}}) = \delta(\boldsymbol{\mathcal{G}}' \otimes V^*) \oplus \mathcal{D}(\boldsymbol{\mathcal{G}}')$$

and, obviously (by last Remark),

$$\mathcal{D}(\boldsymbol{\mathcal{G}}') = \delta(sp_1 \otimes V^*) \oplus \mathcal{D}(\boldsymbol{\mathcal{G}}) = \delta(L_{V^*}) \oplus \delta(sp_1 \otimes V^*)_0 \oplus \mathcal{D}(\boldsymbol{\mathcal{G}})$$

where

$$L_{V^*}=\left\{L^\xi=\sum_\alpha J_\alpha\otimes(\xi\circ J_\alpha),\ \xi\in V^*\right\}$$

and $(sp_1\otimes V^*)_0=\{L\in sp_1\otimes V^*, \mathrm{Tr}(L)=0\}$ where $\mathrm{Tr}(J\otimes\xi)=\xi\circ J$, $J\in sp_1$, $\xi\in V^*$.

Note that the space $\mathrm{Ker}\ \mathrm{Tr}|_{\mathcal{D}(\mathfrak{g}')}=\delta(sp_1\otimes V^*)_0\oplus\mathcal{D}(\mathfrak{g})$ is the space of all traceless tensors from $\mathcal{D}(\mathfrak{g}')$.

We denote by T^S the structure tensor of an (almost) q-like structure S considered as G-structure (see n. 4). We recall that the associated function on the appropriate G-structure P takes values in $\mathcal{D}(\mathfrak{g})$.

THEOREM 4 (See also [5, 11, 12, 14]). *a) Let $H=(J_\alpha)$ be an almost hypercomplex structure on M. Then*

a.1) Its structure tensor T^H is given by

$$T^H=B^H:=-(2/3)\sum_\alpha[J_\alpha,J_\alpha] \tag{7.1}$$

where $[J_\alpha,J_\alpha](X,Y)=(1/4)\{[X,Y]+J_\alpha[J_\alpha X,Y]+J_\alpha[X,J_\alpha Y]-[J_\alpha X,J_\alpha Y]\}$ is the Nijenhuis bracket of $J_\alpha(\alpha=1,2,3)$, and it belongs to the space $\mathrm{Ker}\,\mathrm{Tr}|_{\mathcal{D}(\mathfrak{g}')}$.

a.2) The unique canonical connection ($\mathcal{D}(\mathfrak{g}')$-connection) associated with H is given by

$$\nabla^H_XY=(1/12)\left\{\sum_{(\alpha,\beta,\gamma)}J_\alpha([J_\beta X,J_\gamma Y]+[J_\beta Y,J_\gamma X])+2\sum_\alpha J_\alpha([J_\alpha X,Y]+[J_\alpha Y,X])\right\}+ \tag{7.2}$$

$$+(1/2)B^H(X,Y)+(1/2)[X,Y]$$

where (α,β,γ) indicates sum over cyclic permutations of $(1,2,3)$.

b) Let Q be an almost quaternionic structure on M. Then

b.1) the structure tensor T^Q is given by

$$T^Q=T^H+\sum_\alpha\delta(\tau_\alpha\otimes J_\alpha) \tag{7.3}$$

where $H=(J_\alpha)$ locally generates Q and $\tau_\alpha(\alpha=1,2,3)$ are local 1-forms given by

$$\tau_\alpha(X)=(1/4n-2)\,\mathrm{Tr}(J_\alpha B^H_X)\qquad X\in TM. \tag{7.4}$$

Moreover $\sum_\alpha\tau_\alpha\circ J_\alpha=0$.

b.2) To any almost quaternionic connection ∇ with torsion T one can associate a globally defined $\mathcal{D}$-connection ${}^{\mathrm{Op}}\nabla$, that is almost quaternionic connection with torsion tensor T^Q, locally given by

$${}^{\mathrm{Op}}\nabla_X=\nabla_X+(1/6)\sum_{(\alpha,\beta,\gamma)}[2\varphi_\alpha-\varphi_\beta\circ J_\gamma+\varphi_\gamma\circ J_\beta](X)J_\alpha-p\left[T_X+(1/3)\sum_\alpha T_{J_\alpha X}J_\alpha\right]$$

where

$$p\colon V\otimes V^* \to \mathfrak{g}' = gl_n(\boldsymbol{H}),$$

$$A \mapsto p(A) = (1/4)\left[A - \sum_\alpha J_\alpha A J_\alpha\right],$$

is the natural projection and $\varphi_\alpha (a = 1, 2, 3)$ *are the following local 1-forms* $\varphi_\alpha(X) = (1/2n - 1)\,\mathrm{Tr}\,(J_\alpha T_X)$ $\forall X \in TM$. *Any two* $\mathcal{D}$*-connections are related by formula* $\nabla' = \nabla + S^\xi$, $\xi \in \Lambda^1 M$ (see (5.1)).

Remarks. 1) The first part of statement *a*.1) was proved by E. Bonan [5].

2) The connection ∇^H was defined by M. Obata: it has torsion tensor T^H and is called *Obata connection*.

3) The connection ${}^{\mathrm{Op}}\nabla$ was defined by V. Oproiu [11], and is called *Oproiu connection associated with* ∇.

We indicate here the idea of other proof of *a*.1) based on the following

Lemma [2]. *Let* J *be an almost complex structure on* M *and* ∇ *be a linear connection such that* $\nabla J = 0$, *with torsion tensor* T. *Then* $[J, J] = T^{02}_{(J)} = (1/4)\,[T(\cdot,\cdot) + JT(J\cdot,\cdot) + JT(\cdot, J\cdot) - T(J\cdot, J\cdot)]$ *where* $T^{02}_{(J)}$ *is* $(0, 2)$ *component of the vector valued 2-form* T *with respect to* J *that is* $T^{02}_{(J)}(J\cdot,\cdot) = T^{02}_{(J)}(\cdot, J\cdot) = -JT^{02}_{(J)}(\cdot,\cdot)$.

Now we consider the following G'-equivariant surjective map

$$\chi = \delta \circ (p\otimes 1)\colon V\otimes \Lambda^2 V^* \to V\otimes V^* \otimes V^* \xrightarrow{p\otimes 1} \mathfrak{g}' \otimes V^* \xrightarrow{\delta} \delta(\mathfrak{g}'\otimes V^*)$$

with $\mathrm{Ker}\,\chi = \mathcal{D}(\mathfrak{g}')$. To prove that B^H given by formula (7.1) belongs to $\mathcal{D}(\mathfrak{g}')$ it is sufficient to show that $(p\otimes 1)(B^H) = 0$. By Lemma

$$B^H := -(2/3)\sum_\alpha [J_\alpha, J_\alpha] = -(2/3)\sum_\alpha T^{02}_{(J_\alpha)}$$

where T is the torsion tensor of a connection that preserves $H = (J_\alpha)$. Since for any $X \in TM$ the operator $[T^{02}_{(J_\alpha)}]_X$ anticommutes with J_α its projection $p([T^{02}_{(J_\alpha)}]_X)$ on the space $\mathfrak{g}'$ of operators which commute with $J_\rho\,(\rho = 1, 2, 3)$ vanishes. Hence

$$p(B^H_X) = -(2/3)\sum_\alpha p([T^{02}_{(J_\alpha)}]_X) = 0.$$

A straightforward calculation shows that the connection ∇^H defined by (7.2) is a connection which preserves H and has torsion tensor B^H. This proves the first part of *a*.1) and *a*.2). The last statement of *a*.1) follows from

Lemma. *Nijenhuis tensor* $N_J = [J, J]$ *of an almost complex structure* J *is traceless.*

Statement *b*) was essentially proved by V. Oproiu [11]. Actually it is a straightforward verification that the torsion of the Oproiu connection ${}^{\mathrm{Op}}\nabla$ is given by

$$T^{{}^{\mathrm{Op}}\nabla} \equiv \mathrm{Tor}\,({}^{\mathrm{Op}}\nabla) = T^H + \sum_\alpha \delta(\tau_\alpha \otimes J_\alpha).$$

The equality $T^Q = \mathrm{Tor}\,({}^{\mathrm{Op}}\nabla)$ now follows from the

Lemma. *Denote by* $U = \delta(sp_1 \otimes V^*)_0 \oplus \mathcal{D}(\mathfrak{g})$ *the* G*-module which is sum of two ir-*

reducible G*-modules. Then* $\mathcal{D}(\mathfrak{g}) = \{T \in U \mid \mathrm{Tr}(J_\alpha T_X) = 0,\ \alpha = 1, 2, 3,\ X \in V\}$ *and the projection* $q(T)$ *of a tensor* $T \in U$ *onto* $\mathcal{D}(\mathfrak{g})$ *is given by*

$$q(T) = T + \sum_\alpha \delta(\tau_\alpha \otimes J_\alpha)$$

where $\tau_\alpha(X) = (1/4n - 2)\,\mathrm{Tr}(J_\alpha T_X),\ (\alpha = 1, 2, 3)$.

Indeed q is a projector on the space $D = \{q(T),\ T \in U\}$ and G-module $D \neq 0$, $\delta(sp_1 \otimes V^*)_0$, U; hence $D \equiv \mathcal{D}(\mathfrak{g})$.

8. Explicit formulas for canonical connections of unimodular and Hermitian q-like structures

Using Theorem 4 we derive explicit expressions for the canonical connection ∇^S and the corresponding structure tensor T^S for q-like structures $S = (H, \mathrm{vol})$, (Q, vol), (H, g), (Q, g) on a $4n$-manifold M.

For an almost Hermitian hypercomplex structure (H, g), $H = (J_\alpha)$, we define a $(1, 2)$ tensor field $A = A^{H,g}$ by the formula $A_X = (1/2)g^{-1}\nabla^H_X g$, $X \in TM$. It was proved by E. Bonan [5] that A_X is a symmetric endomorphism commuting with J_α, $\alpha = 1, 2, 3$.

Contracting tensor A we obtain 2-forms: $\omega^{H,g}(X) := \mathrm{Tr}\,A_X$, $\sigma^{H,g}(X) := \mathrm{Tr}(Y \to \to A_Y X)$, $X, Y \in TM$. Then $\nabla^H \mathrm{vol}^g = \omega^{H,g} \otimes \mathrm{vol}^g$.

Proposition 2. *For any vector* $X \in TM$

(8.1) 1) $(\nabla^{H,\mathrm{vol}})_X = (\nabla^H)_X + (1/4n)\,\omega(X)\,\mathrm{Id}, \qquad T^{H,\mathrm{vol}} = T^H + (1/4n)\,\delta(\omega \otimes \mathrm{Id})$

where $\nabla^H_X \mathrm{vol} = \omega(X)\,\mathrm{vol}$;

(8.2) 2) $(\nabla^{Q,\mathrm{vol}})_X = (\nabla^H)_X + \sum_\alpha \tau_\alpha(X) J_\alpha + [1/4(n+1)]\,(S^{\omega^H})_X$,

where $Q = \langle H \rangle$, *that is* H *is a locally defined almost hypercomplex structure that generates* Q, $\nabla^H_X \mathrm{vol} = \omega^H(X)\,\mathrm{vol}$ *and the* τ_α, $\alpha = 1, 2, 3$, *are defined by* (7.4).

3) $T^Q = T^{Q,\mathrm{vol}}$ *for any volume form* vol.

4) [5]. *Canonical connection and structure tensor of almost Hermitian hypercomplex structure* (H, g) *are given by*

(8.3) $$\nabla^{H,g} = \nabla^H + A, \qquad T^{H,g} = T^H + \delta A$$

5) *Canonical connection and structure tensor of almost Hermitian quaternionic structure* (Q, g) *are given by*

(8.4.1) $$(\nabla^{Q,g})_X = (\nabla^{H,g})_X + \sum_\alpha \tau_\alpha(X) J_\alpha + (1/4n)[S^\sigma_X - (S^{g \circ X})_{g^{-1}\sigma}]$$

(8.4.2) $$T^{Q,g} = T^Q + \delta A + (1/n) R_{HP^n}(\cdot,\cdot)(g^{-1}\sigma) = T^{H,g} + \sum_\alpha \delta(\tau_\alpha \otimes J_\alpha) + (1/n) R_{HP^n}(\cdot,\cdot)(g^{-1}\sigma)$$

where $H = (J_\alpha)$ *is a local almost hypercomplex structure that generates* Q, τ_α $(\alpha = 1, 2, 3)$ *are locally defined 1-forms given by* (7.4), $\sigma = \sigma^{H,g}$ *and*

$$R_{HP^n}(X, Y) = (1/4)[S_X^{g\circ Y} - S_Y^{g\circ X}] \tag{8.5}$$

(see also n. 9).

9. Space of curvature tensor of torsionless q-like structures

Let $\mathcal{G} \subset gl_n(V)$ be a space of endomorphisms. Recall that *space* $\mathfrak{R}(\mathcal{G})$ *of curvature tensors of the type* $\mathcal{G}$ is defined as the space of $\mathcal{G}$-valued δ-closed 2-forms, $\mathfrak{R}(\mathcal{G}) = \{R \in \mathcal{G} \otimes \Lambda^2 V^*, \delta R = 0\}$ where δ: $V \otimes V^* \otimes \Lambda^2 V^* \to V \otimes \Lambda^3 V^*$ is the Spencer operator.

The curvature tensor in a point x of any torsionless connection of G-structure π: $P \to M$ belongs to $\mathfrak{R}(\mathcal{G})$.

Now we describe the space $\mathfrak{R}(\mathcal{G})$ for $\mathfrak{g} = sp_1 + gl_n(\boldsymbol{H})$. For any bilinear form B on V we set $R^B(X, Y) = S_X^{B(Y,\cdot)} - S_Y^{B(x,\cdot)}$, $\forall X, Y \in V$ where S is given by (5.1) and we denote by $\mathfrak{R}_{\mathrm{Bil}}$ the space of all such tensors.

Proposition 3 (See [13, 14, 10]). $\mathfrak{R}_{\mathrm{Bil}}$ *is a* $\mathfrak{g}$*-submodule of* $\mathfrak{g}$*-module* $\mathfrak{R}(\mathfrak{g})$ *and a uniquely defined irreducible complementary submodule is* $\mathfrak{R}(sl_n(\boldsymbol{H}))$: $\mathfrak{R}(\mathfrak{g}) \equiv \mathfrak{R}(sp_1 + gl_n(\boldsymbol{H})) = \mathfrak{R}_{\mathrm{Bil}} + \mathfrak{R}(sl_n(\boldsymbol{H}))$.

The $\mathfrak{g}$-module Bil of bilinear forms has the following decomposition into the sum of irreducible modules, [5]: $\mathrm{Bil} = S_\natural^2 + S_{\mathrm{mix}}^2 + \Lambda_\natural^2 + \Lambda_{\mathrm{mix}}^2$ where $S_\natural^2$ (resp., $\Lambda_\natural^2$) is the space of symmetric (resp., skew-symmetric) forms which are Hermitian with respect to any complex structure $J \in sp_1$, and S_{mix}^2, Λ_{mix}^2 are complementary submodules of *mixed forms*. Hence decomposition of $\mathfrak{R}(\mathfrak{g})$ into irreducible submodules may be written as

$$\mathfrak{R}(sp_1 + gl_n(\boldsymbol{H})) = \mathfrak{R}(sl_n(\boldsymbol{H})) + \mathfrak{R}(S_\natural^2) + \mathfrak{R}(S_{\mathrm{mix}}^2) + \mathfrak{R}(\Lambda_\natural^2) + \mathfrak{R}(\Lambda_{\mathrm{mix}}^2).$$

As a simple Corollary we obtain the following decompositions into irreducible $\mathfrak{g}$-submodules:

$$\mathfrak{R}(sp_1 + sl_n(\boldsymbol{H})) = \mathfrak{R}(sl_n(\boldsymbol{H})) + \mathfrak{R}(S_\natural^2) + \mathfrak{R}(S_{\mathrm{mix}}^2), \quad \mathfrak{R}(gl_n(\boldsymbol{H})) = \mathfrak{R}(sl_n(\boldsymbol{H})) + \mathfrak{R}(\Lambda_\natural^2).$$

We indicate also well known decomposition of the space $\mathfrak{R}(sp_1 + sp_n)$ into irreducible $(sp_1 + sp_n)$-submodules: $\mathfrak{R}(sp_1 + sp_n) = \mathfrak{R}(sp_n) + \boldsymbol{R}R_{HP^n}$ where R_{HP^n} is the curvature tensor of the quaternionic projective space HP^n with natural metric, given by (8.5).

A torsionless almost quaternionic connection on a manifold with a quaternionic structure is called *quaternionic*. Curvature tensor of a quaternionic connection belongs to the space $\mathfrak{R}(sp_1 + gl_n(\boldsymbol{H}))$. Its $\mathfrak{R}(sl_n(\boldsymbol{H}))$ component doesn't change under change of quaternionic connection and it is called *Weyl tensor* of quaternionic structure.

Theorem 5. 1) [12] *A quaternionic structure* Q *on a manifold* M *is integrable iff its Weyl tensor vanishes.*

2) *Let* $\mathcal{S}$ *be a* (1*-integrable*) q*-like structure different from quaternionic one. It is integrable iff its canonical connection is flat, that is its torsion and curvature tensors are zero.*

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