Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

Alberto Cialdea

On the finiteness of the energy integral in elastostatics with non absolutely continuous data

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **4** (1993), n.1, p. 35–42.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1993_9_4_1_35_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1993.

Rend. Mat. Acc. Lincei
s. 9, v. 4:35-42 (1993)

Analisi matematica. — *On the finiteness of the energy integral in elastostatics with non absolutely continuous data.* Nota (*) di Alberto Cialdea, presentata dal Socio G. Fichera.

Abstract. — In this paper the main problem of classical elastostatics with non absolutely continuous data is considered. Necessary and sufficient conditions under which the energy integral is finite are given.

Key words: Elasticity; Boundary value problems; Measure theory.

Riassunto. — *Sulla finitezza dell'integrale dell'energia nell'elastostatica con dati non assolutamente continui.* In questo lavoro viene considerato il problema principale dell'elastostatica classica con dati non assolutamente continui. Sono date condizioni necessarie e sufficienti perché l'integrale dell'energia risulti finito.

In papers [1, 2] the main problem of classical elastostatics is considered when non absolutely continuous forces act on the body. As far as existence, uniqueness (up to rigid displacements) and representation of the solution are concerned, a complete theory has been developed. Moreover sufficient conditions under which the energy integral is finite are given. In [1] it is shown by an example that there exist non absolutely continuous data satisfying these conditions, *i.e.* there exist concentrated loads such that the corresponding energy integral is finite.

In the present paper the problem of the finiteness of the energy integral is investigated again and necessary and sufficient conditions are given.

Results given in this paper are also interesting from the point of view of Mathematical Physics. In fact finiteness of the energy integral means that the response of the elastic material under the action of the non absolutely continuous given forces remains in the ambit of classical elasticity of infinitesimal deformations and neither finite deformations nor elastoplastic strains are originated by the given forces.

1. Preliminary results

Let $\Omega \subset \boldsymbol{R}^3$ be the bounded domain representing the natural configuration of an elastic body. We suppose that $\boldsymbol{R}^3 - \overline{\Omega}$ is connected and that the boundary Σ of Ω is a Lyapunov surface, *i.e.* Σ has a uniformly Hölder continuous normal field of some exponent λ $(0 < \lambda \leq 1)$.

Writing $\mu \in M(\overline{\Omega})$ $(\alpha \in M(\Sigma))$ we mean that $\mu = (\mu^1, \mu^2, \mu^3)$ $(\alpha = (\alpha^1, \alpha^2, \alpha^3))$ is a vector measure on $\overline{\Omega}$ (on Σ), *i.e.* μ (α) is a vector valued countably additive set function defined on the family of all the Borel sets contained in $\overline{\Omega}$ (in Σ). This vector measure represents the body (surface) forces acting on $\overline{\Omega}$ (on Σ). In [1, p. 31] it is shown that it is not restrictive to suppose that $\mu(B) = 0$ for every Borel set contained in Σ. We shall suppose that this condition is

(*) Pervenuta all'Accademia il 9 luglio 1992.

satisfied. This means that if some of the given body forces are concentrated on Σ, then we consider them as surface forces.

For any $p > 1$ we denote by $L^p(\Omega)$ (by $L^p(\Sigma)$) the space of the measurable vector-valued functions $u = (u^1, u^2, u^3)$ such that $|u^j|^p$ is integrable over Ω (over Σ). Let us consider the following spaces introduced in [1, 2]:

$$U^p(\Omega) = \left\{ u = (u^1, u^2, u^3) \,|\, \exists\, \varphi = (\varphi^1, \varphi^2, \varphi^3) \in L^p(\Sigma) : u^i(x) = \int_\Sigma \varphi^j(y)\, L_{jy}[s^i(x,y)]\, d\sigma_y,\ x \in \Omega \right\},$$

(where $s^h(x, y)$ are the vectors whose components are the elements of the h-th row of Somigliana's matrix: $\{s_j^i(x, y)\} = \{\delta_{ij}\, |x - y|^{-1} - k[2(1 + k)]^{-1} \partial^2 |x - y| / \partial x_i \partial x_j\}$),

$$U(\Omega) = \bigcap_{1 < p < 2} U^p(\Omega),$$

$$V'(\Omega) = \left\{ v = (v^1, v^2, v^3) \,|\, \exists\, \nu = (\nu^1, \nu^2, \nu^3) \in M(\overline{\Omega}), \exists\, u \in U(\Omega) : v^i(x) = \int_{\overline{\Omega}} s_j^i(x, y)\, d\nu_y^j + u^i(x),\ x \in \Omega \right\}.$$

Let us consider the following boundary value problem:

$$(1.1) \qquad \begin{cases} u \in V'(\Omega), & \\ Eu = \mu & \mu \in M(\overline{\Omega}), \\ Lu = \alpha & \alpha \in M(\Sigma), \end{cases}$$

where: $Eu = \Delta_2 u + k \operatorname{grad} \operatorname{div} u$ $(k > 1/3)$, $Lu = (k - 1) \operatorname{div} u \cdot n + 2 \partial u / \partial n + n \wedge \operatorname{rot} u$, n is the inward unit normal to Σ, $\partial u / \partial n$ is the vector whose components are $\partial u^i / \partial n$. $u \in V'(\Omega)$ is a solution of the BVP (1.1) if

$$(1.2) \qquad \lim_{\rho \to 0^+} \left\{ \int_{\Omega_\rho} u\, Ew\, dx + \int_{\Sigma_\rho} u\, L^\rho w\, d\sigma_\rho \right\} = \int_\Omega w\, d\mu + \int_\Sigma w\, d\alpha, \qquad \forall w \in C^\infty(\boldsymbol{R}^3),$$

where $\Omega_\rho \subset \Omega$ and $\Sigma_\rho = \partial \Omega_\rho$ is some «parallel» surface to Σ tending to Σ when $\rho \to 0^+$ (for the details see either [1] or [2]).

I. *Given* $\mu \in M(\overline{\Omega})$, $\alpha \in M(\Sigma)$ *such that*

$$(1.3) \qquad \int_\Omega (a + b \wedge x)\, d\mu + \int_\Sigma (a + b \wedge x)\, d\alpha = 0 \qquad \forall a, b \in \boldsymbol{R}^3,$$

there exists one solution of BVP (1.1) (i.e. *such that* (1.2) *holds*) *in* V$'(\Omega)$. *The solution is determined up to an additive rigid displacement.*

For the proof of this Theorem, see [2].

II. *If the function u, belonging to* $V'(\Omega)$:

$$u^i(x) = \int_{\overline{\Omega}} s_j^i(x, y)\, d\nu_y^j + \int_\Sigma \varphi^j(y)\, L_{jy}[s^i(x, y)]\, d\sigma_y$$

is solution of BVP (1.1), *then the following Betti formula holds*:

$$(1.4)\qquad \int_\Omega u\, Ew\, dx + \int_\Sigma u\, Lw\, d\sigma = \int_\Omega w\, d\mu + \int_\Sigma w\, d\alpha\,, \qquad \forall w \in C^\infty(\boldsymbol{R}^3)\,,$$

where the trace of u on Σ is:

$$u^i(x) = \int_{\overline{\Omega}} s_j^i(x, y)\, dv_y^j + 2\pi\varphi^i(x) + \int_\Sigma \varphi^j(y)\, L_{jy}[s^i(x, y)]\, d\sigma_y\,.$$

Conversely, let $\Phi \in L^1(\Omega)$, $\Psi \in L^1(\Sigma)$ be such that:

$$\int_\Omega \Phi\, Ew\, dx + \int_\Sigma \Psi\, Lw\, d\sigma = \int_\Omega w\, d\mu + \int_\Sigma w\, d\alpha\,, \qquad \forall w \in C^\infty(\boldsymbol{R}^3)\,,$$

then $\Phi \in V'(\Omega)$, Ψ is the trace of Φ on Σ and Φ is solution of BVP (1.1).

This Theorem was proved in [1] by using a different representation formula. The present result follows immediately from Theorem IV of [2].

Let us denote by $\varepsilon_{ih}(u)$ the linearized strain components and by $\sigma_{ih}(u)$ the stress components: $\varepsilon_{ih}(u) = (\partial u^i/\partial x_h + \partial u^h/\partial x_i)/2$, $\sigma_{ih}(u) = \varepsilon_{ih}(u) + (k-1)\,\delta_{ih}(\varepsilon_{11}(u) + \varepsilon_{22}(u) + \varepsilon_{33}(u))/2$.

The integral

$$(1.5)\qquad \int_\Omega \sum_{i,h}^{1,3} \sigma_{ih}(u)\, \varepsilon_{ih}(u)\, dx$$

is called the *energy integral*. Because of Korn's second inequality (see [3, p. 381-384]) the integral (1.5) is finite if and only if u belongs to $H^1(\Omega)$, *i.e.* u^i $(i = 1, 2, 3)$ and their first derivatives are square-integrable functions on Ω. Then the solution u of *BVP* (1.1) has a finite energy integral if and only if u is solution of the following problem:

$$(1.6)\qquad \begin{cases} u \in H^1(\Omega) \cap V'(\Omega)\,, & \\ Eu = \mu & \mu \in M(\overline{\Omega})\,, \\ Lu = \alpha & \alpha \in M(\Sigma)\,. \end{cases}$$

III. *If u is solution of BVP* (1.6) *then the following Clapeyron's formula holds*:

$$(1.7)\qquad -\int_\Omega \mathcal{E}(u, w)\, dx = \int_\Omega w\, d\mu + \int_\Sigma w\, d\alpha\,, \qquad \forall w \in C^\infty(\boldsymbol{R}^3)\,,$$

where

$$\mathcal{E}(u, w) = 2\sum_{i,h}^{1,3} \sigma_{ih}(u)\, \varepsilon_{ih}(w) = 2\sum_{i,h}^{1,3} \sigma_{ih}(w)\, \varepsilon_{ih}(u)\,.$$

Conversely, let $u \in H^1(\Omega)$ be such that (1.7) *holds; then $u \in V'(\Omega)$ and it is solution of BVP* (1.6).

This Theorem follows immediately from the previous one and from Theorem XII of [1]. Since we shall use this result in the next section, for the sake of completeness we give here a proof. First we observe that

$$(1.8) \qquad -\int_{\Omega} \mathcal{E}(u,w)\,dx = \int_{\Omega} u\,Ew\,dx + \int_{\Sigma} u\,Lw\,d\sigma, \qquad \forall u \in H^1(\Omega),\ \forall w \in C^\infty(\boldsymbol{R}^3).$$

If u is solution of BVP (1.6), then (1.7) follows from (1.4) and (1.8). Conversely, let $u \in H^1(\Omega)$ be such that (1.7) holds; from (1.8) and Theorem II it follows that u is solution of BVP (1.6).

In [1] some sufficient conditions (under which there exists a solution of BVP (1.6)) for μ and α are given.

2. Necessary and sufficient conditions for finiteness of the energy integral

In this Section we shall use the theory of k-forms and k-measures. A summary on this subject may be found in §4 of [1]. By $L^p_k(\Omega)$ we denote the space of differential forms of degree k such that their coefficients are scalar functions belonging to $L^p(\Omega)$.

The next Theorem gives a necessary and sufficient condition for the finiteness of the energy integral. If μ and α are the given forces, we shall set $\overline{\mu}(B) = \mu(B \cap \Omega) + \alpha(B \cap \Sigma)$ for every Borel set of $\boldsymbol{R}^3$. $\overline{\mu}$ represents the *total force* acting on the body.

IV. *Given* $\mu \in M(\overline{\Omega})$, $\alpha \in M(\Sigma)$ *such that* (1.3) *holds, there exists a solution of BVP* (1.6) *if and only if the components* $\overline{\mu}^h$ *of the total force* $\overline{\mu}$ ($\overline{\mu}^h$ *being viewed as 3-measures*) *are the differentials of 2-forms belonging to* $L^2_2(\Omega)$, i.e. *if and only if there exist* $\gamma^h \in L^2_2(\Omega)$ $(h = 1,2,3)$ *such that*:

$$(2.1) \qquad \int_{\overline{\Omega}} w_h\,d\overline{\mu}^h = -\int_{\Omega} dw_h \wedge \gamma^h, \qquad \forall w \in C^\infty(\boldsymbol{R}^3).$$

If u is solution of BVP (1.6), then Clapeyron's formula (1.7) holds. This implies that

$$\left| \int_{\overline{\Omega}} w_h\,d\overline{\mu}^h \right| \le C \sum_{h=1}^{3} \|dw_h\|_{L^2_1(\Omega)} \qquad \forall w \in C^\infty(\boldsymbol{R}^3).$$

Then we may extend the functional $\int_{\overline{\Omega}} w_h\,d\overline{\mu}^h$ to the quotient space $H^1(\Omega)/\boldsymbol{R}^3$ equipped with the scalar product

$$(v,w) = \int_{\Omega} dv_h \wedge * dw_h\,.$$

By the Riesz Representation Theorem, there exists $\Gamma \in H^1(\Omega)$ such that

$$\int_{\overline{\Omega}} w_h\,d\overline{\mu}^h = \int_{\Omega} dw_h \wedge * d\Gamma^h \qquad \forall w \in C^\infty(\boldsymbol{R}^3).$$

Assuming $\gamma^h = - * d\Gamma^h$, we obtain (2.1).

Conversely, let us suppose (1.3) and (2.1) hold. If we define $F(w) = \int_{\overline{\Omega}} w_h\,d\overline{\mu}^h$, we have that $F(w)$ can be extended to a linear and continuous functional on $H^1(\Omega)$. Moreover, because of Korn's second inequality, we have

$$\sum_{h=1}^{3} \|dw_h\|_{L_1^2(\Omega)} \leq C\left(\int_\Omega \mathcal{E}(w, w)\, dx\right)^{1/2},$$

for any $w \in H^1(\Omega)$ such that:

$$\int_\Omega \left(\frac{\partial w_i}{\partial x_j} - \frac{\partial w_j}{\partial x_i}\right) dx = 0 \qquad (i, j = 1, 2, 3). \tag{2.2}$$

Then

$$|F(w)| \leq C'\left(\int_\Omega \mathcal{E}(w, w)\, dx\right)^{1/2},$$

for any $w \in H^1(\Omega)$ such that (2.2) holds. On the other hand, given $v \in H^1(\Omega)$, define

$$b_i = \frac{1}{2\,\mathrm{meas}(\Omega)} \int_\Omega \delta_{ijk}^{123} \frac{\partial v_j}{\partial x_k}\, dx.$$

Thus $w = v + b \wedge x$ satisfies (2.2). Keeping in mind (1.3) we have

$$|F(v)| = |F(w)| \leq C'\left(\int_\Omega \mathcal{E}(w, w)\, dx\right)^{1/2} = C'\left(\int_\Omega \mathcal{E}(v, v)\, dx\right)^{1/2} \qquad \forall v \in H^1(\Omega). \tag{2.3}$$

Let us consider the quotient space $H^1(\Omega)/\{a + b \wedge x \mid a, b \in \boldsymbol{R}^3\}$ equipped with the scalar product

$$((u, w)) = \int_\Omega \mathcal{E}(u, w)\, dx;$$

from (1.3), (2.3) it follows that $F(v)$ is a linear and continuous functional on this space. By the Riesz Representation Theorem, there exists $u \in H^1(\Omega)$ such that

$$F(w) = \int_{\overline{\Omega}} w_h\, d\overline{\mu}^h = -\int_\Omega \mathcal{E}(u, w)\, dx \qquad \forall w \in C^\infty(\boldsymbol{R}^3).$$

From Theorem III it follows that u is solution of BVP (1.6).

The next Theorem provides another characterization for the surface measures in order to have a finite energy integral, when there are no body forces.

We denote by $H_1^{1/2}(\Sigma)$ the space of the differential forms ψ of degree one defined on Σ such that ψ is the restriction on Σ of a differential form $\Psi \in H_1^1(\Omega)$ (*i.e.* the coefficients and their first derivatives of Ψ are square-integrable functions on Ω). We recall that if Ψ is a smooth differential form defined in $\overline{\Omega}$, say $\Psi = \Psi_h\, dx^h$, the restriction $\Psi|_\Sigma$ is the 1-form given in local coordinates by

$$\Psi_h[x(t)]\,[\partial x^h/\partial t^j]\,dt^j.$$

If $\Psi \in H_1^1(\Omega)$ we may define $\Psi|_\Sigma$ by using usual density arguments. We equip $H_1^{1/2}(\Sigma)$

with the following norm:

$$\|\psi\|_{H_1^{1/2}(\Sigma)} = \inf_{\substack{\Psi \in H_1^1(\Omega) \\ \Psi|_\Sigma = \psi}} \|\Psi\|_{H_1^1(\Omega)} .$$

V. *Let $\alpha \in M(\Sigma)$ be such that*

$$\int_\Sigma (a + b \wedge x)\, d\alpha = 0 \qquad \forall a, b \in \boldsymbol{R}^3 .$$

There exists a solution of the BVP

$$\begin{cases} u \in H^1(\Omega) \cap V'(\Omega), \\ Eu = 0, \\ Lu = \alpha, \end{cases} \tag{2.4}$$

if and only if the components of α are the differentials on Σ of 1-forms belonging to $H_1^{1/2}(\Sigma)$, i.e. *if and only if there exist $\psi^h \in H_1^{1/2}(\Sigma)$ $(h = 1, 2, 3)$ such that*

$$\int_\Sigma w_h\, d\alpha^h = - \int_\Sigma dw_h \wedge \psi^h , \qquad \forall w \in C^\infty(\boldsymbol{R}^3). \tag{2.5}$$

If (2.5) is satisfied, we have

$$\int_\Sigma w_h\, d\alpha^h = - \int_\Sigma dw_h \wedge \psi^h = \int_\Omega dw_h \wedge d\Psi^h , \qquad \forall w \in C^\infty(\boldsymbol{R}^3),$$

where $\Psi^h \in H_1^1(\Omega)$ are such that $\Psi^h|_\Sigma = \psi^h$. Then (2.1) holds, where $\gamma^h = -d\Psi^h$. From Theorem IV it follows that there exists a solution of BVP (2.4).

Conversely, let u be a solution of BVP (2.4). By Theorem IV there exist $\gamma^h \in L_2^2(\Omega)$ such that

$$\int_\Sigma w_h\, d\alpha^h = - \int_\Omega dw_h \wedge \gamma^h , \qquad \forall w \in C^\infty(\boldsymbol{R}^3).$$

In particular

$$\int_\Omega dw_h \wedge \gamma^h = 0, \qquad \forall w \in \mathring{C}^\infty(\Omega),$$

i.e. $d\gamma^h = 0$ $(h = 1, 2, 3)$. Then there exist $\Gamma^h \in H_1^1(\Omega)$ such that $\gamma^h = d\Gamma^h$. The proof of this may be found in [5] (Theorem 3.4 (¹); see also Remark 3.10). Consequently

$$\int_\Sigma w_h\, d\alpha^h = - \int_\Omega dw_h \wedge d\Gamma^h = \int_\Sigma dw_h \wedge \Gamma^h , \qquad \forall w \in C^\infty(\boldsymbol{R}^3).$$

Assuming $\psi^h = -\Gamma^h|_\Sigma$ we obtain the result.

In [1] we have given the following sufficient condition: *if the Hölder exponent of Σ is $\lambda > 1/2$ and if $\alpha^h = d\psi^h$, where the coefficients of ψ are in $C^{\lambda'}(\Sigma)$* (i.e. *they are*

(¹) Actually these Authors do not use the differential forms, but the above-mentioned Theorem can be immediately restated using them.

Hölder continuous functions of exponent λ') *with* $\lambda' > 1/2$, *then there exists a solution of BVP* (2.4).

Now this condition can be obtained from Theorem V. This will follow immediately from the next Theorem

VI. *If the Hölder exponent of* Σ *is* $\lambda > 1/2$ *and the coefficients of the* 1*-form* ψ *are in* $C^{\lambda'}(\Sigma)$ $(\lambda' > 1/2)$, *then* $\psi \in H_1^{1/2}(\Sigma)$.

For the sake of simplicity we shall suppose that $1/2 < \lambda = \lambda' < 1$, which is obviously not restrictive.

We may write ψ in local coordinates: $\psi = \psi_j(t)\, dt^j$. Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$ be the solution of the system

$$\begin{cases} \gamma_h(x) \dfrac{\partial x^h}{\partial t_j}(t) = \psi_j(t) & j = 1, 2, \\ \gamma_h(x) n^h(x) = 0, \end{cases} \tag{2.6}$$

where $x = x(t)$. This system has one and only one solution in any point $x \in \Sigma$ and the values $\gamma_h(x)$ do not depend on the local coordinates we use. In fact if $x = x(t) = \tilde{x}(\tilde{t})$, $\psi = \tilde{\psi}_j(t)\, d\tilde{t}^j$ and $\gamma(x)$ is solution of (2.6), then $\gamma(x)$ is solution of the system

$$\begin{cases} \gamma_h(x) \dfrac{\partial \tilde{x}^h}{\partial \tilde{t}_j}(\tilde{t}) = \tilde{\psi}_j(\tilde{t}) & j = 1, 2, \\ \gamma_h(x) n^h(x) = 0. \end{cases}$$

We have $\psi = \gamma_h(x)\, dx^h$ $(x \in \Sigma)$ and $\gamma_h(x)$ are Hölder continuous functions of exponent λ. Let now $u_h(x)$ be the double layer (harmonic) potentials such that $u_h(x) = \gamma_h(x)$, $x \in \Sigma$ $(h = 1, 2, 3)$. In other words, let

$$u_h(x) = \int_\Sigma \varphi_h(y) \frac{\partial}{\partial n_y} \frac{1}{|x - y|}\, d\sigma_y$$

where $\varphi_h(x)$ is the solution of the integral equation

$$2\pi \varphi_h(x) + \int_\Sigma \varphi_h(y) \frac{\partial}{\partial n_y} \frac{1}{|x - y|}\, d\sigma_y = \gamma_h(x), \qquad x \in \Sigma.$$

Because of known Theorems (see [6, p. 240]) the functions $\varphi_h(x)$ are Hölder continuous with exponent λ.

Now, in order to prove that $u_h(x) \in H^1(\Omega)$ we may repeat the reasoning employed in [1, p. 47-48]. Namely let $\varphi_h^*(x)$ be a function belonging to $C^\lambda(\overline{\Omega})$ such that $\varphi_h^*(x) = \varphi_h(x)$, $x \in \Sigma$ (for the construction of such a function, see [4, p. 383]). We have

$$\frac{\partial u_h}{\partial x_k} = \int_\Sigma [\varphi_h(y) - \varphi_h^*(x)] \frac{\partial}{\partial n_y} \frac{\partial}{\partial x_k} \frac{1}{|x - y|}\, d\sigma_y$$

and then

$$\left| \frac{\partial u_h}{\partial x_k} \right| \le C \int_{\Sigma} |x-y|^{\lambda-3} d\sigma_y \qquad \forall x \in \Omega \,.$$

It follows (see [7, p. 806])

$$\int_{\Omega} \left| \frac{\partial u_h}{\partial x_k} \right|^2 dx \le C^2 \int_{\Omega} dx \int_{\Sigma} |x-y|^{\lambda-3} d\sigma_y \int_{\Sigma} |x-w|^{\lambda-3} d\sigma_w =$$

$$= C^2 \int_{\Sigma} d\sigma_y \int_{\Sigma} d\sigma_w \int_{\Omega} |x-y|^{\lambda-3} |x-w|^{\lambda-3} dx \le K \int_{\Sigma} d\sigma_y \int_{\Sigma} |y-w|^{2\lambda-3} d\sigma_w \,.$$

Since $2\lambda - 3 > -2$, $u_h(x) \in H^1(\Omega)$. Consequently the form $u_h(x)\,dx^h$ belongs to $H_1^1(\Omega)$ and it is such that $u_h(x)\,dx^h|_{\Sigma} = \psi$, *i.e.* $\psi \in H_1^{1/2}(\Sigma)$.

References

[1] A. Cialdea, *Elastostatics with non absolutely continuous data.* Journ. of Elasticity, 23, 1990, 13-51.
[2] A. Cialdea, *A complement to the theory of elastostatics with non absolutely continuous data.* Rend. Mat., 7, 9, 1989, 1-8.
[3] G. Fichera, *Existence theorems in elasticity.* In: *Mechanics of solids.* Vol. 2, Springer-Verlag, Berlin - Heidelberg - New York - Tokyo 1984, 347-389.
[4] G. Fichera - L. De Vito, *Funzioni analitiche di una variabile complessa.* 3ª ed., Veschi, Roma 1971.
[5] V. Girault - P. A. Raviart, *Finite element methods for Navier-Stokes equations.* Springer-Verlag, Berlin - Heidelberg - New York - Tokyo 1986.
[6] V. D. Kupradze (ed.), *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity.* North-Holland, Amsterdam 1979.
[7] M. Picone, *Appunti di analisi superiore.* Rondinella, Napoli 1940.

Dipartimento di Matematica
Università degli Studi di Roma «La Sapienza»
Piazzale A. Moro, 2 - 00185 Roma