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Holomorphic isometries of Cartan domains of type four

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Geometria. — *Holomorphic isometries of Cartan domains of type four.* Nota (*) del Socio EDOARDO VESENTINI.

ABSTRACT. — The holomorphic isometries for the Kobayashi metric of Cartan domains of type four are characterized.

KEY WORDS: Cartan domain; Kobayashi metric; Holomorphic isometry; Complex extreme point.

RIASSUNTO. — *Isometrie olomorfe di domini di Cartan del quarto tipo.* Si caratterizzano le isometrie olomorfe per la metrica di Kobayashi dei domini di Cartan del quarto tipo.

Let $\mathcal{L}(\mathcal{K})$ be the complex Banach space of all bounded linear operators on a complex Hilbert space $\mathcal{K}$. A *Cartan factor of type four* is a closed subspace $\mathcal{H}$ of $\mathcal{L}(\mathcal{K})$ which is invariant under the adjunction $*$ in $\mathcal{L}(\mathcal{K})$ and such that $X \in \mathcal{H}$ implies that X^2 is a scalar multiple of the identity I on $\mathcal{K}$:

$$X^2 = cI \tag{1}$$

for some $c \in \boldsymbol{C}$. The open unit ball D for the norm $|||\ |||$ of $\mathcal{H}$ is called a *Cartan domain of type four*. It is a bounded domain on which the group Aut D of all holomorphic automorphisms of D acts transitively. These facts imply that the Kobayashi and Carathéodory differential metrics on D coincide. Let Iso D be the semigroup of all holomorphic maps of D into D which are isometries for these differential metrics. The invariance properties of these metrics imply that Aut D is a subgroup of Iso D: a proper subgroup if $\dim_{\boldsymbol{C}} \mathcal{H} = \infty$.

The group Aut D was determined by U. Hirzebruch when $\mathcal{H}$ has finite dimension and by L. A. Harris [2], when $\dim_{\boldsymbol{C}} \mathcal{H} = \infty$ (cf. [4] for further details and for bibliographical references).

In this *Note* the semigroup Iso D will be determined. Since Aut D is known [2, 4] and acts transitively on D, the main thrust in the paper will be concentrated in characterizing the isotropy semigroup $(\text{Iso } D)_0$ of 0 in Iso D (Theorem I). It will be shown, incidentally, that $(\text{Iso } D)_0$ is linear, or, better to say, every element of $(\text{Iso } D)_0$ is the restriction to D of a continuous linear operator on $\mathcal{H}$. Hence H. Cartan's linearity theorem (cf. *e.g.* [1]) extends from $(\text{Aut } D)_0$ to $(\text{Iso } D)_0$ in the case of the Cartan domain D: a fact which does not hold for all Cartan domains in infinite dimensions, as examples show [3].

In the proof of Theorem I, the structure of complex discs affinely imbedded in the boundary of the Cartan domain will play a crucial rôle.

1. As a consequence of (1), for $X, Y \in \mathcal{H}$, $XY + YX$ is a scalar multiple of I. Setting, for X, Y in $\mathcal{H}$, $XY^* + Y^*X = 2(X|Y)I$ the function $X, Y \to (X|Y) \in \boldsymbol{C}$ is a posi-

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tive-definite inner product on $\mathcal{H}$, defining a norm $\| \ \|$ which is equivalent to the norm $\||\ \||$ of $\mathcal{H}$ as a subspace of $\mathcal{L}(\mathcal{H})$.

Hence the identity map of $\mathcal{H}$ onto itself is a continuous isomorphism of $\mathcal{H}$, endowed with the norm $\||\ \||$, onto $\mathcal{H}$ equipped with the Hilbert space norm $\| \ \|$.

Changing notations, denoting by $x, y, z, \dots$ the elements of $\mathcal{H}$ and by $x \to \bar{x}$ the conjugation defined by the adjunction in $\mathcal{L}(\mathcal{H})$, the Cartan domain D is expressed [2, 4] by

$$D = \{x \in \mathcal{H}: \|x\|^2 < (1 + |(x|\bar{x})|^2)/2 < 1\}. \tag{2}$$

Since D is the open unit ball for the norm $\||\ \||$, the Kobayashi differential metric at the center 0 of D coincides with $\||\ \||$. This latter norm is related to $\| \ \|$ by the formula [2, 4]:

$$\||v\||^2 = \|v\|^2 + \sqrt{\|v\|^4 - |(v|\bar{v})|^2} \qquad (v \in \mathcal{H}). \tag{3}$$

The boundary ∂D of D consists of the points of the closure $\bar{D}$ of D at which at least one of the inequalities in (2) becomes an equality. The Schwarz inequality implies that $x \in \partial D$ if, and only if, $\|x\| \leqslant 1$ and

$$\|x\|^2 = (1 + |(x|\bar{x})|^2)/2. \tag{4}$$

Proposition 1. *For any* $x \in \mathcal{H}$, *for which* (4) *holds*

$$y = \bar{x} - (\bar{x}|x)x \tag{5}$$

is – up to constant factor – the unique vector such that

$$\|x + \zeta y\|^2 = (1 + |(x + \zeta y|\overline{x + \zeta y})|^2)/2 \tag{6}$$

for all ζ *in a neighborhood of* 0 *in* $\boldsymbol{C}$.

Proof. The equality (6) is satisfied by all ζ in a neighborhood of 0 if, and only if, (4) holds together with the following conditions:

$$(y|\bar{y}) = 0, \tag{7}$$

$$\sqrt{2}\,|(x|\bar{y})| = \|y\|, \tag{8}$$

$$(x|y) = (x|\bar{x})(\bar{x}|y). \tag{9}$$

Let e_1 and e_2 be two orthonormal real vectors such that x and $\bar{x}$ are contained in the two dimensional complex subspace of $\mathcal{H}$ spanned by e_1 and e_2. Let $\{e_j\}$ be an orthonormal base of $\mathcal{H}$, whose elements are all real vectors, containing $e_1 = e_{j_1}$ and $e_2 = e_{j_2}$. There exist $\alpha_1, \alpha_2, \beta_j$ in $\boldsymbol{C}$ such that $x = \alpha_1 e_1 + \alpha_2 e_2$, $y = \sum \beta_j e_j$. Hence

$$\|x\|^2 = |\alpha_1|^2 + |\alpha_2|^2 \tag{10}$$

$$(x|\bar{x}) = \alpha_1^2 + \alpha_2^2 \tag{11}$$

and (4), (7), (8), (9) become

$$|\alpha_1|^2 + |\alpha_2|^2 = (1 + |\alpha_1^2 + \alpha_2^2|^2)/2, \tag{4'}$$

$$\sum \beta_j^2 = 0, \tag{7'}$$

$$2|\alpha_1 \beta_{j_1} + \alpha_2 \beta_{j_2}|^2 = \sum |\beta_j|^2, \qquad (\alpha_1 - \overline{\alpha_1}(\alpha_1^2 + \alpha_2^2))\,\overline{\beta_{j_1}} + (\alpha_2 - \overline{\alpha_2}(\alpha_1^2 + \alpha_2^2))\,\overline{\beta_{j_2}} = 0. \tag{8'}$$

This latter equation is equivalent to

$$\overline{\beta_{j_1}} = \lambda(\alpha_2 - \overline{\alpha_2}(\alpha_1^2 + \alpha_2^2)), \quad \overline{\beta_{j_2}} = -\lambda(\alpha_1 - \overline{\alpha_1}(\alpha_1^2 + \alpha_2^2)), \tag{12}$$

for some $\lambda \in \boldsymbol{C}$, so that

$$\alpha_1\beta_{j_1} + \alpha_2\beta_{j_2} = \overline{\lambda}(\alpha_1\overline{\alpha_2} - \overline{\alpha_1}\alpha_2), \tag{13}$$

and, by (10), (11), (4),

$$|\beta_{j_1}|^2 + |\beta_{j_2}|^2 = 2|\lambda|^2(\|x\|^2 - 1)^2. \tag{14}$$

Setting $\alpha_1 = |\alpha_1| e^{i\theta_1}$, $\alpha_2 = |\alpha_2| e^{i\theta_2}$, with $\theta_1, \theta_2 \in \boldsymbol{R}$, (4′) reads now $4|\alpha_1|^2|\alpha_2|^2 \sin^2(\theta_1 - \theta_2) = (\|x\|^2 - 1)^2$. Thus, by (13) and (14),

$$2|\alpha_1\beta_{j_1} + \alpha_2\beta_{j_2}|^2 = 8|\lambda|^2|\alpha_1|^2|\alpha_2|^2\sin^2(\theta_1 - \theta_2) = 2|\lambda|^2(\|x\|^2 - 1)^2 = |\beta_{j_1}|^2 + |\beta_{j_2}|^2,$$

and (8′) implies that y is uniquely defined, up to a constant factor $\lambda \in \boldsymbol{C}$, by

$$y = \beta_{j_1}e_1 + \beta_{j_2}e_2, \tag{15}$$

where β_{j_1} and β_{j_2} are given by (12).

Finally, by (9), (10), (11) and (4), $(y|\overline{y}) = \overline{\lambda}^2(1 - 2\|x\|^2 + |(x|\overline{x})|^2)(\overline{x}|x) = 0$, and conditions (7), (8), (9) are all satisfied.

If $\alpha_1\overline{\alpha_2} - \overline{\alpha_1}\alpha_2 \neq 0$, the linear system

$$\begin{cases} a\alpha_1 + b\overline{\alpha_1} = \overline{\alpha_2} - (\overline{x}|x)\,\alpha_2 \\ a\alpha_2 + b\overline{\alpha_2} = -\overline{\alpha_1} + (\overline{x}|x)\,\alpha_1 \end{cases}$$

has the unique solution $a = ((1 - |(x|\overline{x})|^2)/2(\alpha_1\overline{\alpha_2} - \overline{\alpha_1}\alpha_2))(\overline{x}, x)$, $b = (|(x|\overline{x})|^2 - 1)/2(\alpha_1\overline{\alpha_2} - \overline{\alpha_1}\alpha_2)$, and the vector (15) is collinear to the vector y expressed by (5). If $\alpha_1\overline{\alpha_2} - \overline{\alpha_1}\alpha_2 = 0$, then $\theta_2 = \theta_1 + k\pi$ for some $k \in \boldsymbol{Z}$. Hence $(x|\overline{x}) = \alpha_1^2 + \alpha_2^2 = (|\alpha_1|^2 + |\alpha_2|^2)e^{2i\theta_1} = \|x\|^2 e^{2i\theta_1}$.

Thus, by (4), $|(x|\overline{x})| = \|x\| = 1$, and since $\alpha_1 - \overline{\alpha_1}(x|\overline{x}) = 0$, $\alpha_2 - \overline{\alpha_2}(x|\overline{x}) = 0$ then the vectors expressed by (5) and (15) both vanish.

That completes the proof of Proposition 1.

Note that, by (4),

$$\|\overline{x} - (\overline{x}|x)x\|^2 = ((1 - |(x|\overline{x})|^2)^2)/2. \tag{16}$$

2. Going back to the Cartan domain D defined by (2), Proposition 1 yields

COROLLARY 2. *For any $x \in \partial D$ and any $\varepsilon > 0$ there exists a unique $y \in \mathcal{H}$ (defined up to a suitable constant factor) such that $x + \zeta y \in \partial D$ for all $\zeta \in \boldsymbol{C}$ with $|\zeta| < \varepsilon$. The vector y is expressed by* (5).

By (16), $y = 0$ if, and only if, $|(x|\overline{x})| = 1$, *i.e.*, by (4), $\|x\| = 1$. This provides a new proof of the fact, established by L. A. Harris in [2] using a different argument, whereby the complex extreme points of the closure $\overline{D}$ of D are those points of ∂D such that $\|x\| = 1$ or, equivalently, $|(x|\overline{x})| = 1$.

Let $A \in \mathcal{L}(\mathcal{H})$ be a linear isometry for the norm $|||\ |||$. Since D is the open unit ball for $|||\ |||$, then $AD \subset D$ and $A(\partial D) \subset \partial D$. For any $z \in \mathcal{H}\setminus\{0\}$, there is a unique $t > 0$ such that $x = tz \in \partial D$. Let $\varepsilon > 0$, and let y be the vector defined (up to a suitable constant factor) by Corollary 2. Since $A(x + \zeta y) = Ax + \zeta Ay \in \partial D$ whenewer $|\zeta| < \varepsilon$, then by Corollary 2 and by (5), Ay is proportional to the vector $\overline{Ax} - (\overline{Ax}\,|Ax)Ax$.

Since A is an isometry for $|||\ |||$, then $|||Ay||| = |||y|||$. But, being $(y\,|\,\bar{y}) = 0$ and $(Ay\,|\,\overline{Ay}) = 0$, (3) yields $|||y|||^2 = 2\|y\|^2$, $|||Ay|||^2 = 2\|Ay\|^2$, and therefore $\|Ay\| = \|y\|$, *i.e.*, by (16) $|(Ax\,|\,\overline{Ax})| = |(x\,|\,\bar{x})|$ or, equivalently,

$$|(Az\,|\,\overline{Az})| = |(z\,|\,\bar{z})| \quad \text{for all } z \in \mathcal{H}. \tag{17}$$

As a consequence, if $(z\,|\,\bar{z}) = 0$, then $(Az\,|\,\overline{Az}) = 0$.

LEMMA 3. *Let A be a bounded linear operator in a complex Hilbert space $\mathcal{H}$ endowed with a conjugation $z \to \bar{z}$. If $(z\,|\,\bar{z}) = 0$ implies that $(Az\,|\,\overline{Az}) = 0$, there exists $\alpha \in C$ such that ${}^tAA = \alpha I$.*

Here the transposed operator ${}^tA$ is defined by $\overline{{}^tAv} = A^*\bar{v}$.

PROOF. Let $v \in \mathcal{H}$ be such that $(v\,|\,\bar{v}) \neq 0$. For $u \in \mathcal{H}$ and $\zeta \in C$

$$(\zeta u + v\,|\,\overline{\zeta u + v}) = \zeta^2(u\,|\,\bar{u}) + 2\zeta(u\,|\,\bar{v}) + (v\,|\,\bar{v}), \tag{18}$$

$$(A(\zeta u + v)\,|\,\overline{A(\zeta u + v)}) = \zeta^2(Au\,|\,\overline{Au}) + 2\zeta(Au\,|\,\overline{Av}) + (Av\,|\,\overline{Av}),$$

and whenever $(\zeta u + v\,|\,\overline{\zeta u + v}) = 0$, then $(A(\zeta u + v)\,|\,\overline{A(\zeta u + v)}) = 0$.

Let $\mathcal{S}$ be the dense set in $\mathcal{H}$ consisting of those points u for which the roots of the polynomial (18) are distinct, *i.e.* $\mathcal{S} = \{u \in \mathcal{H}\colon (u\,|\,\bar{v})^2 \neq (u\,|\,\bar{u})(v\,|\,\bar{v})\}$. For every $u \in \mathcal{S}$ there exists $\alpha \in C$ such that

$$(Au\,|\,\overline{Au}) = \alpha(u\,|\,\bar{u}), \quad (Au\,|\,\overline{Av}) = \alpha(u\,|\,\bar{v}), \quad (Av\,|\,\overline{Av}) = \alpha(v\,|\,\bar{v}).$$

The first and third conditions imply that α is independent of $u \in \mathcal{S}$ and v. Since $\mathcal{S}$ is dense, these conditions are satisfied by all u and v in $\mathcal{H}$. The second equation reads then ${}^tAA = \alpha I$. QED.

Going back to the isometry A, this Lemma implies the existence of $\alpha \in C$ such that ${}^tAA = \alpha I$. By (17), $|\alpha| = 1$ and thus, by (4), $\|Ax\|^2 = (1 + |(Ax\,|\,\overline{Ax})|^2)/2 = (1 + |(x\,|\,\bar{x})|^2)/2 = \|x\|^2$ for all $x \in \partial D$, and therefore also for all $x \in \mathcal{H}$, proving thereby that A is a linear isometry for the norm $\|\ \|$. Choosing a square root of α, the operator $A' = A/\sqrt{\alpha}$ is a linear isometry for both the norms $\|\ \|$ and $|||\ |||$, for which

$${}^tA'A' = I. \tag{19}$$

The operators $A_1' = (A' + \overline{A'})/2$, $A_2' = (A' - \overline{A'})/2i$ are real, and, for any real vector $v \in \mathcal{H}$, $(A'v\,|\,\overline{A'v}) = \|A_1'v\|^2 - \|A_2'v\|^2 + 2i(A_1'v\,|A_2'v)$, $\|A'v\|^2 = \|A_1'v\|^2 + \|A_2'v\|^2$.

Since, by (19), $(A'v\,|\,\overline{A'v}) = ({}^tA'A'v\,|v) = \|v\|^2 = \|A'v\|^2$, then $A_2'v = 0$ for all real $v \in \mathcal{H}$, and therefore $A_2' = 0$, proving thereby that A' is a linear real isometry of the complex Hilbert space $\mathcal{H}$.

Viceversa, if A' is a linear real isometry of the Hilbert space $\mathcal{H}$, then, by (19) and

(3), $|||A'z|||^2 = \|A'z\|^2 + \sqrt{\|A'z\|^4 - |(A'z\,|\,\overline{A'z})|^2} = \|z\|^2 + \sqrt{\|z\|^4 - |(z\,|\,\bar{z})|^2} =$ $= |||z|||^2$ for all $z \in \mathcal{H}$.

In conclusion, the following Theorem has been established.

THEOREM I. *Every linear real isometry of the complex Hilbert space $\mathcal{H}$ is an isometry for the norm $|||\ |||$. Viceversa, if A is a linear isometry for $|||\ |||$, there exists a constant $\alpha \in \boldsymbol{C}$, with $|\alpha| = 1$, such that αA is a linear real isometry of the complex Hilbert space $\mathcal{H}$.*

This Theorem was established by L. A. Harris in [2] using a different argument, under the additional hypothesis that A be invertible in $\mathcal{L}(\mathcal{H})$.

REMARK. Strictly similar considerations to those developed in n. 2 yield a generalization of Theorem I to the case in which A is a bounded linear map of a complex Hilbert space $\mathcal{H}$ into a complex Hilbert space $\mathcal{H}'$, and both $\mathcal{H}$ and $\mathcal{H}'$ are endowed with conjugations and with norms $|||\ |||$ defined by (3) in terms of their respective Hilbert space norms:

If A is a real $\|\ \|$-isometry, then A is a $|||\ |||$-isometry. Viceversa, if A is a $|||\ |||$-isometry, there is $\alpha \in \boldsymbol{C}$, with $|\alpha| = 1$, such that αA is a real isometry.

3. In this Section, H. Cartan's linearity theorem will be shown to hold for holomorphic isometries of Cartan domains of type four (1).

Let $h \in \operatorname{Iso} D$ be such that $h(0) = 0$. The differential of h at $0, A = dh(0) \in \mathcal{L}(\mathcal{H})$, is a linear isometry for the Kobayashi differential metric at 0. By Theorem I there is some $\varphi \in \boldsymbol{R}$ such that $e^{i\varphi}A$ is a real linear isometry in the complex Hilbert space $\mathcal{H}$. Thus there is a holomorphic function $l\colon D \to \mathcal{H}$ – expressed in D by the normally convergent power series expansion $l(x) = P_2(x) + P_3(x) + \ldots$, where $P_n\colon \mathcal{H} \to \mathcal{H}$ is a continuous homogeneous polynomial of degree $n = 2, 3, \ldots$ – such that $h(\zeta x) = \zeta A x + l(\zeta x)$ for all $x \in D$ and all $\zeta \in \boldsymbol{C}$ with $|\zeta| \leq 1$.

Hence

$$\|h(\zeta x)\|^2 = |\zeta|^2 \|Ax\|^2 + 2\operatorname{Re}(\zeta (Ax\,|\,l(\zeta x))) + \|l(\zeta x)\|^2 ,$$

where

$$(Ax\,|\,l(\zeta x)) = \bar{\zeta}^2 (Ax\,|\,P_2(x)) + \bar{\zeta}^3 (Ax\,|\,P_3(x)) + \ldots,$$

$$\|l(\zeta x)\|^2 = |\zeta|^4 \{\|P_2(x)\|^2 + 2\operatorname{Re}(\bar{\zeta}(P_2(x)\,|\,P_3(x))) + |\zeta|^2 \|P_3(x)\|^2 + \ldots\}.$$

Setting $\zeta = \rho e^{i\theta}$ with $\rho \geq 0$ and $\theta \in \boldsymbol{R}$, integration with respect to $(1/2\pi)\, d\theta$ from 0 to

(1) An example constructed in [2] shows that Cartan's theorem does not hold for all Cartan domains.

2π yields

$$\frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\rho e^{i\theta}\left(Ax \mid l(\rho e^{i\theta}x)\right)\right) d\theta = 0\,, \tag{20}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \|l(\rho e^{i\theta}x)\|^2 d\theta = \|P_2(x)\|^2 + \|P_3(x)\|^2 + \dots \tag{21}$$

for all $x \in D$.

Since D is the unit ball for the norm $|||\ |||$, the Kobayashi distance of $x \in D$ from 0 is given by $\omega(0, |||x|||)$ where ω is the Poincaré distance in the open unit disc of $\boldsymbol{C}$ (cf. *e.g.* [1]). Hence

$$|||f(\zeta x)||| = |||\zeta x||| \tag{22}$$

for all $x \in D$ and all $\zeta \in \boldsymbol{C}$ with $|\zeta| \leqslant 1$. Since $e^{i\varphi}A$ is a linear real isometry for $\|\ \|$, then (3) yields, for all real $x \in \mathcal{H}$, $|||\zeta x|||^2 = \|\zeta x\|^2$, $|||A\zeta x|||^2 = \|A\zeta x\|^2 = \|\zeta x\|^2$, and (22) gives

$$2\operatorname{Re}\left(\zeta(Ax \mid l(\zeta x))\right) + \|l(\zeta x)\|^2 + \sqrt{\|h(\zeta x)\|^4 - |(h(\zeta x) \mid \overline{h(\zeta x)})|^2} = 0\,.$$

Since the last summand is non-negative, (20) and (21) imply that

$$P_2(x) = P_3(x) = \dots = 0 \tag{23}$$

for all real $x \in D$ and therefore all real $x \in \mathcal{H}$. Let $\mathcal{H}''$ be the closed linear real subspace of $\mathcal{H}$ consisting of all real vectors.

LEMMA 4. *Let g be a holomorphic map of a domain $D \subset \mathcal{H}$ into a complex Banach space $\mathcal{E}$. If $D \cap \mathcal{H}'' \neq \emptyset$ and if $g = 0$ on $D \cap \mathcal{H}''$, then $g = 0$ on D.*

This Lemma, well known for scalar valued holomorphic functions, extends trivially to the case of $g\colon D \to \mathcal{E}$ as a consequence of the Hahn-Banach theorem.

Thus (23) implies that $P_2 = P_3 = \dots = 0$, and in conclusion the following theorem holds.

THEOREM II. *If $h \in \operatorname{Iso} D$ is such that $h(0) = 0$, there exist a linear real isometry A of the Hilbert space $\mathcal{H}$ and $\varphi \in \boldsymbol{R}$, such that h is the restriction to D of the linear operator $e^{i\varphi}A$.*

This Theorem provides a complete description of $\operatorname{Iso} D$, as will be shown now.

4. Assuming in $\boldsymbol{C}^2$ the canonical conjugation, the Hilbert space direct sum $\mathcal{H} \oplus \boldsymbol{C}^2$ is endowed with a conjugation leaving $\mathcal{H}$ and $\boldsymbol{C}^2$ invariant, and whose restrictions to $\mathcal{H}$ and to $\boldsymbol{C}^2$ coincide with the given ones.

Let $J \in \mathcal{L}(\mathcal{H} \oplus C^2)$ be the operator defined by the matrix

$$J = \begin{pmatrix} I & 0 \\ 0 & -I_2 \end{pmatrix},$$

where I and I_2 stand for the identity operators in $\mathcal{H}$ and in C^2. For $M \in \mathcal{L}(\mathcal{H})$ and $G \in \mathcal{L}(\mathcal{H} \oplus C^2)$, ${}^tM \in \mathcal{L}(\mathcal{H})$ and ${}^tG \in \mathcal{L}(\mathcal{H} \oplus C^2)$ will denote the transposed operators in $\mathcal{H}$ and in $\mathcal{H} \oplus C^2$ respectively.

Let Λ be the semigroup consisting of all linear, real, continuous operators G: $\mathcal{H} \oplus C^2 \to \mathcal{H} \oplus C^2$ such that ${}^tGJG = J$, and let Γ be the maximum subgroup of Λ consisting of all $G \in \Lambda$ which are invertible in $\mathcal{L}(\mathcal{H} \oplus C^2)$. Every $G \in \Lambda$ is represented by a matrix

$$G = \begin{pmatrix} M & B_1 & B_2 \\ (\cdot|C_1) & E_{11} & E_{12} \\ (\cdot|C_2) & E_{21} & E_{22} \end{pmatrix}$$

where $M \in \mathcal{L}(\mathcal{H})$ is a real operator, B_1, B_2, C_1, C_2 are real vectors in $\mathcal{H}$, and $E_{11}, E_{12}, E_{21}, E_{22}$ are real scalars.

As was shown in [4], the set $\Lambda_0 = \{G \in \Lambda\colon E_{11}E_{22} - E_{12}E_{21} > 0\}$ is a subsemigroup of Λ_0. Therefore $\Gamma_0 := \Lambda_0 \cap \Gamma$ is a subgroup of Γ.

For $G \in \mathcal{L}(\mathcal{H} \oplus C^2)$, let δG be the continuous quadratic polynomial $\mathcal{H} \to C$ defined on $z \in \mathcal{H}$ by $\delta G(z) = 2(z|C_1 - C_2) + (E_{11} - E_{22} + i(E_{12} + E_{21}))(z|\bar{z}) + E_{11} + E_{22} + i(E_{21} - E_{12})$.

If $G \in \Lambda_0$, there is a neighborhood V of $\overline{D}$ such that $\delta G(z) \neq 0$ for all $z \in V$ [4]. Let $\widehat{G}$ be the holomorphic map of D into $\mathcal{H}$ defined on $x \in D$ by $\widehat{G}(x) = (2Mx + (1 + (x|\bar{x}))B_1 - i(1 - (x|\bar{x}))B_2)/\delta G(x)$.

As was shown in [4], $\widehat{G}(D) \subset D$ and $\widehat{G} \in \operatorname{Iso} D$ for all $G \in \Lambda_0$. The map $G \to \widehat{G}$ defines a homomorphism ϕ: $\Lambda_0 \to \operatorname{Iso} D$.

THEOREM III. *The homomorphism* ϕ *maps* Λ_0 *surjectively onto* $\operatorname{Iso} D$.

PROOF. Let $f \in \operatorname{Iso} D$ and let $x_0 = f(0)$. Since $\phi(\Gamma_0)$ acts transitively on D [4], there is some $G \in \Gamma_0$ for which $\widehat{G}(0) = x_0$. The map $h = \widehat{G}^{-1} \circ f \in \operatorname{Iso} D$ fixes 0. By Theorem II there exist a linear, real isometry A of $\mathcal{H}$ and $\theta \in R$, such that $h(x) = e^{i\theta}Ax$, for all $x \in D$.

Let $H \in \Lambda_0$ be defined by

$$H = \begin{pmatrix} A & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Then $\delta H(x) = 2e^{-i\theta}x$, and therefore $\widehat{H}(x) = e^{i\theta}Ax = h(x)$, for all $x \in D$.

Then $f = \widehat{G} \circ \widehat{H} = \widehat{H \circ G}$. QED

The kernel of the homomorphism $\Lambda_0 \to \operatorname{Iso} D$ consists of $\pm$ the identity operator on $\mathcal{H} \oplus C^2$. Theorem III implies that $\phi(\Gamma_0) = \operatorname{Aut} D$ as was shown in [4].

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