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Variational inequalities and rearrangements

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Equazioni a derivate parziali. — *Variational inequalities and rearrangements.* Nota di Angelo Alvino, Silvano Matarasso e Guido Trombetti, presentata (*) dal Socio E. Magenes.

Abstract. — We give comparison results for solutions of variational inequalities, related to general elliptic second order operators, involving solutions of symmetrized problems, using Schwarz spherical symmetrization.

Key words: Schwarz symmetrization; Comparison results; Variational elliptic inequalities.

Riassunto. — *Disequazioni variazionali e riordinamenti*. Si danno risultati di confronto per soluzioni di disequazioni variazionali, relative ad operatori ellittici del secondo ordine, riconducendosi a un problema a simmetria radiale con l'ausilio della simmetrizzazione di Schwarz.

1. Introduction

Let A be a second order differential operator defined by

$$Au = -(a_{ij}(x)\,u_{x_i})_{x_j} + (b_i(x)\,u)_{x_i} + d_i(x)\,u_{x_i} + c_0(x)\,u\,,$$

where we use the standard convention on repeated subscripts. The coefficients belong to $L^\infty(\Omega)$ ($\Omega =$ open bounded subset of $\mathbf{R}^N$) and satisfy the following conditions:

(1) $$a_{ij}(x)\,\xi_i\,\xi_j \geq |\xi|^2 \qquad \forall \xi \in \mathbf{R}^N\,,$$

(2) $$\sum_i |b_i(x) + d_i(x)|^2 \leq R^2 \qquad R \geq 0\,,$$

(3) $$(b_i(x))_{x_i} + c_0(x) \geq c(x) \qquad \text{on} \qquad \mathcal{D}'(\Omega)\,,\quad c(x) \in L^\infty(\Omega)\,.$$

Let $u \in H_0^1(\Omega)$ a solution of the variational inequality

(4) $$a(u, v-u) \geq \int_\Omega f(v-u) \qquad \forall v \in H_0^1(\Omega),\quad u, v \geq 0\,,$$

where $f \in L^2(\Omega)$ and $a(.,.)$ is the bilinear form

$$a(\phi, \psi) = \int_\Omega a_{ij}\,\phi_{x_i}\,\psi_{x_j} - \int_\Omega b_i\,\phi\psi_{x_i} + \int_\Omega d_i\,\phi_{x_i}\,\psi + \int_\Omega c_0\,\phi\psi\,.$$

Besides we consider the following symmetrized operator ($\Delta =$ laplacian)

$$A^{\#}U = -\Delta U + R|x|^{-1}x_i\,U_{x_i} + c_{\#}(x)\,U$$

(*) Nella seduta del 9 maggio 1992.

and the related variational inequality ($\forall V \in H_0^1(\Omega^\#)$, $U, V \geq 0$)

$$
(5) \qquad a^\#(U, V-U) = \\
= \int_{\Omega^\#} \left\{ U_{x_i}(V-U)_{x_i} + \frac{R}{|x|} x_i U_{x_i}(V-U) + c_\# U(V-U) \right\} \geq \int_{\Omega^\#} f^\#(V-U),
$$

where $\Omega^\#$ is the ball of $\mathbf{R}^N$ centered in O, such that $|\Omega^\#| = |\Omega| = \text{meas}\,\Omega, f^\#(x)$ is the spherical decreasing rearrangement of f, $c_\#(x)$ is the spherical increasing rearrangement of c.

We assume that (5) has an unique spherical decreasing solution U; then we can «compare» the solution u of (4) with the solution U of the simpler problem (5). To be more specific, if u^*, U^* denote the decreasing rearrangements of u, U respectively, we get in particular

$$
(6) \qquad \int_0^s \exp(-RC_N^{-1/N}\sigma^{1/N})\, u^*(\sigma)\, d\sigma \leq \int_0^s \exp(-RC_N^{-1/N}\sigma^{1/N})\, U^*(\sigma)\, d\sigma, \quad s \in [0, |\Omega|],
$$

where C_N is the measure of the unit N-ball. This comparison result provides optimal bounds for norms of the solution u of variational inequality (4) in terms of similar norms of the solution U of problem (5); in particular (6) implies

$$
\sup_{x \in \Omega} u(x) = u^*(0) \leq U^*(0) = \sup_{x \in \Omega^\#} U(x).
$$

We point out that, if $R = 0$, (6) becomes

$$
(7) \qquad \int_0^s u^* \leq \int_0^s U^*, \qquad \forall s \in [0, |\Omega|],
$$

that implies (see [2] for example)

$$
\int_\Omega F(u) \leq \int_{\Omega^\#} F(U)
$$

for all convex, nonnegative, Lipschitz continuous function F such that $F(0) = 0$. Moreover the stronger inequality

$$
(8) \qquad u^*(s) \leq U^*(s),
$$

can be established when $0 \leq s \leq |\{x \in \Omega\colon c(x) \leq 0\}|$. Therefore if $c \leq 0$, from (8) we can derive an optimal lower bound for the coincidence set of u

$$
(9) \qquad |u = 0| \geq |U = 0|;
$$

if $c = 0$, $|U = 0|$ can be evaluated: $|U = 0| = |\Omega| - \bar{s}$, where $\bar{s}$ is the unique solution of the equation in s

$$
\int_0^s \exp(-RC_N^{-1/N}\sigma^{1/N}) f^*(\sigma)\, d\sigma = 0.
$$

As usual the procedure for obtaining comparison results as (6), (7) or (8) can be split into two steps. At first, integrating on the level sets of the solution u to (4) an ordinary integro-differential inequality satisfied by the rearrangement u^* of u rises. The principal tools we use at this stage are the isoperimetric inequality [12], a coarea formula [16], Hardy inequality on rearrangements and so on. Then we handle the integro-differential inequality in such a way to derive, via a maximum principle, the desired estimations.

This method was firstly developed by Talenti [25] who studied an elliptic equation without lower order terms; afterwords the method has been fitted to more general cases: see Alvino-Trombetti [5-7], Bandle [8], Chiti [11], P. L. Lions [22], Talenti [26], Alvino-Lions-Trombetti [1, 3, 4], Ferone-Posteraro [15], Giarrusso-Trombetti [17], Trombetti-Vasquez [27].

Finally we mention the papers of Bandle-Mossino [9], Maderna-Salsa [23] who earlier established comparison results for solutions to variational inequalities. For variational parabolic inequalities see also Diaz-Mossino [14].

2. Preliminary results

If $\phi \in L^1(\Omega)$ we write $|\phi > t| := |\{x \in \Omega\colon \phi(x) > t\}|$, $t \in \mathbf{R}$; then we set $\mu_\phi(t) = |\phi > t|$ $t \in \mathbf{R}$ (distribution function of ϕ), $\phi^*(s) = \sup\{t \mid \mu_\phi(t) \geq s\}$ $s \in [0, |\Omega|]$ (decreasing rearrangement of ϕ), $\phi_*(s) = \phi^*(|\Omega| - s)$ $s \in [0, |\Omega|]$ (increasing rearrangement of ϕ), $\phi^\#(x) = \phi^*(C_N |x|^N)$, $x \in \Omega^\#$ (spherical symmetric decreasing rearrangement of ϕ), $\phi_\#(x) = \phi_*(C_N |x|^N)$, $x \in \Omega^\#$ (spherical symmetric increasing rearrangement of ϕ). If $\phi = \phi^+ - \phi^-$, where ϕ^+, ϕ^- are the positive and negative part of ϕ, we have $\phi^* = \phi^{+*} - \phi^-_*$ and $\phi_* = \phi^+_* - \phi^{-*}$. The distribution function $\mu_\phi(t)$ maps the interval $]\text{ess.inf}\,\phi, \text{ess.sup}\,\phi[$ into $]0, |\Omega|[$. If μ_ϕ is strictly decreasing and continuous, ϕ^* is the inverse function of μ_ϕ; generally ϕ^* is the smallest decreasing function from $[0, |\Omega|]$ such that $\phi^*(\mu_\phi(t)) \geq t$ for every $t \in \mathbf{R}$. A basic property of ϕ^*, as well as of any other type of rearrangement, is that ϕ^* and ϕ have the same distribution function. Consequently

$$\int_\Omega F(\phi) = -\int_{-\infty}^{+\infty} F(t)\, d\mu_\phi(t) = \int_{\Omega^\#} F(\phi^\#) = \int_0^{|\Omega|} F(\phi^*),$$

whenever F is non negative and convex; in particular we have

$$\|u\|_{L^p} = \|u^\#\|_{L^p} = \|u^*\|_{L^p} \qquad \forall p \in [1, +\infty].$$

For the main properties of rearrangements we refer to [2, 8, 13, 18, 19, 24, 26]. We just recall two results we will employ later on.

Lemma 2.1 (see [18]). *If f, g are measurable functions on Ω, then*

$$\int_0^{|\Omega|} f^* g_* \leq \int_\Omega fg \leq \int_0^{|\Omega|} f^* g^*. \tag{10}$$

(10) is known as Hardy inequality.

LEMMA 2.2 *If*

$$\int_a^s \phi \leq \int_a^s \psi, \qquad \text{on} \quad [a, b]$$

and $h \geq 0$ is a decreasing function on $[a, b]$ then

$$\int_a^s \phi h \leq \int_a^s \psi h, \qquad \text{on} \quad [a, b].$$

For simplicity we use the following notations

$$p(s) := (NC_N^{1/N})^{-2} s^{2/N-2}, \qquad e(s) := \exp(RC_N^{-1/N} s^{1/N}), \qquad \beta(s) = e(s)p(s),$$

$$F_u(s) = \int_0^s e^{-1}(f^* - c_* u^*).$$

The following Theorem provides the basic inequality in the subsequent developments.

THEOREM 2.1. *Let u be solution of* (4) *; then (a.e. on* $[0, |u > 0|]$)

$$-du^*/ds \leq \beta(s) F_u(s). \tag{11}$$

For the proof we could refer to [3]; however for the sake of completeness we give a sketch of it. Consider the functions

$$\phi_h(x) := \begin{cases} h & t + h < u(x), \\ u(x) - t & t < u(x) \leq t + h, \\ 0 & u(x) \leq t, \end{cases}$$

with $h \geq 0$, $t \in]0, \sup u[$. We have $u \pm \phi_h \geq 0$ so we can replace the test function v in (4) by the functions $u \pm \phi_h$; we obtain

$$\frac{1}{h} a(u, \phi_h) = \frac{1}{h} \int_\Omega f\phi_h .$$

By ellipticity condition (1), letting h go to zero, we get

$$-\frac{d}{dt} \int_{u>t} |\nabla u|^2 \leq -\frac{d}{dt} \int_{u>t} b_j u_{x_j} u + \int_{u>t} b_j u_{x_j} - \tag{12}$$

$$- \int_{u>t} c_0 u - \int_{u>t} (b_j + d_j) u_{x_j} + \int_{u>t} f.$$

Since (see [3])

$$-\frac{d}{dt} \int_{u>t} b_j u_{x_j} u \leq t \int_{u>t} (c_0 - c),$$

setting $\phi(x) = \max\{u(x) - t, 0\}$, by and (3) and Hardy inequality (10), we get

$$(13)\qquad -\frac{d}{dt}\int_{u>t} b_j u_{x_j} u + \int_{u>t} b_j u_{x_j} - \int_{u>t} c_0 u \leqslant$$

$$\leqslant \int_{\Omega} (b_j \phi_{x_j} - c_0 \phi + c\phi) - \int_{u>t} cu \leqslant -\int_{u>t} cu \leqslant -\int_0^{\mu(t)} c_* u^*$$

where $\mu(t) = \mu_u(t)$ is the distribution function of u. Moreover

$$(14)\qquad NC_N^{1/N}\mu(t)^{1-1/N} \leqslant (-\mu'(t))^{1/2}\left(-\frac{d}{dt}\int_{u>t} |\nabla u|^2\right)^{1/2},$$

$$(15)\qquad \left|\int_{u>t} (b_j + d_j) u_{x_j}\right| \leqslant \frac{R}{NC_N^{1/N}} \int_t^{+\infty} \mu(s)^{-1+1/N}(-\mu'(s))\left(-\frac{d}{ds}\int_{u>s} |\nabla u|^2\right),$$

$$(16)\qquad \int_{u>t} f \leqslant \int_0^{\mu(t)} f^*;$$

(14) is a consequence of the isoperimetric inequality [12], Fleming-Risheal coarea formula [16], Schwarz inequality (see [25] for a complete proof); (16) can be easily deduced from (10); for (15) we refer to [3].

From (12), (13), (15), (16), we obtain

$$-\frac{d}{dt}\int_{u>t} |\nabla u|^2 \leqslant$$

$$\leqslant \frac{R}{NC_N^{1/N}} \int_t^{+\infty} \mu(s)^{-1+1/N}(-\mu'(s))\left(-\frac{d}{ds}\int_{u>t} |\nabla u|^2\right) + \int_0^{\mu(t)} (f^* - c_* u^*).$$

By Gronwall lemma

$$-\frac{d}{dt}\int_{u>t} |\nabla u|^2 \leqslant e(\mu(t)) \int_t^{+\infty} e^{-1}(\mu)\{f^*(\mu) - c_*(\mu) u^*(\mu)\}(-\mu');$$

hence, from (14) we get

$$(17)\qquad [-\mu'(t)]^{-1} \leqslant \beta(\mu(t)) F_u(\mu(t)).$$

By standard arguments (see [26] for instance), (17) can be written in terms of the «inverse» function u^* of μ; then we get (11).

Remark 2.1. *From* (11) *it follows*

$$(18)\qquad F_u \geqslant 0 \quad \text{on} \quad [0, |u > 0|].$$

When does (11) become an equality? A close analysis of the proof shows it happens when $A = A^\#$ and the problem (5) has a spherical decreasing solution $U = U^\#$.

This circumstance is linked to spectral properties of the operator $A^{\#}$: we suppose (see Proposition 2.1) the operator $A^{\#}$ to satisfy conditions that imply the following property

$$A^{\#} V \geqslant 0, \qquad V \geqslant 0 \quad \text{on } \partial\Omega^{\#} \Rightarrow V \geqslant 0 \quad \text{on } \Omega^{\#} . \tag{19}$$

PROPOSITION 2.1. *Let us assume $c \geqslant 0$ or, if $c^- \not\equiv 0$, let any one of the following equivalent three conditions be satisfied*

i) there exists a non negative function $H \not\equiv 0$ such that the Dirichlet problem

$$A^{\#} Z = H, \qquad Z \in H_0^1(\Omega^{\#}),$$

has a non negative solution Z;

ii) the first eigenvalue λ_1 of the problem

$$A^{\#} \varphi = \lambda c^{-\#} \varphi, \qquad \varphi \in H_0^1(\Omega^{\#}), \tag{20}$$

is positive;

iii) there exists $\alpha > 0$ such that

$$\int_{\Omega^{\#}} e^{-R|x|} (|\nabla\varphi|^2 + c_{\#} \varphi^2) \geqslant \alpha \int_{\Omega^{\#}} e^{-R|x|} |\nabla\varphi|^2, \qquad \forall\varphi \in H_0^1(\Omega^{\#}). \tag{21}$$

Then the problem (5) *has a unique solution $U = U^{\#}$. Moreover property* (19) *holds.*

REMARK 2.2. *We observe that*

$$A^{\#} Z = H \Leftrightarrow -(e^{-R|x|} Z_{x_i})_{x_i} + c_{\#} e^{-R|x|} Z = e^{-R|x|} H;$$

hence the first eigenvalue λ_1 of the problem (20) *is real and*

$$\lambda_1 = \min_{\phi \in H_0^1} \frac{\displaystyle\int_{\Omega^{\#}} e^{-R|x|} (|\nabla\phi|^2 + c_{\#} \phi^2)\, dx}{\displaystyle\int_{\Omega^{\#}} e^{-R|x|} c^{-\#} \phi^2\, dx}. \tag{22}$$

Proposition 2.1 is proved in Appendix; now we point out that the above arguments yield the following result:

THEOREM 2.2. *If one of the conditions in Proposition* 2.1 *is verified, then*

$$-dU^*/ds = \beta(s) F_U(s), \quad \text{a.e.} \quad \text{on} \quad [0, |U > 0|] \tag{23}$$

where $U(= U^{\#})$ is the unique solution of (5).

REMARK 2.3. *From regularity results* (see [10] for instance), *the solution U of* (5) *belongs to $H^2(\Omega^{\#})$ and then $U^* \in \mathcal{C}^1(]0, |\Omega|])$. Consequently if $|U > 0| < |\Omega|$ then $U^{*\prime}(|U > 0|) = 0$ and from* (23) *it follows*

$$F_U(|U > 0|) = 0. \tag{24}$$

REMARK 2.4. *We have*

$$f^*(s) < 0 \quad \text{on } [|U > 0|, |\Omega|]. \tag{25}$$

Namely if $|U>0| \leqslant \inf\{s: c_*(s)>0\} = s_1$, *then from* (24)

$$\int_0^{|U>0|} e^{-1} f^* = \int_0^{|U>0|} e^{-1} c_* U^* \leqslant 0;$$

so f^* *cannot be non negative on* $[0, |U>0|]$. *If* $|U>0| > s_1$, *from* (23) *we have*

$$\begin{cases} -(\beta^{-1} U^{*\prime})' + e^{-1} c_* U^* = e^{-1} f^* & \text{on }]s_1, |U>0|[, \\ U^*(|U>0|) = U^{*\prime}(|U>0|) = 0, \end{cases}$$

while $e^{-1}c_* > 0$ *and* $U^* > 0$ *on* $]s_1, |U>0|[$; *by maximum principle* f^* *cannot be non negative on* $]s_1, |U>0|[$. *Then we get* (25).

REMARK 2.5. *If* $c=0$, *from the Remark* 2.3, *either* $|U>0| = |\Omega|$ *or* $|U>0|$ *is the unique solution of the equation in* s

$$F(s) = \int_0^s e^{-1} f^* = 0.$$

If u *is solution of* (4), *by* (18), $F(|u>0|) \geqslant 0$, *and then*

$$|u>0| \leqslant |U>0|, \tag{26}$$

from which the optimal bound (9) *for the coincidence set of* u.

With Theorem 2.1 and Theorem 2.2 as a starting point we can establish the following crucial inequality.

LEMMA 2.3. *Let* u, U *solutions of* (4), (5), *respectively; assume that one of the conditions in Proposition* 2.1 *is fulfilled; then, if* $w = u^* - U^*$,

$$w' - \beta(s)\int_0^s e^{-1} c_* w \geqslant 0, \quad \text{a.e. on } [0, |u>0|]. \tag{27}$$

PROOF. From (11) and (23) it follows

$$w' - \beta(s)\int_0^s e^{-1} c_* w \geqslant 0, \quad 0 \leqslant s \leqslant M, \tag{28}$$

where $M = \min\{|u>0|, |U>0|\}$. If $|u>0| \leqslant |U>0|$ (27) is trivial. Otherwise we set

$$\tilde{f}(s) := \begin{cases} f^*(s), & 0 \leqslant s \leqslant |U>0|, \\ 0, & |U>0| < s \leqslant |u>0|; \end{cases}$$

by virtue of (24), (23) becomes

$$-\frac{dU^*}{ds} = \beta(s)\int_0^s e^{-1}[\tilde{f} - c_* U^*] \quad \text{a.e. on } [0, |u>0|].$$

Hence by (11)

$$w' - \beta(s) \int_0^s e^{-1} c_* w \geqslant \begin{cases} 0 & 0 \leqslant s \leqslant |U > 0|, \\ -\beta(s) \displaystyle\int_{|U>0|}^{s} e^{-1} f^* & |U > 0| < s \leqslant |u > 0|. \end{cases}$$

From Remark 2.4 we have (27).

3. Comparison results

As pointed out in the introduction the kind of comparison results we can derive from inequality (27) depends on the sign of the function $c(x)$. At first we consider the cases $c \equiv 0$, $c \geqslant 0$, $c \leqslant 0$; although these cases fall within a more general one (see Theorem 3.4), we prefer to give direct, simpler proofs.

When $c \equiv 0$ we have

THEOREM 3.1. *Let u be a solution of* (4) *where the coefficients of operator A satisfy* (1), (2) *and* (3) *with $c \equiv 0$. Then we have*

$$u^* \leqslant U^* \quad \text{on} \quad [0, |\Omega|]. \tag{29}$$

We can assume $f^+ \not\equiv 0$, otherwise (29) is trivial for $u = U = 0$. We have

$$F_u(s) = F_U(s) = F(s) = \int_0^s e^{-1} f^*;$$

from (11), (26), (23) we get

$$u^*(s) \leqslant \int_s^{|u>0|} \beta F \leqslant \int_s^{|U>0|} \beta F = U^*(s)$$

that is (29).

REMARK 3.1. *The previous result is a slight generalization of a result of* [9, 23] *concerning the case $R = 0$.*

The following two theorems are concerned with the cases $c \geqslant 0$ and $c \leqslant 0$ respectively.

THEOREM 3.2. *Let u be a solution of* (4) *and assume that operator A satisfies* (1), (2) *and* (3) *with $c \geqslant 0$. Then we have*

$$u^* \leqslant U^*, \quad \text{on} \quad [0, \bar{s}], \tag{30}$$

$$\int_{\bar{s}}^{s} e^{-1} u^* \leqslant \int_{\bar{s}}^{s} e^{-1} U^*, \quad \text{on } [\bar{s}, |\Omega|], \tag{31}$$

where $\bar{s} = |\{s: c_*(s) = 0\}|$.

THEOREM 3.3. *Let u be a solution of* (4) *and assume that operator A satisfies* (1), (2), *and* (3) *with $c \leqslant 0$; if one of the conditions i), ii), iii) of Proposition* 2.1. *is verified, then*

$$u^* \leqslant U^*, \qquad \text{on} \quad [0, |\Omega|]. \tag{32}$$

PROOF OF THEOREM 3.2. If $|u > 0| \leqslant \bar{s}$, (31) is trivial. Then let it be $|u > 0| > \bar{s}$; setting

$$W(s) = \int_{\bar{s}}^{s} e^{-1} c_* w, \qquad s \in [\bar{s}, |u > 0|],$$

where $w = u^* - U^*$, by (27) we have

$$\begin{cases} -(ec_*^{-1} W')' + \beta W \leqslant 0, & \text{on} \quad]\bar{s}, |u > 0|[, \\ W(\bar{s}) = 0, \quad W'(|u > 0|) \leqslant 0. \end{cases}$$

Hence, by maximum principle, we get $W(s) \leqslant 0$ on $[\bar{s}, |u > 0|]$, that is

$$\int_{\bar{s}}^{s} e^{-1} c_* u^* \leqslant \int_{\bar{s}}^{s} e^{-1} c_* U^*, \qquad s \in [\bar{s}, |u > 0|]; \tag{33}$$

by Lemma 2.2, we deduce (31). From (31) we have $u^*(\bar{s}) \leqslant U^*(\bar{s})$; since $w' \geqslant 0$ on $[0, \bar{s}]$, we get (30).

PROOF OF THEOREM 3.3. We assume $c < 0$; otherwise we replace c by $c - \varepsilon$ with $\varepsilon > 0$ and gain the result getting ε go to zero. Setting

$$W(s) = \int_0^s \delta w,$$

where $w = u^* - U^*$ and $\delta = -e^{-1} c_* (> 0)$, from (27) we have

$$\begin{cases} -(\delta^{-1} W')' - \beta W \leqslant 0, & \text{on} \quad [0, |u > 0|], \\ W(0) = 0, \quad W'(|u > 0|) \leqslant 0. \end{cases} \tag{34}$$

At first we show that

$$W(s) \leqslant 0 \qquad s \in [0, |u > 0|]. \tag{35}$$

If (35) does not hold, as $w(|u > 0|) \leqslant 0$, there exists $\tilde{s} \in]0, |u > 0|]$ such that $w(\tilde{s}) = 0$ and $w(s) \leqslant 0$ for $s \in [\tilde{s}, |u > 0|]$; hence $W'(\tilde{s}) = 0$ and $W^+ \not\equiv 0$ on $[0, \tilde{s}]$. The first eigenvalue $\tilde{\lambda}$ of the problem

$$-(\delta^{-1} Z')' - \beta Z = \lambda \beta Z, \qquad Z(0) = Z'(\tilde{s}) = 0,$$

is the same as the first eigenvalue of the problem

$$A^{\#} \varphi = \lambda c^{-\#} \varphi, \qquad \varphi \in H_0^1(\tilde{B}),$$

where $\tilde{B}$ is the ball centered in O such that $|\tilde{B}| = \tilde{s}$. The conditions on $A^{\#}$ (see *ii*) in Proposition 2.1 and Remark 2.2) yield that $\tilde{\lambda}$ is positive. Hence by (34), using variational characterization of $\tilde{\lambda}$,

$$0 \leq \tilde{\lambda} \int_0^{\tilde{s}} \beta (W^+)^2 \leq \int_0^{\tilde{s}} \left[\delta^{-1} \left(\frac{dW^+}{ds} \right)^2 - \beta (W^+)^2 \right] \leq 0$$

and then $W^+ = 0$ that is absurd.

From (35), integrating (34) on $[s, |u > 0|]$, we easily obtain (32).

In the following Theorem we make no assumption concerning the sign of the function c. We assume $c^+, c^- \not\equiv 0$ in order not to fall in previous cases.

THEOREM 3.4. *Let u be a solution of* (4), *with operator A satisfying* (1), (2), (3) *and one of the conditions i), ii), iii) of Proposition* 2.1, *then*

$$u^* \leq U^*, \quad on \quad [0, \bar{s}], \tag{36}$$

$$\int_{\bar{s}}^{s} e^{-1} u^* \leq \int_{\bar{s}}^{s} e^{-1} U^*, \quad on \quad [\bar{s}, |\Omega|], \tag{37}$$

where $\bar{s} = |\{s\colon c_*(s) \leq 0\}|$.

From (27) we get

$$-w' \leq -\beta \int_0^s e^{-1} \{ c_*^+ + c^{-*} \} w + 2\beta \int_0^s e^{-1} c^{-*} w;$$

if we set $\delta = e^{-1}(c_*^+ + c^{-*})$ and assume for sake of simplicity

$$|\{x \in \Omega\colon c = 0\}| = 0, \tag{38}$$

the function

$$W(s) = \int_0^s e^{-1} (c_*^+ + c^{-*}) w,$$

verifies the following conditions

$$\begin{cases} -(\delta^{-1} W')' + \beta W \leq TW, \\ W(0) = 0, \qquad W'(|u > 0|) \leq 0, \end{cases} \tag{39}$$

where T is the operator defined by

$$T(\varphi) := \begin{cases} 2\beta(s)\, \varphi(s), & 0 \leq s \leq \bar{s}, \\ 2\beta(s)\, \varphi(\bar{s}), & \bar{s} < s < |\Omega|. \end{cases}$$

Our first goal is to show that

$$W \leq 0 \quad \text{on} \quad [0, |u > 0|]. \tag{40}$$

Suppose that this is not the case; then for some $\tilde{s} \in]0, |u > 0|]$ we have $W'(\tilde{s}) = 0$, and $W^+ \not\equiv 0$ on $[0, \tilde{s}]$, and

(41) $$\begin{cases} -(\delta^{-1} W')' + \beta W \leqslant TW \\ W(0) = 0, \qquad W'(\tilde{s}) = 0, \end{cases}$$

Let G be the Green operator of the problem

$$-(\delta^{-1} Z')' + \beta Z = g, \qquad Z(0) = Z'(\tilde{s}) = 0;$$

obviously G is a linear, positive, compact operator from $\mathcal{C}^0$ in $\mathcal{C}^0$. By (41) we have

(42) $$W \leqslant G(T(W)) = K(W),$$

where $K = G \circ T$ is a linear, positive, compact operator acting on the space $\mathcal{C}^0$. Now we use some properties on positive operators (see [20] for an exhaustive treatment). From (42), by using theorem 2.5 of [20], the operator K has an unique (within the norm) positive characteristic vector ϕ, $K\phi = \mu\phi$ with $\mu \geqslant 1$. However, if $\tilde{\lambda}_1$ is the first eigenvalue of problem (20) with $\Omega^\#$ replaced by the ball centered in O whose measure is $\tilde{s}$ and φ is a relative eigenfunction, we have (see also (23))

$$-\frac{d\varphi^*}{ds} = \beta \int_0^s \{(1 + \tilde{\lambda}_1) c^{-*} - c_*^+\} \varphi^* e^{-1};$$

it follows

$$-(\delta^{-1} \Phi')' + \beta\Phi = (2 + \tilde{\lambda}_1)\beta \int_0^s c^{-*} \varphi^* e^{-1},$$

with

$$\Phi(s) := \int_0^s \delta\varphi^*;$$

thus

$$\Phi = (1 + 2^{-1}\tilde{\lambda}_1) K\Phi.$$

Hence Φ is a positive characteristic vector of K and then $1 + \tilde{\lambda}_1/2 = \mu^{-1}$. Since $\tilde{\lambda}_1 > 0$ (see *ii*) in Proposition 2.1 and Remark 2.2), we get $\mu < 1$ that is absurd. Then we have (40), from which we deduce

(43) $$w(0) \leqslant 0$$

that is

$$\sup_{x \in \Omega} u(x) = u^*(0) \leqslant U^*(0) = \sup_{x \in \Omega^\#} U^\#(x).$$

Moreover, from (27) we have, if $s \in [0, \tilde{s}]$,

(44) $$w' \geqslant -\beta(s)\, W(s) \geqslant 0.$$

So in order to obtain (36), we have to show that $w(\bar{s}) \leqslant 0$. Namely, if $w(\bar{s}) > 0$, since $w(|u>0|) \leqslant 0$, by (43) there exists s', $s'' \in [0, |u>0|]$ such that $s' < \bar{s} < s''$ and

$$(45) \qquad w(s') = w(s'') = 0\,, \qquad w(s) > 0 \quad \text{on} \quad]s', s''[.$$

Hence, from (27), if $s > \bar{s}$,

$$w'(s) \geqslant \beta(s)\left(-W(\bar{s}) + \int_{\bar{s}}^{s} e^{-1} c_*^+ w\right)$$

and then from (40) and (45) we have $w'(s) \geqslant 0$ on $[\bar{s}, s'']$, that is absurd. Now we show the validity of (37). From (27), if $\bar{s} < s \leqslant |u>0|$, we have

$$w' + \beta(s)\, W(\bar{s}) - \beta(s) \int_{\bar{s}}^{s} e^{-1} c_*^+ w \geqslant 0\,,$$

and then by (40)

$$w' - \beta(s) \int_{\bar{s}}^{s} e^{-1} c_*^+ w \geqslant 0\,, \qquad \text{on} \quad [\bar{s}, |u>0|]\,.$$

Setting

$$\overline{W}(s) = \int_{\bar{s}}^{s} e^{-1} c_*^+ w = \int_{\bar{s}}^{s} \delta w\,,$$

we have

$$(46) \qquad \begin{cases} -(\delta^{-1}\overline{W}')' + \beta\overline{W} \leqslant 0\,, & \text{on} \quad]\bar{s}, |u>0|[\,, \\ \overline{W}(\bar{s}) = 0\,, \quad \overline{W}'(|u>0|) \leqslant 0 \end{cases}$$

and then, by maximum principle $\overline{W}(s) \leqslant 0$ on $[\bar{s}, |u>0|]$, that is (37).

Thus Theorem is proved under condition (38); let us show that the result is valid without this supplementary hypothesis. Proceeding as in the proof of Theorem 3.3 we replace the coefficient $c_\#$ in (5) by

$$c_\varepsilon(x) = \begin{cases} c_\#(x) - \varepsilon & \text{if } c_\#(x) \leqslant 0\,, \\ c_\#(x) & \text{if } c_\#(x) > 0\,, \end{cases}$$

and consider the perturbed problem

$$a_\varepsilon^\#(U^\varepsilon, V - U^\varepsilon) = \int_{\Omega^\#} \left[U_{x_i}^\varepsilon (V - U^\varepsilon)_{x_i} + \frac{R}{|x|} x_i U_{x_i}^\varepsilon (V - U^\varepsilon) + c_\varepsilon U^\varepsilon (V - U^\varepsilon) \right] \geqslant$$

$$\geqslant \int_{\Omega^\#} f^\#(V - U^\varepsilon)\,, \qquad \forall V \in H_0^1(\Omega^\#)\,, \qquad \text{with } U^\varepsilon,\ V \geqslant 0\,.$$

The above arguments yield (36), (37), with the rearrangement of U^ε instead of U^*; getting ε go to zero we get the result.

Appendix

A. Proof of Proposition 2.1.

i) $\Rightarrow$ *ii)* We have

$$\lambda_1 \int_{\Omega^\#} c^{-\#} e^{-R|x|} \phi_1 Z = \int_{\Omega^\#} e^{-R|x|} H\phi_1 ,$$

where ϕ_1 is the first eigenfunction; we conclude observing that ϕ_1 does not change sign on $\Omega^\#$.

ii) $\Rightarrow$ *iii)* From Remark 2.2 we have

$$(\lambda_1 + 1) \int_{\Omega^\#} e^{-R|x|} c^{-\#} \phi^2 \leqslant \int_{\Omega^\#} e^{-R|x|} (|\nabla\phi|^2 + c_\#^+ \phi^2) , \qquad \phi \in H_0^1$$

and then, with $0 < \alpha < 1$,

$$\int_{\Omega^\#} e^{-R|x|} (|\nabla\phi|^2 + c_\# \phi^2) - \alpha \int_{\Omega^\#} e^{-R|x|} |\nabla\phi|^2 =$$

$$= (1-\alpha) \int_{\Omega^\#} e^{-R|x|} |\nabla\phi|^2 + \int_{\Omega^\#} e^{-R|x|} (c_\#^+ - c^{-\#}) \phi^2 \geqslant$$

$$\geqslant (\lambda_1 - \alpha - \alpha\lambda_1) \int_{\Omega^\#} e^{-R|x|} c^{-\#} \phi^2 + \alpha \int_{\Omega^\#} e^{-R|x|} c_\#^+ \phi^2 .$$

Then we conclude choosing $0 < \alpha < \lambda_1/(1+\lambda_1)$.

iii) $\Rightarrow$ *i)* By coercivity we have existence and unicity of the solution of

$$\begin{cases} -(e^{-R|x|} Z_{x_i})_{x_i} + c_\# e^{-R|x|} Z = e^{-R|x|} H , \\ Z \in H_0^1 , \end{cases} \tag{47}$$

and hence of $A^\# Z = H$, $Z \in H_0^1$. By (47) we have, with $H \geqslant 0$,

$$\alpha \int_{\Omega^\#} e^{-R|x|} |\nabla Z^-|^2 \leqslant \int_{\Omega^\#} e^{-R|x|} [|\nabla Z^-|^2 + c_\# |Z^-|^2] = -\int_{\Omega^\#} e^{-R|x|} H Z^- \leqslant 0 .$$

Then $Z^- = 0$. With similar arguments we can prove (19).

Since U is solution of (5) iff U is solution of the variational inequality

$$\int_{\Omega^\#} e^{-R|x|} [U_{x_i} (\phi - U)_{x_i} + c_\# U(\phi - U)] \geqslant \int_{\Omega^\#} e^{-R|x|} f^\# (\phi - U) ,$$

by *iii)* we obtain the existence and unicity of solution U (spherically symmetric). Now we show $U = U^*$, that is U is also spherically decreasing. For sake of simplicity we assume $f^\#$ and $c_\#$ sufficiently smooth. Deriving with respect to $\rho = |x|$ the equation

$$A^\# U = f^\# , \qquad \text{on} \qquad \{x\colon U > 0\} ,$$

we obtain setting $v = U_\rho$

$$v_{\rho\rho} - \frac{n-1}{\rho} v_\rho + R v_\rho + \left(\frac{n-1}{\rho^2} + c_\#\right) v = f_\rho^\# - c_{\#\rho} U. \tag{48}$$

We note that $f_\rho^\# - c_{\#\rho} U \leq 0$ and $v(0) = 0$, $v(|U > 0|) \leq 0$. The operator at left side in (48) is $A^\# + (n-1)\,|x|^{-2}$ and obviously it has the property (19); then $v = U_\rho \leq 0$.

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