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The multiple layer potential for the biharmonic equation in n variables

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Analisi matematica. — *The multiple layer potential for the biharmonic equation in* n *variables.* Nota di Alberto Cialdea, presentata (*) dal Socio G. Fichera.

Abstract. — The definition of multiple layer potential for the biharmonic equation in $\boldsymbol{R}^n$ is given. In order to represent the solution of Dirichlet problem by means of such a potential, a singular integral system, whose symbol determinant identically vanishes, is considered. The concept of bilateral reduction is introduced and employed for investigating such a system.

Key words: Singular integral systems; Potential theory; Biharmonic problem.

Riassunto. — *Il potenziale di multiplo strato per l'equazione biarmonica in* n *variabili*. Viene data la definizione di potenziale di multiplo strato per l'equazione biarmonica in $\boldsymbol{R}^n$. Volendo rappresentare la soluzione del problema di Dirichlet per mezzo di tale potenziale, si ottiene un sistema di equazioni integrali singolari, il cui determinante simbolico si annulla identicamente. Il concetto di riduzione bilatera viene introdotto ed impiegato per studiare tale sistema.

1. Introduction

The multiple layer potential for elliptic equations of higher order in two variables was introduced in [1]. As Agmon remarks in his paper, his method cannot be used for higher order equations in more then two variables.

Recently an alternative concept of multiple layer potential, for the same equation considered by Agmon, has been given in [4]. In the present paper it is shown how this approach may be used in the case of an higher order equation in any number of variables. To this end the biharmonic equation in n variables is considered and the definition of multiple layer potential, extending the one in [4], is used (see § 3). When we try to solve Dirichlet problem by means of such a potential, we obtain a multidimensional singular integral system, whose symbol determinant, differently from the case $n = 2$ (see [4]), identically vanishes. Therefore, the usual regularization theory, which is used for studying singular integral systems, turns out to be insufficient. The concept of *bilateral reduction* is introduced in § 2 and employed in § 4 for investigating the singular integral system we have. Finally Dirichlet problem is solved by means of such a potential (§ 5).

Although the equation considered in this paper is very particular, it is very likely that the theory may be extended to more general cases, as our present research work in this field seems to prove. Let us observe that the aim of this paper (as well as of papers [3, 4] and of next papers) is not to give a new solution of some classical problem, which is nowadays solved by standard methods, but to investigate how classical algorithms of analysis, which seem to be applicable only to very particular cases (for instance harmonic problem) can be generalized in such a way to have a general range of applicability.

(*) Nella seduta del 24 aprile 1992.

2. Some results of functional analysis

Let us indicate by B and B' two Banach spaces. We say that a linear and continuous operator S: $B \to B'$ *can be reduced on the left* (*on the right*) if there exists a linear and continuous operator $S': B' \to B$ such that $S'S = I + \mathcal{T}$ $(SS' = I + \mathcal{T})$, where I is the identity and $\mathcal{T}$: $B \to B(\mathcal{T}$: $B' \to B')$ is a completely continuous operator. The properties of such operators are well known. Here we list the main ones:

i) the dimension of the kernel $\mathcal{N}(S)$ (the codimension of the range $S(B)$) is finite;

ii) the range $S(B)$ is closed in B';

iii) there exists a solution $\varphi \in B$ of the equation $S\varphi = \psi (\psi \in B')$ if and only if $\langle \gamma, \psi \rangle = 0$ for any $\gamma \in B'^*$ such that $S^* \gamma = 0$ (B'^* is the topological dual space of B' and $S^*: B'^* \to B^*$ is the adjoint of S).

For the proofs of these theorems see [5, 7, 13].

I. *If* S: $B \to B'$, S': $B' \to B$ *are linear and continuous operators such that* $S'S = I + \mathcal{T}$, $\mathcal{T}$ *being completely continuous, then the following decomposition holds*: [1]

$$B' = S(B_\rho) \oplus \mathcal{N}(S') \oplus \Gamma \tag{2.1}$$

where B_ρ *is a subspace of* B *having finite codimension and* Γ *is a subspace of* B' *having finite dimension.*

Let $z_1, \ldots, z_\rho$ be linearly independent vectors of B such that [2] $[z_1, \ldots, z_\rho] = \mathcal{N}(I + \mathcal{T})$. It is well known that there exist $\zeta_1, \ldots, \zeta_\rho \in B^*$ such that $\langle \zeta_h, z_k \rangle = \delta_{hk}$ $(h, k = 1, \ldots, \rho)$ (see [7, p. 94]). We shall set $B_\rho = \{u \in B \mid \langle \zeta_h, u \rangle = 0,\ h = 1, \ldots, \rho\}$.

Let $\tau_1, \ldots, \tau_\rho$ be linearly independent vectors of B^* such that $[\tau_1, \ldots, \tau_\rho] = \mathcal{N}(I + \mathcal{T}^*)$. Let $\eta_1, \ldots, \eta_s$ $(s \le \rho)$ be linearly independent vectors of B'^* such that $[\eta_1, \ldots, \eta_s] = [S'^* \tau_1, \ldots, S'^* \tau_\rho]$. There exist $e_1, \ldots, e_s \in B'$ such that $\langle \eta_h, e_k \rangle = \delta_{hk}$ $(h, k = 1, \ldots, s)$. We define $\Gamma = [e_1, \ldots, e_s]$.

If ψ is a vector of B', let us set

$$\Psi = \psi - \sum_{h=1}^{s} \langle \eta_h, \psi \rangle e_h . \tag{2.2}$$

We have: $\langle \eta_k, \Psi \rangle = 0$ $(k = 1, \ldots, s)$ and then $\langle \tau_h, S'\Psi \rangle = \langle S'^* \tau_h, \Psi \rangle = 0$ $(h = 1, \ldots, \rho)$. From the theory of Fredholm equations it follows that there exists a solution $\alpha \in B$ of the equation

$$\alpha + \mathcal{T}\alpha = S'\Psi . \tag{2.3}$$

It is easy to prove that there exists one and only one $\alpha \in B_\rho$ solution of the eq. (2.3). If $\beta = \Psi - S\alpha$, then $S'\beta = S'(\Psi - S\alpha) = S'\Psi - \alpha - \mathcal{T}\alpha = 0$, *i.e.* $\beta \in \mathcal{N}(S')$. From

(1) (2.1) means that for any $\psi \in B'$, there exist $\alpha \in B_\rho, \beta \in \mathcal{N}(S'), \gamma \in \Gamma$ such that $\psi = S\alpha + \beta + \gamma$, $S\alpha, \beta$ and γ being uniquely determined by ψ.

(2) By $[z_1, \ldots, z_\rho]$ we denote the subspace spanned by $z_1, \ldots, z_\rho$.

(2.2) it follows that

$$\psi = S\alpha + \beta + \sum_{h=1}^{s} \langle \eta_h, \psi \rangle e_h .$$

In order to complete the proof, it will suffice to show that if

$$(2.4) \qquad S\alpha + \beta + \gamma = 0, \qquad \alpha \in B_\rho, \ \ \beta \in \mathfrak{N}(S'), \ \ \gamma \in \Gamma,$$

then $S\alpha = \beta = \gamma = 0$. Indeed, from (2.4) it follows that $\alpha + \mathcal{C}\alpha + S'\gamma = 0$ and then $\langle S'^* \tau_h, \gamma \rangle = \langle \tau_h, S'\gamma \rangle = 0$ $(h = 1, \ldots, \rho)$, *i.e.* $\langle \eta_k, \gamma \rangle = 0$ $(k = 1, \ldots, s)$. This implies $\gamma = 0$ and then $\alpha + \mathcal{C}\alpha = 0$. Since $\alpha \in B_\rho$ it follows $\alpha = 0$ and (because of (2.4)) $\beta = 0$. Let us remark that we have shown something more, namely that even $\alpha \in B_\rho$ is uniquely determined by ψ.

II. *Let S: $B \to B'$, S': $B' \to B$ be linear and continuous operators such that $S'S = I + \mathcal{C}$, $\mathcal{C}$ being completely continuous. There exists a constant K such that if $\psi = S\alpha + \beta + \mu$, $\alpha \in B_\rho$, $\beta \in \mathfrak{N}(S')$, $\gamma \in \Gamma$, then* (3)

$$(2.5) \qquad \|\alpha\| + \|\gamma\| \le K\|S'\psi\|.$$

Assume the contrary. There exists a sequence $\{\psi_n\} \subset B'$ such that: $\psi_n = S\alpha_n + \beta_n + \gamma_n$, $\alpha_n \in B_\rho$, $\beta_n \in \mathfrak{N}(S')$, $\gamma_n \in \Gamma$, $\|\alpha_n\| + \|\gamma_n\| = 1$, $\|S'\psi_n\| \to 0$. Since $\dim \Gamma$ is finite, there exists a subsequence $\{\gamma_{n_k}\}$ and $\gamma_0 \in \Gamma$ such that $\gamma_{n_k} \to \gamma_0$; since $\mathcal{C}$ is completely continuous we may suppose $\{\alpha_{n_k}\}$ is such that $\mathcal{C}\alpha_{n_k}$ is convergent. On the other hand $S'\psi_{n_k} = \alpha_{n_k} + \mathcal{C}\alpha_{n_k} + S'\gamma_{n_k} \to 0$; this implies that $\{\alpha_{n_k}\}$ is a convergent sequence, *i.e.* there exists $\alpha_0 \in B_\rho$ such that $\alpha_{n_k} \to \alpha_0$; moreover $\alpha_0 + \mathcal{C}\alpha_0 + S'\gamma_0 = 0$, *i.e.* $S\alpha_0 + \gamma_0 \in \mathfrak{N}(S')$. Because of Theorem I, we have $\alpha_0 = 0$, $\gamma_0 = 0$, and then $\alpha_{n_k} \to 0$, $\gamma_{n_k} \to 0$. This is a contradiction, because $\|\alpha_{n_k}\| + \|\gamma_{n_k}\| = 1$.

We say that a *bilateral reduction of a linear and continuous operator S: $B \to B'$ is given, if B'' is a Banach space, S_1: $B'' \to B$, S_2: $B' \to B''$ are linear and continuous operators and*

$$(2.6) \qquad S_2 S S_1 = I + \mathcal{C},$$

where $\mathcal{C}$: $B'' \to B''$ is a completely continuous operator.

We remark that it is always possible to give trivial bilateral reductions for an operator $S \not\equiv 0$. For example, if $Su_1 = v_1$, $v_1 \ne 0$, we may take $B'' = \boldsymbol{R}$, $S_1 a = au_1$, S_2 a linear and continuous functional: $B' \to \boldsymbol{R}$ such that $S_2 v_1 = 1$. We have $S_2 S S_1 a = a$, $\forall a \in \boldsymbol{R}$. However considering bilateral reductions may be useful in order to prove that the range of an operator S is closed. This is showed by the following theorem:

III. *If S is a linear and continuous operator and the bilateral reduction* (2.6) *is given, then the range $S(B)$ is closed in B' if and only if $S[\mathfrak{N}(S_2 S)]$ is closed in B'.*

First assume $\mathfrak{N}(S) = \{0\}$.

If $S(B)$ is closed, then there exists a constant K such that $\|u\| \le K\|Su\|$, $\forall u \in B$. In particular, $\|u\| \le K\|Su\|$, $\forall u \in \mathfrak{N}(S_2 S)$, *i.e* $S[\mathfrak{N}(S_2 S)]$ is closed in B'. Conversely, since

(3) Obviously $\|\alpha\|$ is the norm in B, $\|\gamma\|$ is the norm in B', etc.

$S_2 S$ reduces S_1 on the left, by using Theorem I we have

$$B = S_1(B''_\rho) \oplus \mathcal{N}(S_2 S) \oplus \Gamma \tag{2.7}$$

where B''_ρ is a subspace of B'' and Γ is a finite dimensional subspace of B. Let $\{u_n\}$ be a sequence in $S(B)$ such that $u_n \to u_0$. We have to show that $u_0 \in S(B)$. Since $u_n = S\psi_n$, $\psi_n \in B$, from (2.7) it follows that $\psi_n = S_1 \alpha_n + \beta_n + \gamma_n$, $\alpha_n \in B''_\rho$, $\beta_n \in \mathcal{N}(S_2 S)$, $\gamma_n \in \Gamma$. Because of (2.5) we have $\|\alpha_n - \alpha_{n+p}\| + \|\gamma_n - \gamma_{n+p}\| \leq K\|S_2 S(\psi_n - \psi_{n+p})\| = K\|S_2(u_n - u_{n+p})\|$. Since $u_n \to u_0$, $\{\alpha_n\}$ and $\{\gamma_n\}$ are Cauchy sequences and therefore there exist $\alpha_0 \in B''_\rho$, $\gamma_0 \in \Gamma$ such that $\alpha_n \to \alpha_0$, $\gamma_n \to \gamma_0$. On the other hand $S\psi_n = SS_1\alpha_n + S\beta_n + S\gamma_n$, *i.e.* $S\beta_n = u_n - SS_1\alpha_n - S\gamma_n$. It follows that $\{S\beta_n\}$ is a convergent sequence in $S[\mathcal{N}(S_2 S)]$. Since $S[\mathcal{N}(S_2 S)]$ is closed and $\mathcal{N}(S) = \{0\}$, we have that $\{\beta_n\}$ is a Cauchy sequence and there exists $\beta_0 \in \mathcal{N}(S_2 S)$ such that $\beta_n \to \beta_0$. So we have showed that $\psi_n = S_1\alpha_n + \beta_n + \gamma_n$ tends to $\psi_0 = S_1\alpha_0 + \beta_0 + \gamma_0$. It follows that $u_n = S\psi_n \to S\psi_0$. But since $u_n \to u_0$, we have $u_0 = S\psi_0$.

If $N(S) \neq \{0\}$ we define (4): $\tilde{B} = B/\mathcal{N}(S)$; $\tilde{S}_1 \colon B'' \to \tilde{B}$, $\tilde{S}_1\alpha = [S_1\alpha]$; $\tilde{S}\colon \tilde{B} \to B'$, $\tilde{S}[\psi] = S\psi$. It is obvious that

$$S(B) = \tilde{S}(\tilde{B}); \tag{2.8}$$

$$S_2\tilde{S}\tilde{S}_1\alpha = S_2\tilde{S}[S_1\alpha] = S_2 S S_1\alpha = \alpha + \mathcal{C}\alpha. \tag{2.9}$$

Moreover $u \in \tilde{S}[\mathcal{N}(S_2\tilde{S})] \Leftrightarrow u = \tilde{S}[\psi]$, $S_2\tilde{S}[\psi] = 0 \Leftrightarrow u = S\psi$, $S_2 S\psi = 0 \Leftrightarrow u \in S[\mathcal{N}(S_2 S)]$, *i.e.*

$$\tilde{S}[\mathcal{N}(S_2\tilde{S})] = S[\mathcal{N}(S_2 S)]. \tag{2.10}$$

Since $\mathcal{N}(\tilde{S}) = \{[0]\}$, (2.9) implies that $\tilde{S}(\tilde{B})$ is closed if and only if $\tilde{S}[\mathcal{N}(S_2\tilde{S})]$ is closed; the result in full generality follows from (2.8), (2.10).

IV. *If S is a linear and continuous operator and the bilateral reduction* (2.6) *is given, then the range $S(B)$ is closed in B' if and only if $S^*[\mathcal{N}(S_1^* S^*)]$ is closed in B^*.*

This Theorem follows immediately from the previous one, because $S(B)$ is closed in B' if and only if $S^*(B'^*)$ is closed in B^*, and if (2.6) holds then $S_1^* S^* S_2^* = I + \mathcal{C}^*$, where $\mathcal{C}^* \colon B''^* \to B''^*$ is completely continuous.

Because of these Theorems, if we want to prove that the range of an operator S is closed (and consequently that the alternative theorem holds) it will be sufficient to find a space B'' and a couple of operators S_1, S_2 such that (2.6) holds and such that we are able to prove that $S[\mathcal{N}(S_2 S)]$ (or $S^*[\mathcal{N}(S_1^* S^*)]$) is closed. We rematk that, while the reduction on the left may be applied only if dim $\mathcal{N}(S)$ is finite and the reduction on the right only if codim $S(B)$ is finite, the technique here proposed may be applied even if dim $\mathcal{N}(S) =$ codim $S(B) = \infty$. In the next section, by means of bilateral reduction, we shall obtain an existence theorem for a multidimensional singular integral system, having infinite eigensolutions and infinite compatibility conditions and whose symbol determinant identically vanishes.

(4) $B/\mathcal{N}(S)$ is the usual quotient space whose elements are denoted by $[\psi]$.

Finally we recall that in the classical theory of reduction, as far as the closure of the range is concerned, completely continuous operators are negligible. Unfortunately, in Theorems III, IV they are not negligible: if S(B) is closed, it may happen that $(S + \mathscr{C})\ (B)$ ($\mathscr{C}$ being completely continuous) is not closed. It is very easy to give examples.

3. The multiple layer potential

Let Ω be a bounded domain of $\boldsymbol{R}^n$ such that $\boldsymbol{R}^n - \overline{\Omega}$ is connected and $\Sigma = \partial\Omega$ is a Lyapunov boundary. It means that Σ has a uniformly Hölder continuous normal field of some exponent $\mu\,(0 < \mu \leq 1)$. $\nu = (\nu_1, \ldots, \nu_n)$ denotes the *outward unit normal* to Σ.

Let us consider the fundamental solution for biharmonic equation

$$F(x, y) = \begin{cases} [c_n (n-2)(n-4)]^{-1} |x-y|^{4-n} & n = 3, 5, 6, \ldots, \\ -(2c_4)^{-1} \log |x-y| & n = 4, \end{cases}$$

where c_n is the hypersurface measure of the unit sphere of $\boldsymbol{R}^n$.

The integral

$$u(x) = \int_\Sigma \varphi_h(y) \frac{\partial}{\partial \nu_y} \frac{\partial}{\partial y_h} F(x, y)\, d\sigma_y \tag{3.1}$$

will be called a *(biharmonic) multiple layer potential*. We shall need the following technical lemmas

V. *If φ is an integrable (over Σ) real function and x is a Lebesgue point for φ, then*

$$\lim_{x' \to x} \int_\Sigma \varphi(y) \frac{\partial^3}{\partial y_h \partial y_k \partial y_j} F(x', y)\, d\sigma_y = \tag{3.2}$$
$$= \nu_h(x)\, \nu_k(x)\, \nu_j(x)\, \varphi(x) + \int_\Sigma \varphi(y) \frac{\partial^3}{\partial y_h \partial y_k \partial y_j} F(x, y)\, d\sigma_y,$$

where the limit denotes the internal angular boundary value (see [12, p. 293]) and the last integral is understood in the sense of the principal value (see [12, Chapter IV]).

Let $s_0(x, y)$ be the fundamental solution for Laplace equation: $s_0(x, y) = -[(n - 2)c_n]^{-1} |x-y|^{2-n}$, $n = 3, 4, 5, \ldots$. It is very well known that

$$\lim_{x' \to x} \int_\Sigma \varphi(y) \frac{\partial}{\partial y_h} s_0(x', y)\, d\sigma_y = \frac{1}{2} \nu_h(x)\, \varphi(x) + \int_\Sigma \varphi(y) \frac{\partial}{\partial y_h} s_0(x, y)\, d\sigma_y. \tag{3.3}$$

On the other hand we have

$$\frac{\partial}{\partial \nu_y} \frac{\partial^2}{\partial y_k \partial y_j} F(x, y) = \tag{3.4}$$
$$= \frac{1}{c_n} \left\{ \left[\delta_{jk} - n \frac{(y_j - x_j)(y_k - x_k)}{|y-x|^2} \right] \frac{(y_s - x_s)\, \nu_s(y)}{|y-x|^n} + \frac{(y_j - x_j)\, \nu_k(y) + (y_k - x_k)\, \nu_j(y)}{|y-x|^n} \right\}.$$

Since

$$(3.5)\qquad \left[\delta_{jk} - n\,\frac{(y_j - x_j)(y_k - x_k)}{|y - x|^2}\right]\frac{(y_s - x_s)\,\nu_s(y)}{|y - x|^n} =$$

$$= \left(\nu_i(y)\frac{\partial}{\partial y_k} - \nu_k(y)\,\frac{\partial}{\partial y_i}\right)\frac{(y_i - x_i)(y_j - x_j)}{|y - x|^n},$$

by using the same techniques employed in [12, Ch. V, §1, 2] in the case $n = 3$, we get

$$(3.6)\qquad \lim_{x' \to x} \int_{\Sigma} \varphi(y)\left[\delta_{jk} - n\,\frac{(y_j - x_j')(y_k - x_k')}{|y - x'|^2}\right]\frac{(y_s - x_s')\,\nu_s(y)}{|y - x'|^n}\,d\sigma_y =$$

$$= \int_{\Sigma} \varphi(y)\left[\delta_{jk} - n\,\frac{(y_j - x_j)(y_k - x_k)}{|y - x|^2}\right]\frac{(y_s - x_s)\,\nu_s(y)}{|y - x|^n}\,d\sigma_y .$$

Moreover, since

$$\frac{1}{c_n}\int_{\Sigma} \varphi(y)\,\frac{(y_j - x_j)\,\nu_k(y) + (y_k - x_k)\,\nu_j(y)}{|y - x|^n}\,d\sigma_y =$$

$$= \int_{\Sigma} \varphi(y)\left[\nu_k(y)\frac{\partial}{\partial y_j} + \nu_j(y)\frac{\partial}{\partial y_k}\right]s_0(x, y)\,d\sigma_y ,$$

it follows from (3.3)

$$\lim_{x' \to x}\frac{1}{c_n}\int_{\Sigma} \varphi(y)\,\frac{(y_j - x_j')\,\nu_k(y) + (y_k - x_k')\,\nu_j(y)}{|y - x'|^n}\,d\sigma_y =$$

$$= \nu_k(x)\,\nu_j(x)\,\varphi(x) + \frac{1}{c_n}\int_{\Sigma} \varphi(y)\,\frac{(y_j - x_j)\,\nu_k(y) + (y_k - x_k)\,\nu_j(y)}{|y - x|^n}\,d\sigma_y ,$$

and because of (3.4), (3.6)

$$\lim_{x' \to x}\int_{\Sigma} \varphi(y)\frac{\partial}{\partial \nu_y}\,\frac{\partial^2}{\partial y_k\,\partial y_j}\,F(x', y)\,d\sigma_y = \nu_k(x)\,\nu_j(x)\,\varphi(x) + \int_{\Sigma} \varphi(y)\frac{\partial}{\partial \nu_y}\,\frac{\partial^2}{\partial y_k\,\partial y_j}\,F(x, y)\,d\sigma_y .$$

Finally, observing that $\partial/\partial y_h = \nu_h\,\partial/\partial\nu + \nu_i\,[\nu_i\,\partial/\partial y_h - \nu_h\,\partial/\partial y_i]$ and

$$\lim_{x' \to x}\int_{\Sigma} \varphi(y)\,\nu_h(y)\frac{\partial}{\partial \nu_y}\,\frac{\partial^2}{\partial y_k\,\partial y_j}\,F(x', y)\,d\sigma_y =$$

$$= \nu_h(x)\,\nu_k(x)\,\nu_j(x)\,\varphi(x) + \int_{\Sigma} \varphi(y)\,\nu_h(y)\frac{\partial}{\partial \nu_y}\,\frac{\partial^2}{\partial y_k\,\partial y_j}\,F(x, y)\,d\sigma_y ,$$

$$\lim_{x'\to x} \int_\Sigma \varphi(y)\, \nu_i(y) \left[\nu_i(y) \frac{\partial}{\partial y_h} - \nu_h(y) \frac{\partial}{\partial y_i} \right] \frac{\partial^2}{\partial y_k \partial y_j} F(x', y)\, d\sigma_y =$$

$$= \int_\Sigma \varphi(y)\, \nu_i(y) \left[\nu_i(y) \frac{\partial}{\partial y_h} - \nu_h(y) \frac{\partial}{\partial y_i} \right] \frac{\partial^2}{\partial y_k \partial y_j} F(x, y)\, d\sigma_y ,$$

we obtain (3.2).

VI. *Let φ be an Hölder continuous function belonging to $C^\mu(\Sigma)$ (μ is the Hölder exponent of the normal field on Σ). Then the function*

$$\int_\Sigma \varphi(y) \frac{\partial^2}{\partial y_h \partial y_k} F(x, y)\, d\sigma_y$$

belongs to $W^{2,p}(\Omega)$ for any p such that

$$1 \le p < (1-\mu)^{-1} . \tag{3.7}$$

It will suffice to show that the function

$$w(x) \equiv \int_\Sigma \varphi(y) \frac{\partial^4}{\partial x_i \partial x_j \partial y_h \partial y_k} F(x, y)\, d\sigma_y$$

belongs to $L^p(\Omega)$.

At first, let us prove that, if $\varphi \in C^\mu(\Sigma)$, the function

$$g(x) \equiv \int_\Sigma \varphi(y) \frac{\partial^2}{\partial y_h \partial y_k} s_0(x, y)\, d\sigma_y$$

belongs to $L^p(\Omega)$. Indeed we may write

$$g(x) = \int_\Sigma \varphi(y)\, \nu_h(y) \frac{\partial}{\partial \nu_y} \frac{\partial}{\partial y_k} s_0(x, y)\, d\sigma_y + \tag{3.8}$$

$$+ \int_\Sigma \varphi(y)\, \nu_s(y) \left[\nu_s(y) \frac{\partial}{\partial y_h} - \nu_h(y) \frac{\partial}{\partial y_s} \right] \frac{\partial}{\partial y_k} s_0(x, y)\, d\sigma_y .$$

Let $\varphi_h^*(x)$ be a function belonging to $C^\mu(\overline{\Omega})$ such that $\varphi_h^*(x) = \varphi(x)\, \nu_h(x)$, $x \in \Sigma$ (for the construction of such a function, see [10, p. 383]).

Since

$$\int_\Sigma \frac{\partial}{\partial \nu_y} \frac{\partial}{\partial y_k} s_0(x, y)\, d\sigma_y = - \frac{\partial}{\partial x_k} \int_\Sigma \frac{\partial}{\partial \nu_y} s_0(x, y)\, d\sigma_y = 0 ,$$

$$\int_\Sigma \left[\nu_s(y) \frac{\partial}{\partial y_h} - \nu_h(y) \frac{\partial}{\partial y_s} \right] \frac{\partial}{\partial y_k} s_0(x, y)\, d\sigma_y = 0 ,$$

we may write

$$g(x) = \int_{\Sigma} [\varphi_h^*(y) - \varphi_h^*(x)] \frac{\partial}{\partial \nu_y} \frac{\partial}{\partial y_k} s_0(x, y)\, d\sigma_y +$$

$$+ \int_{\Sigma} [\varphi_s^*(y) - \varphi_s^*(x)] \left[\nu_s(y) \frac{\partial}{\partial y_h} - \nu_h(y) \frac{\partial}{\partial y_s} \right] \frac{\partial}{\partial y_k} s_0(x, y)\, d\sigma_y .$$

If follows

$$|g(x)| \le C \int_{\Sigma} |x - y|^{\mu - n} d\sigma_y \qquad \forall x \in \Omega . \tag{3.9}$$

Because of (3.7), there exists α such that $1 - \mu + (n-1)/p < \alpha < n/p$ and then

$$\int_{\Omega} |g(x)|^p dx \le C^p \int_{\Omega} dx \left(\int_{\Sigma} |x - y|^{\mu - n} d\sigma_y \right)^p \le$$

$$\le C^p \int_{\Omega} dx \int_{\Sigma} |x - y|^{-\alpha p} d\sigma_y \left(\int_{\Sigma} |x - t|^{(\mu + \alpha - n) q} d\sigma_t \right)^{p/q}$$

where $q = p/(p-1)$. Since $(n - \mu - \alpha) q < n - 1$, $\alpha p < n$, we have

$$\int_{\Omega} |g(x)|^p dx \le K \int_{\Sigma} d\sigma_y \int_{\Omega} |x - y|^{-\alpha p} dx < \infty .$$

As far as $w(x)$ is concerned, in a similar way we may write

$$w(x) = \int_{\Sigma} \varphi(y) \nu_i(y) \frac{\partial}{\partial \nu_y} \frac{\partial^3}{\partial y_j \partial y_h \partial y_k} F(x, y)\, d\sigma_y + \tag{3.10}$$

$$+ \int_{\Sigma} \varphi(y) \nu_s(y) \left[\nu_s(y) \frac{\partial}{\partial y_i} - \nu_i(y) \frac{\partial}{\partial y_s} \right] \frac{\partial^3}{\partial y_j \partial y_h \partial y_k} F(x, y)\, d\sigma_y .$$

The first integral is equal to

$$\int_{\Sigma} [\varphi_i^*(y) - \varphi_i^*(x)] \frac{\partial}{\partial \nu_y} \frac{\partial^3}{\partial y_j \partial y_h \partial y_k} F(x, y)\, d\sigma_y - \tag{3.11}$$

$$- \varphi_i^*(x) \frac{\partial}{\partial x_j} \int_{\Sigma} \frac{\partial}{\partial \nu_y} \frac{\partial^2}{\partial y_h \partial y_k} F(x, y)\, d\sigma_y .$$

On the other hand, in view of (3.4), (3.5),

$$\int_{\Sigma} \frac{\partial}{\partial \nu_y} \frac{\partial^2}{\partial y_h \partial y_k} F(x, y)\, d\sigma_y = \frac{1}{c_n} \int_{\Sigma} \left(\nu_i(y) \frac{\partial}{\partial y_k} - \nu_k(y) \frac{\partial}{\partial y_i} \right) \frac{(y_i - x_i)(y_h - x_h)}{|y - x|^n} d\sigma_y +$$

$$+ \int_{\Sigma} \left[\nu_k(y) \frac{\partial}{\partial y_h} + \nu_h(y) \frac{\partial}{\partial y_k} \right] s_0(x, y)\, d\sigma_y ;$$

but the first integral on the right hand side vanishes and the second one belongs to $W^{1,p}(\Omega)$ in view of the first part of this Lemma. Arguing as before, from (3.10), (3.11) it follows that $w(x)$ belongs to $L^p(\Omega)$.

Obviously the Lemma we have just proved implies that, if $\varphi_1, \dots, \varphi_n$ belong to $C^\mu(\Sigma)$, the multiple layer potential (3.1) belongs to $W^{2,p}(\Omega)$ for any p such that (3.7) holds.

4. A BILATERAL REDUCTION

Let $1 < p < \infty$. Let us denote by $L^p(\Sigma)$ the vector space of all measurable real functions such that $|u|^p$ is integrable over Σ. Given $g_k \in L^p(\Sigma)$ we shall consider the system $\partial u/\partial x_k = g_k$, $(k = 1, \dots, n)$, where u is the multiple layer potential (3.1). This is a singular integral system which may be written in the following way in view of Lemma V:

$$(4.1) \qquad -\nu_k(x)\nu_h(x)\varphi_h(x) + \\ + \int_\Sigma \varphi_h(y)\nu_j(y) \frac{\partial^3}{\partial x_k \partial y_h \partial y_j} F(x,y)\, d\sigma_y = g_k(x) \qquad (k = 1, \dots, n).$$

In order to investigate the symbol determinant of this system, let us observe that $\nu_j(y)\ \partial^3(F(x,y))/\partial x_k \partial y_h \partial y_j = (c_n)^{-1}[\nu_k(y)(y_h - x_h) + \nu_h(y)(y_k - x_k)]\ |y-x|^{-n} + \mathcal{O}(|x-y|^{\mu-n+1}) = (c_n)^{-1}[\nu_k(x)(y_h - x_h) + \nu_h(x)(y_k + x_k)]\,|y-x|^{-n} + \mathcal{O}(|x-y|^{\mu-n+1})$. Then, assuming local coordinates at every point $x \in \Sigma$ where the x_n-axis is pointed along $\nu(x)$ and indicating by $\alpha_1, \dots, \alpha_n$ the components of the vector $(\varphi_1, \dots, \varphi_n)$ in local coordinates, the left hand side of (4.1) may be written as

$$\frac{1}{c_n}\int_\Sigma \frac{\eta_k}{r^n}\alpha_n(y)\, d\sigma_y + T_k(x) \qquad k = 1, \dots, n-1$$

$$-\alpha_n(y) + \frac{1}{c_n}\int_\Sigma \frac{\eta_h\alpha_h(y) + \eta_n\alpha_n(y)}{r^n}\, d\sigma_y + T_n(x) \qquad k = n$$

where $\eta_k = y_k - x_k$, $r = |y-x|$ and T_k are weakly singular integral operators. Therefore the matrix symbol $\{\sigma_{ij}\}$ in the new coordinates is such that $\sigma_{ij} = 0$ if $1 \le i, j \le n-1$. This implies that the determinant vanishes, because $n > 2$. Since the range of the symbol determinant is invariant under the transformation we did (see [13, pp. 251-254 and p. 387]), the symbol determinant of (4.1) identically vanishes.

Moreover system (4.1) has infinite eigensolutions and infinite compatibility conditions (see the remark before Theorem XI).

Let $L_1^p(\Sigma)$ be the vector space of the differential forms of degree 1 defined on Σ such that their coefficients are integrable functions belonging to $L^p(\Sigma)$ in any admissible local system of coordinate. Let us introduce the new unknowns: $\psi = \varphi_h\nu_h$,

$\Phi = \varphi_h\, dy^h$. If we consider the tangential operators $M^{j_1 \dots j_{n-2}} u = *_\Sigma (du \wedge dx^{j_1} \dots dx^{j_{n-2}})$ (if λ is a $(n-1)$-form on Σ, say $\lambda = \lambda_0 d\sigma$, λ_0 being a scalar function, then $*_\Sigma \lambda = \lambda_0$) and we observe that

$$\frac{\partial}{\partial x_h} = \nu_h \frac{\partial}{\partial \nu} - \frac{1}{(n-2)!}\, \delta^{1 \dots\dots n}_{h i_2 \dots i_n}\, \nu_{i_2} M^{i_3 \dots i_n}$$

(the summation being extended to every ordered set $i_2, \dots, i_n$ of integers such that $1 \leq i_h \leq n$), then the multiple layer potential (3.1) may be written as

$$u(x) = \int_\Sigma \psi(y)\, \nu_j(y)\, \nu_h(y) \frac{\partial^2}{\partial y_h\, \partial y_j} F(x, y)\, d\sigma_y + \tag{4.2}$$

$$+ \frac{1}{(n-2)!} \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}} \left[\frac{\partial}{\partial y_j} F(x, y) \right] \nu_j(y)\, dy^{j_1} \dots dy^{j_{n-2}}$$

and the system (4.1)

$$- \nu_k(x)\, \psi(x) + \int_\Sigma \psi(y)\, \nu_j(y)\, \nu_h(y) \frac{\partial^3}{\partial x_k\, \partial y_h\, \partial y_j} F(x, y)\, d\sigma_y + \tag{4.3}$$

$$+ \frac{1}{(n-2)!} \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}} \left[\frac{\partial^2}{\partial x_k\, \partial y_j} F(x, y) \right] \nu_j(y)\, dy^{j_1} \dots dy^{j_{n-2}} = g_k(x) \quad (k = 1, \dots, n).$$

Let us consider now the linear and continuous operator $S\colon L^p(\Sigma) \times L_1^p(\Sigma) \to L^p(\Sigma) \times L_1^p(\Sigma)$ defined as follows

$$S(\psi, \Phi) = \Bigg(-\psi(x) + \int_\Sigma \psi(y)\, \nu_j(y)\, \nu_h(y) \frac{\partial}{\partial \nu_x} \frac{\partial^2}{\partial y_h\, \partial y_j} F(x, y)\, d\sigma_y +$$

$$+ \frac{1}{(n-2)!} \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}} \left[\frac{\partial}{\partial \nu_x} \frac{\partial}{\partial y_j} F(x, y) \right] \nu_j(y)\, dy^{j_1} \dots dy^{j_{n-2}},$$

$$\int_\Sigma \psi(y)\, \nu_j(y)\, \nu_h(y)\, d_x \left[\frac{\partial^2}{\partial y_h\, \partial y_j} F(x, y) \right] d\sigma_y +$$

$$+ \frac{1}{(n-2)!} \int_\Sigma \Phi(y) \wedge M_y^{j_1 \dots j_{n-2}} \left[d_x \frac{\partial}{\partial y_j} F(x, y) \right] \nu_j(y)\, dy^{j_1} \dots dy^{j_{n-2}} \Bigg).$$

Keeping in mind that $\nu_k(x)\, dx^k = 0$, it is possible to show that the system (4.1) (or (4.3)) is equivalent to the equation

$$S(\psi, \Phi) = (g_k \nu_k,\, g_k\, dx^k). \tag{4.4}$$

VII. *We have:* (5)

$$\nu_j(y)\,\nu_h(y)\frac{\partial}{\partial \nu_x}\,\frac{\partial^2}{\partial y_h\,\partial y_j}\,F(x,y) = \mathcal{O}(|x-y|^{\mu-n+1});$$

$$M_y^{j_1\dots j_{n-2}}\left[\frac{\partial}{\partial \nu_x}\,\frac{\partial}{\partial y_j}\,F(x,y)\right]\nu_j(y) = M_x^{j_1\dots j_{n-2}}[s_0(x,y)] + \mathcal{O}(|x-y|^{\mu-n+1});$$

$$\nu_j(y)\,\nu_h(y)\,d_x\left[\frac{\partial^2}{\partial y_h\,\partial y_j}\,F(x,y)\right] = d_x[s_0(x,y)] + \mathcal{O}(|x-y|^{\mu-n+1});$$

$$M_y^{j_1\dots j_{n-2}}\left[d_x\,\frac{\partial}{\partial y_j}\,F(x,y)\right]\nu_j(y) = \mathcal{O}(|x-y|^{\mu-n+1})$$

for x, y varying on Σ.

Taking into account that $\nu_k(x)(x_k-y_k) = \mathcal{O}(|x-y|^{1+\mu})$, we have

$$\nu_j(y)\,\nu_h(y)\frac{\partial}{\partial \nu_x}\,\frac{\partial^2}{\partial y_h\,\partial y_j}\,F(x,y) = \nu_j(y)\,\nu_h(y)\,\nu_k(x)(c_n)^{-1}\,\{[\delta_{jh}(x_k-y_k) +$$
$$+\delta_{kh}(x_j-y_j) + \delta_{jk}(x_h-y_h)]\,|x-y|^{-n} - n\,(x_k-y_k)(x_j-y_j)(x_h-y_h)\,|x-y|^{-n-2}\} =$$
$$= \mathcal{O}(|x-y|^{\mu-n+1});$$

$$M_y^{j_1\dots j_{n-2}}\left[\frac{\partial}{\partial \nu_x}\,\frac{\partial}{\partial y_j}\,F(x,y)\right]\nu_j(y) =$$
$$= M_y^{j_1\dots j_{n-2}}[-\delta_{jk}s_0(x,y) - (c_n)^{-1}(x_k-y_k)(x_j-y_j)\,|x-y|^{-n}]\,\nu_k(x)\,\nu_j(y) =$$
$$= -\,M_y^{j_1\dots j_{n-2}}[s_0(x,y)][\nu_j(x)-\nu_j(y)]\,\nu_j(y) - M_y^{j_1\dots j_{n-2}}[s_0(x,y)] + \mathcal{O}(|x-y|^{\mu-n+1}) =$$
$$= -\,M_y^{j_1\dots j_{n-2}}[s_0(x,y)] + \mathcal{O}(|x-y|^{\mu-n+1}) = M_x^{j_1\dots j_{n-2}}[s_0(x,y)] + \mathcal{O}(|x-y|^{\mu-n+1});$$

$$\nu_j(y)\,\nu_h(y)\,d_x\left[\frac{\partial^2}{\partial y_h\,\partial y_j}\,F(x,y)\right] = \nu_j(y)\,\nu_h(y)(c_n)^{-1}\{[\delta_{jh}(x_k-y_k) + \delta_{kh}(x_j-y_j) +$$
$$+\delta_{jk}(x_h-y_h)]\,|x-y|^{-n} - n\,(x_k-y_k)(x_j-y_j)(x_h-y_h)\,|x-y|^{-n-2}\}\,dx^k =$$
$$= (c_n)^{-1}(x_k-y_k)\,|x-y|^{-n}dx^k + \mathcal{O}(|x-y|^{\mu-n+1}) = d_x[s_0(x,y)] + \mathcal{O}(|x-y|^{\mu-n+1}).$$

To prove the last formula, we observe that $\nu_k(y)\,dx^k = [\nu_k(y)-\nu_k(x)]\,dx^k = \mathcal{O}(|x-y|^{\mu})$ and $M^{j_1\dots j_{n-2}}(y_j)\,\nu_j(y) = 0$; then

$$M_y^{j_1\dots j_{n-2}}\left[d_x\,\frac{\partial}{\partial y_j}\,F(x,y)\right]\nu_j(y) =$$
$$= M_y^{j_1\dots j_{n-2}}[-\delta_{jk}s_0(x,y) - (c_n)^{-1}(x_k-y_k)(x_j-y_j)\,|x-y|^{-n}]\,\nu_j(y)\,dx^k =$$
$$= -M_y^{j_1\dots j_{n-2}}[s_0(x,y)]\,\nu_k(y)\,dx^k - (c_n)^{-1}M_y^{j_1\dots j_{n-2}}[(x_k-y_k)\,|x-y|^{-n}](x_j-y_j)\,\nu_j(y)\,dx^k +$$
$$+(c_n)^{-1}(x_k-y_k)\,|x-y|^{-n}M^{j_1\dots j_{n-2}}(y_j)\,\nu_j(y)\,dx^k = \mathcal{O}(|x-y|^{\mu-n+1}).$$

(5) In the last two formulas $\mathcal{O}(|x-y|^{\mu-n+1})$ means a differential form whose coefficients are $\mathcal{O}(|x-y|^{\mu-n+1})$.

It will be convenient to consider the following linear operators:

$$J\colon L^p(\Sigma) \to L_1^p(\Sigma), \qquad J\varphi(x) = \int_\Sigma \varphi(y)\, d_x[s_0(x, y)]\, d\sigma_y\,,$$

$$J'\colon L_1^p(\Sigma) \to L^p(\Sigma), \qquad J'\,\Psi(z) = *_\Sigma \int_\Sigma \Psi(x) \wedge d_z[s_{n-2}(z, x)]\,,$$

where $s_{n-2}(z, x)$ is the Hodge form (see [11]):

$$\sum_{j_1 < \ldots < j_{n-2}} s_0(z, x)\, dz^{j_1} \ldots dz^{j_{n-2}} dx^{j_1} \ldots dx^{j_{n-2}}\,.$$

It is proved in [2] (Theorem I) that J' reduces J, namely

$$J'J\varphi(z) = -\frac{1}{4}\,\varphi(z) + \int_\Sigma \varphi(y)\, L(z, y)\, d\sigma_y \equiv -\frac{1}{4}\,\varphi + L\varphi, \tag{4.5}$$

where $L(x, y)$ is a kernel having a weak singularity.

The following Theorem provides a useful bilateral reduction for S.

VIII. *Let* S_1, S_2 *be the following linear and continuous operators*

$$S_1\colon L^p(\Sigma) \times L^p(\Sigma) \to L^p(\Sigma) \times L_1^p(\Sigma), \qquad S_1(\psi, \varphi) = (\psi, -4J\varphi)$$

$$S_2\colon L^p(\Sigma) \times L_1^p(\Sigma) \to L^p(\Sigma) \times L^p(\Sigma), \qquad S_2(\psi, \Psi) = (-4J'\,\Psi, \psi - 4J'\,\Psi)\,.$$

Then $S_2 S S_1 = I + \mathscr{C}$, *where* $\mathscr{C}\colon L^p(\Sigma) \times L^p(\Sigma) \to L^p(\Sigma) \times L^p(\Sigma)$ *is completely continuous.*

First of all we observe that, since

$$\begin{aligned} d_z[s_{n-2}(z, x)] &= \sum_{j_1 < \ldots < j_{n-2}} \frac{\partial}{\partial z_k} s_0(z, x)\, dz^k dz^{j_1} \ldots dz^{j_{n-2}} dx^{j_1} \ldots dx^{j_{n-2}} = \\ &= \sum_{j_1 < \ldots < j_{n-2}} \delta^{1 \ldots\ldots n}_{hkj_1 \ldots j_{n-2}} \frac{\partial}{\partial z_k} s_0(z, x)\, \nu_h(z)\, d\sigma_z\, dx^{j_1} \ldots dx^{j_{n-2}} = \\ &= \sum_{j_1 < \ldots < j_{n-2}} \delta^{1 \ldots\ldots n}_{hkj_1 \ldots j_{n-2}} M_z^{j_1 \ldots j_{n-2}}[s_0(z, x)]\, d\sigma_z\, dx^{j_1} \ldots dx^{j_{n-2}} = \\ &= \frac{1}{(n-2)!} M_z^{j_1 \ldots j_{n-2}}[s_0(z, x)]\, d\sigma_z\, dx^{j_1} \ldots dx^{j_{n-2}}\,, \end{aligned}$$

we may write

$$\frac{1}{(n-2)!} \int_\Sigma \Phi(x) \wedge M_z^{j_1 \ldots j_{n-2}}[s_0(z, x)]\, dx^{j_1} \ldots dx^{j_{n-2}} = *_\Sigma \int_\Sigma \Phi(x) \wedge d_z[s_{n-2}(z, x)] = J'\,\Phi.$$

So, keeping in mind Lemma VII, we have $S(\psi, \Phi) = (-\psi + \mathscr{C}_{11}\psi + J'\,\Phi + \mathscr{C}_{12}\Phi,$ $J\psi + \mathscr{C}_{21}\psi + \mathscr{C}_{22}\Phi)$, where $\mathscr{C}_{11}\colon L^p(\Sigma) \to L^p(\Sigma)$, $\mathscr{C}_{12}\colon L_1^p(\Sigma) \to L^p(\Sigma)$, $\mathscr{C}_{21}\colon L^p(\Sigma) \to L_1^p(\Sigma)$, $\mathscr{C}_{22}\colon L_1^p(\Sigma) \to L_1^p(\Sigma)$ are completely continuous. In view of (4.5) we have: $SS_1(\psi, \varphi) = S(\psi, -4J\varphi) = (-\psi + \mathscr{C}_{11}\psi - 4J'J\varphi - 4\mathscr{C}_{12}J\varphi,\ J\psi + \mathscr{C}_{21}\psi - 4\mathscr{C}_{22}J\varphi) = (-\psi + \mathscr{C}_{11}\psi + \varphi - 4L\varphi - 4\mathscr{C}_{12}J\varphi,\ J\psi + \mathscr{C}_{21}\psi - 4\mathscr{C}_{22}J\varphi)$. Then: $S_2SS_1(\psi, \varphi) = S_2(-\psi + \mathscr{C}_{11}\psi + \varphi - 4L\varphi - 4\mathscr{C}_{12}J\varphi,\ J\psi + \mathscr{C}_{21}\psi - 4\mathscr{C}_{22}J\varphi) = (-4J'(J\psi + \mathscr{C}_{21}\psi - 4\mathscr{C}_{22}J\varphi),\ -\psi + \mathscr{C}_{11}\psi + \varphi - 4L\varphi - 4\mathscr{C}_{12}J\varphi - 4J'(J\psi + \mathscr{C}_{21}\psi - 4\mathscr{C}_{22}J\varphi)) = (\psi -$

$- 4L\psi - 4J' \mathcal{C}_{21}\psi + 16J' \mathcal{C}_{22} J\varphi, \quad \mathcal{C}_{11}\psi + \varphi - 4L\varphi - 4\mathcal{C}_{12} J\varphi - 4L\psi - 4J' \mathcal{C}_{21}\psi +$
$+ 16J' \mathcal{C}_{22} J\varphi) = (\psi, \varphi) + \mathcal{C}(\psi, \varphi)$.

IX. *$S(L^p(\Sigma) \times L_1^p(\Sigma))$ is closed in $L^p(\Sigma) \times L_1^p(\Sigma)$.*

Because of Theorems III and VIII, it is sufficient to show that $S[\mathfrak{N}(S_2 S)]$ is closed in $L^p(\Sigma) \times L_1^p(\Sigma)$. If we denote by u the multiple layer potential (4.2), we may write: $S(\psi, \Phi) = (\partial u/\partial\nu, du)$, $S_2 S(\psi, \Phi) = (-4J' du, \partial u/\partial\nu - 4J' du)$. Since $\mathfrak{N}(S_2 S) = \{(\psi, \Phi) \in L^p(\Sigma) \times L_1^p(\Sigma) \mid \partial u/\partial\nu = J' du = 0\}$, $S[\mathfrak{N}(S_2 S)]$ is constituted by the vectors $(0, du)$ (u being given by (4.2)) such that $J' du = 0$. Then $S[\mathfrak{N}(S_2 S)] \subseteq \{(0, dH) \mid H \in W^{1,p}(\Sigma), J' dH = 0\}$. On the other hand, $H \in W^{1,p}(\Sigma)$ if and only if there exists $h \in L^p(\Sigma)$ such that

$$H(x) = \int_\Sigma h(y) s_0(x, y)\, d\sigma_y, \qquad x \in \Sigma$$

(this is a consequence of the results contained in §2 of [2]). Since $dH = Jh$, we may write

$$S[\mathfrak{N}(S_2 S)] \subseteq \{(0, Jh) \mid h \in L^p(\Sigma),\ J' Jh = 0\}. \tag{4.6}$$

But, because of (4.5), the $\dim \{h \in L^p(\Sigma) \mid J' Jh = 0\}$ is finite. This implies $\dim S[\mathfrak{N}(S_2 S)] < \infty$ and then $S[\mathfrak{N}(S_2 S)]$ is closed.

The proof of the Theorem is now complete, but actually we may prove that

$$S[\mathfrak{N}(S_2 S)] = \{(0, 0)\}. \tag{4.7}$$

Indeed we have (see [2, Theorem I])

$$J' Jh = -\frac{1}{4} h(z) + \int_\Sigma h(y)\, d\sigma_y \int_\Sigma \frac{\partial}{\partial\nu_x} s_0(x, y) \frac{\partial}{\partial\nu_z} s_0(z, x)\, d\sigma_x =$$
$$= \frac{\partial}{\partial\nu}\left[u(z) + \int_\Sigma s_0(z, x) \frac{\partial}{\partial\nu_x} u(x)\, d\sigma_x\right]$$

where

$$u(z) = \int_\Sigma h(y) s_0(z, y)\, d\sigma_y .$$

Therefore $J' Jh = 0$ if and only if

$$u(z) + \int_\Sigma s_0(z, x) \frac{\partial}{\partial\nu_x} u(x)\, d\sigma_x = c, \qquad z \in \Omega \tag{4.8}$$

(because the left hand side belongs to a class of harmonic functions in which the usual uniqueness theorems hold; see [2, 9]). If

$$u(z) + \int_\Sigma s_0(z, x) \frac{\partial}{\partial\nu_x} u(x)\, d\sigma_x = 0, \qquad z \in \Omega,$$

then

$$\int_{\Sigma} s_0(z, x)\left[h(x) + \frac{\partial}{\partial \nu_x} u(x)\right] d\sigma_x = 0, \qquad z \in \Omega.$$

But this implies

$$h(x) + \frac{\partial}{\partial \nu_x} u(x) = \frac{1}{2} h(z) + \int_{\Sigma} h(y) \frac{\partial}{\partial \nu_x} s_0(x, y)\, d\sigma_y = 0 \tag{4.9}$$

almost everywhere on Σ, *i.e.*

$$\lim_{x'' \to x} \frac{\partial}{\partial \nu} \int_{\Sigma} h(y)\, s_0(x'', y)\, d\sigma_y = 0$$

where the limit denotes the external angular boundary value. It follows from (4.9) that $h \in C^{\mu}(\Sigma)$ and from the last formula (by using usual arguments in potential theory)

$$\int_{\Sigma} h(y)\, s_0(x, y)\, d\sigma_y = 0, \qquad x \in \boldsymbol{R}^n - \overline{\Omega}$$

and then $h = 0$. Therefore the only solution of $J'Jh = 0$ is $h(x) = c\, h_0(x)$, where $h_0(x)$ is the function such that

$$\int_{\Sigma} h_0(y)\, s_0(x, y)\, d\sigma_y = 1, \qquad x \in \Omega.$$

Hence $Jh = du = 0$, $\forall h \in \mathcal{N}(J'J)$. Now (4.7) follows from (4.6).

5. The biharmonic problem

Let us consider the boundary conditions

$$\left.\frac{\partial u}{\partial x_k}\right|_{\Sigma} = g_k \qquad (k = 1, \ldots, n) \tag{5.1}$$

where g_k are given in $L^p(\Sigma)$. The multiple layer potential satisfies (5.1) if and only if it is solution of the system (4.1). Let $K(\varphi_1, \ldots, \varphi_n)$ be the operator given by the left hand side of (4.1). It is very easy to prove that, because of Theorem IX, $K([L^p(\Sigma)]^n)$ is closed in $[L^p(\Sigma)]^n$ (see (4.4)). Then there exists a multiple layer potential solution of (5.1) if and only if $(g_1, \ldots, g_n)$ is orthogonal to any eigensolution $(\xi_1, \ldots \xi_n)$ of the adjoint system $K^*(\xi_1, \ldots, \xi_n) = 0$, *i.e.*

$$-\nu_h(y)\, \nu_k(y)\, \xi_k(y) + \int_{\Sigma} \xi_k(x)\, \nu_j(y) \frac{\partial^3}{\partial x_k\, \partial y_h\, \partial y_j} F(x, y)\, d\sigma_x = 0 \qquad (h = 1, \ldots, n). \tag{5.2}$$

In other words, for any $(\xi_1, \ldots, \xi_n)$ such that

$$\lim_{y'' \to y} \frac{\partial}{\partial \nu} \frac{\partial}{\partial y_h} \int_{\Sigma} \xi_k(x) \frac{\partial}{\partial x_k} F(x, y'')\, d\sigma_x = 0, \qquad (h = 1, \ldots, n)$$

where the limit denotes the external angular boundary value (this relation may be proved by using the same technique employed in Lemma V).

X. *There exist only a finite number of linearly independent simple layer potentials*

$$v(y) = \int_{\Sigma} \xi_k(x) \frac{\partial}{\partial x_k} F(x, y)\, d\sigma_x \tag{5.3}$$

such that

$$\lim_{y'' \to y} \frac{\partial}{\partial \nu} \frac{\partial}{\partial y_h} v(y'') = 0, \qquad (h = 1, \ldots, n). \tag{5.4}$$

If we set $\lambda = \xi_k \nu_k$, $\Xi = \xi_k dx^k$, the simple layer potential (5.3) may be written as

$$v(y) = \int_{\Sigma} \lambda(x) \frac{\partial}{\partial \nu_x} F(x, y)\, d\sigma_x + \frac{1}{(n-2)!} \int_{\Sigma} \Xi(x) \wedge M_x^{j_1 \ldots j_{n-2}} [F(x, y)]\, dx^{j_1} \ldots dx^{j_{n-2}}.$$

On the other hand in [3] it is showed that there exists $\xi \in L^p(\Sigma)$ such that

$$\int_{\Sigma} \Xi(x) \wedge M_x^{j_1 \ldots j_{n-2}} [F(x, y)]\, dx^{j_1} \ldots dx^{j_{n-2}} =$$

$$= \int_{\Sigma} J\xi(x) \wedge M_x^{j_1 \ldots j_{n-2}} [F(x, y)]\, dx^{j_1} \ldots dx^{j_{n-2}}, \qquad \forall y \in \boldsymbol{R}^n.$$

The conditions (5.4) become

$$-\nu_h(y)\lambda(y) + \int_{\Sigma} \lambda(x) \frac{\partial}{\partial \nu_x} \frac{\partial}{\partial \nu_y} \frac{\partial}{\partial y_h} F(x, y)\, d\sigma_x +$$

$$+ \frac{1}{(n-2)!} \int_{\Sigma} J\xi(x) \wedge M_x^{j_1 \ldots j_{n-2}} \left[\frac{\partial}{\partial \nu_y} \frac{\partial}{\partial y_h} F(x, y) \right] dx^{j_1} \ldots dx^{j_{n-2}} = 0 \qquad (h = 1, \ldots, n).$$

These equations are equivalent to the system

$$-\lambda(y) + \int_{\Sigma} \lambda(x) \nu_j(y) \nu_h(y) \frac{\partial}{\partial \nu_x} \frac{\partial^2}{\partial y_h \partial y_j} F(x, y)\, d\sigma_x +$$

$$+ \frac{1}{(n-2)!} \int_{\Sigma} J\xi(x) \wedge M_x^{j_1 \ldots j_{n-2}} \left[\frac{\partial^2}{\partial y_h \partial y_j} F(x, y) \right] \nu_h(y) \nu_j(y)\, dx^{j_1} \ldots dx^{j_{n-2}} = 0,$$

$$\int_{\Sigma} \lambda(x)\, d_y \left[\frac{\partial}{\partial \nu_x} \frac{\partial}{\partial \nu_y} F(x, y) \right] d\sigma_x +$$

$$+ \frac{1}{(n-2)!} \int_{\Sigma} J\xi(x) \wedge M_x^{j_1 \ldots j_{n-2}} \left[d_y \frac{\partial}{\partial y_j} F(x, y) \right] \nu_j(y)\, dx^{j_1} \ldots dx^{j_{n-2}} = 0.$$

By using the same technique employed in Lemma VII and with the help of (4.5), it is possible to write this system as

$$\begin{cases} -\lambda + \mathcal{R}_{11}\lambda - \dfrac{1}{4}\xi + \mathcal{R}_{12}\xi = 0 \\ J\lambda + \mathcal{R}_{21}\lambda + \mathcal{R}_{22}\xi = 0 \end{cases}$$

where $\mathcal{R}_{11}$, $\mathcal{R}_{12}$: $L^p(\Sigma) \to L^p(\Sigma)$, $\mathcal{R}_{21}$, $\mathcal{R}_{22}$: $L^p(\Sigma) \to L_1^p(\Sigma)$ are completely continuous. From the last equation we deduce: $-\lambda/4 + L\lambda + J'\mathcal{R}_{21}\lambda + J'\mathcal{R}_{22}\xi = 0$, and then the $\dim\{(\lambda, \xi) \in L^p(\Sigma) \times L^p(\Sigma) \mid (5.4) \text{ holds}\}$ must be finite. This implies the result.

We say that $(\xi_1, \ldots, \xi_n)$ is *an eigensolution of the first kind* of (5.2) if

$$\int_\Sigma \xi_k(x) \frac{\partial}{\partial x_k} F(x, y)\, d\sigma_x = 0, \qquad \forall y \in \boldsymbol{R}^n,$$

i.e.

$$\int_\Sigma \xi_k \frac{\partial f}{\partial x_k}\, d\sigma = 0, \qquad \forall f \in C^\infty(\boldsymbol{R}^n).$$

An eigensolution that is not of the first kind is called *of the second kind.*

REMARK. Now we are in a position to prove that system (4.1) has infinite eigensolutions and infinite compatibility conditions. Indeed it is obvious that there are infinite linearly independent eigensolutions of the first kind of (5.2); then there are infinite compatibility conditions of system (4.1). Analogously there exist infinite eigensolutions of the equation

$$\tilde{K}(\xi_1, \ldots, \xi_n) \equiv \nu_h(y)\,\nu_k(y)\,\xi_k(y) + \int_\Sigma \xi_k(x)\,\nu_j(y) \frac{\partial^3}{\partial x_k\, \partial y_h\, \partial y_j} F(x, y)\, d\sigma_x = 0 \qquad (h = 1, \ldots, n),$$

because $\tilde{K}(\xi_1, \ldots, \xi_n)$ is the following internal angular boundary value

$$\lim_{y' \to y} \frac{\partial}{\partial \nu} \frac{\partial}{\partial y_h} \int_\Sigma \xi_k(x) \frac{\partial}{\partial x_k} F(x, y')\, d\sigma_x = 0, \qquad (h = 1, \ldots, n).$$

If $\dim \mathcal{N}(K)$ were finite, K would admit a left regularization, because its range $K([L^p(\Sigma)]^n)$ is closed (see [7, p. 162]). On the other hand: $\tilde{K} = -K + (\tilde{K} + K)$ and $(\tilde{K} + K)$ is completely continuous. Thus $\tilde{K}$ should admit a left regularization, which is impossible, because $\dim \mathcal{N}(\tilde{K})$ is infinite. It follows that system (4.1) has infinite eigensolutions.

XI. *The number of linearly independent eigensolutions of the second kind of* (5.2) *is finite and it is equal to the number of linearly independent simple layer potentials* (5.3) *such that* (5.4) *holds.*

Let s be the number of linearly independent simple layer potentials (5.3) such that (5.4) holds. Let us suppose that there exist $s + 1$ linearly independent eigensolutions of the second kind of (5.2): $(\xi_1^j, \ldots, \xi_n^j)$ $(j = 1, \ldots, s+1)$. The functions

$$v^j(y) = \int_\Sigma \xi_k^j(x) \frac{\partial}{\partial x_k} F(x, y)\, d\sigma_x \tag{5.5}$$

are $s+1$ simple layer potential satisfying (5.4) and then they must be linearly dependent, *i.e.* there exist constants $c_1, \ldots, c_{s+1}$ which are not simultaneously vanishing and such that $c_j v^j(y) \equiv 0$. This means $(c_j \xi_1^j, \ldots, c_j \xi_n^j)$ is an eigensolution of the first kind. This is a contradiction.

On the other hand, let $v^1(y), \ldots, v^s(y)$ be s linearly independent simple layer potentials (5.5) solutions of (5.4). If $(\xi_1^j, \ldots, \xi_n^j)$ $(j = 1, \ldots, s)$ were linearly independent, there would exist constants $c_1, \ldots, c_s$ which are not simultaneously vanishing and such that $(c_j \xi_1^j, \ldots, c_j \xi_n^j) = (0, \ldots, 0)$. Hence $c_j v^j(y) \equiv 0$, which is impossible.

From now on we shall suppose that the Hölder exponent μ *of the normal field on* Σ *is such that* $1/2 < \mu < 1$. Let us consider the following BVP

$$\begin{cases} u \in H^2(\Omega) \cap C^4(\Omega) & \\ \Delta_2 \Delta_2 u = 0 & \text{in } \Omega \\ \left.\dfrac{\partial u}{\partial x_k}\right|_\Sigma = g_k & (k = 1, \ldots, n); \end{cases} \tag{5.6}$$

the functions g_k $(k = 1, \ldots, n)$ are given in $C^\mu(\Sigma)$ and they satisfy the natural compatibility conditions:

$$\int_\Sigma g_k\, dx^k \wedge \vartheta = 0 \tag{5.7}$$

for any smooth $(n-2)$-form ϑ such that $d\vartheta = 0$. This is equivalent to say that $g_k dx^k$ is homologous to zero (see [6, pp. 218-219]).

XII. *Given* $g_k \in C^\mu(\Sigma)$ *such that* $g_k dx^k$ *is homologous to zero, there exists a solution of BVP* (5.6) *and it is determined up to an additive constant. It belongs to* $C^{1,\mu}(\overline{\Omega})$ *and it may be represented in the following way*:

$$u(x) = \int_\Sigma \varphi_h(y) \frac{\partial}{\partial \nu_y} \frac{\partial}{\partial y_h} F(x, y)\, d\sigma_y + a_j F(x, y^j) + c$$

where c, $a_1, \ldots, a_s$ *are constants and* $y^1, \ldots, y^s$ *are in* $\boldsymbol{R}^n - \overline{\Omega}$. *If there are no eigensolutions of the second kind of* (5.2), *then the term* $a_j F(x, y^j)$ *may be suppressed.*

It is possible to show that (5.7) holds if and only if $\int_\Sigma g_k \xi_k d\sigma = 0$, for any $(\xi_1, \ldots, \xi_n)$ eigensolution of the first kind of (5.2). If there are no eigensolutions of the second kind, there exists a multiple layer potential $u(x)$ solution of (5.6). If we write it as (4.2), (ψ, Φ) is solution of (4.4). From (2.7) it follows that

$$(\psi, \Phi) = S_1 \alpha + \beta + \gamma, \tag{5.8}$$

where $\alpha \in L^p(\Sigma) \times L^p(\Sigma)$, $\beta \in \mathcal{N}(S_2 S)$, $\gamma \in \Gamma = [e_1, \ldots, e_s]$, S_1, S_2 are the operators introduced in Theorem VIII. It is an easy matter to verify that we may choose $e_1, \ldots, e_s$ belonging to $C^{\mu}(\Sigma) \times C_1^{\mu}(\Sigma)$. Moreover, if (ψ, Φ) is solution of (4.4), we have: $\alpha + \mathscr{C}\alpha + S_2 S\gamma = S_2 S(g_k \nu_k, g_k dx^k)$. This implies $\alpha \in C^{\mu}(\Sigma) \times C^{\mu}(\Sigma)$, because of known theorems (see [12]). Then a solution of (4.4) is given by (5.8), where $\alpha \in C^{\mu}(\Sigma) \times C^{\mu}(\Sigma)$, $\gamma \in C^{\mu}(\Sigma) \times C_1^{\mu}(\Sigma)$, and $\beta \in \mathfrak{N}(S_2 S)$. But, because of (4.7), $\mathfrak{N}(S) = \mathfrak{N}(S_2 S)$; this means that if (5.8) is solution of (4.7), then $S_1 \alpha + \gamma$ is still solution of the same equation. Consequently there exists a solution of (4.4) belonging to $C^{\mu}(\Sigma) \times C_1^{\mu}(\Sigma)$. Because of Theorem VI, the multiple layer potential $u(x)$ belongs to $H^2(\Omega)$. It is then solution of (5.6). Moreover, in view of known theorems (see [12]), it belongs to $C^{1,\mu}(\overline{\Omega})$. If u' is another solution of (5.6), we have $u - u' \in H^2(\Omega) \cap C^4(\Omega)$, $\Delta_2 \Delta_2 (u - u') = 0$ in Ω, $\partial(u - u')/\partial x_k |_{\Sigma} = 0$, $(k = 1, \ldots, n)$. This implies $u - u' = c$ (see [8]).

Let us suppose now $(\xi_1^j, \ldots, \xi_n^j)$ $(j = 1, \ldots, s)$ are eigensolutions of the second kind. From the proof of X, it follows that we may choose $(\xi_1^j, \ldots, \xi_n^j)$ in $[C^{\mu}(\Sigma)]^n$. There exist $y^1, \ldots, y^s \in \boldsymbol{R}^n - \overline{\Omega}$ such that

$$\det\{v^j(y^i)\}_{i, j = 1, \ldots, s} \neq 0 \tag{5.9}$$

(v^j given by (5.5)). Assume the contrary. Let $t < s$ be the rank of $\{v^j(y^i)\}$. Let $c_1, \ldots, c_{t+1}$ be an eigensolution of the homogeneous system $\sum_{j=1}^{t+1} c_j v^j(y^i) = 0$ $(i = 1, \ldots, s)$. The function $w(x) \equiv \sum_{j=1}^{t+1} c_j v^j(x)$ vanishes in $\boldsymbol{R}^n - \overline{\Omega}$ and thus it is solution of the problem: $w \in C^{2+\mu}(\overline{\Omega}) \cap C^4(\Omega)$, $\Delta_2 \Delta_2 w = 0$ in Ω, $\partial w/\partial x_k |_{\Sigma} = 0$, $(k = 1, \ldots, n)$. Then $w(x) \equiv 0$ in Ω and consequently $\left(\sum_{j=1}^{t+1} c_j \xi_1^j, \ldots, \sum_{j=1}^{t+1} c_j \xi_n^j\right)$ is an eigensolution of the first kind. This is a contradiction. Let us set

$$\tilde{g}_k(x) = g_k(x) - \sum_{j=1}^{s} a_j \partial[F(x, y^j)]/\partial x_k,$$

where $y^1, \ldots, y^s$ satisfy (5.9) and $(a_1, \ldots, a_s)$ is solution of the system

$$\sum_{i=1}^{s} a_i v^j(y^i) = \int_{\Sigma} \xi_k^j g_k \, d\sigma \qquad (j = 1, \ldots, s).$$

It follows that $(\tilde{g}_1, \ldots, \tilde{g}_n) \in [C^{\mu}(\Sigma)]^n$ is orthogonal to any eigensolution of system (5.2) and then there exists a multiple layer potential $u(x)$ solution of (5.6) where g_k is replaced by $\tilde{g}_k$. Arguing as before we obtain the result.

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