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General construction of Banach-Grassmann algebras

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Fisica matematica. — *General construction of Banach-Grassmann algebras.* Nota di VLADIMIR G. PESTOV, presentata (*) dal Corrisp. C. Cercignani.

ABSTRACT. — We show that a free graded commutative Banach algebra over a (purely odd) Banach space E is a Banach-Grassmann algebra in the sense of Jadczyk and Pilch if and only if E is infinite-dimensional. Thus, a large amount of new examples of separable Banach-Grassmann algebras arise in addition to the only one example previously known due to A. Rogers.

KEY WORDS: Banach-Grassmann Algebras; Superanalysis; Graded algebras.

RIASSUNTO. — *Costruzione generale di algebre di Banach-Grassmann.* Si mostra che un'algebra di Banach libera graduato-commutativa su uno spazio di Banach E puramente dispari è un'algebra di Banach-Grassmann nel senso di Jadczyk e Pilch se e solo se E ha dimensione infinita. È quindi possibile ottenere un gran numero di nuovi esempi di algebre di Banach-Grassmann separabili, in aggiunta all'unico esempio precedentemente noto, dovuto ad A. Rogers.

1. INTRODUCTION

In superanalysis one is supposed to have at hand a ground algebra serving as a supply of odd (anticommuting) constants [1-7]. For general reasons, such an algebra Λ is assumed to be a Hausdorff topological associative unital graded commutative algebra [4], and as a rule, a locally convex one [6]. A natural requirement that any «supernumber», x, should decompose into the body (number) part, x_B, and the soul (nilpotent) part, x_S, imposes upon Λ the condition of being a local algebra [5]. The property of convergence of the so-called superfield expansion [7] (= Grassmann analytic continuation [8]) at least in the analytic case actually restricts the class of ground algebras to the complete locally multiplicatively convex algebras in the sense of [9, 10], and this way one comes to the notion of a graded local Arens-Michael, or GLAM, algebra ([11]; cf. also [12]). Numerous examples of GLAM algebras can be found in [11, 13]; all concrete algebras of «supernumbers» [1-8] fit into that class.

Among particularly convenient properties of ground algebras is the Jadczyk-Pilch self-duality property introduced in [14] for graded commutative Banach algebras. The Banach algebras with that property – the so-called Banach-Grassmann algebras – have become popular recently [15, 16]. However, the Rogers algebra B_∞ [17] still remains actually the only example of a Banach-Grassmann algebra. There are also some other examples [13, 18] but they are unseparable and thus «too big».

The present *Note* adds a large amount of new examples of separable Banach-Grassmann algebras. They are just exterior algebras over Banach spaces endowed with a relevant norm and completed after that. We call this construction «a free graded commutative Banach algebra over a Banach space». It was proposed by us ear-

(*) Nella seduta del 14 marzo 1992.

lier [13]; here we study the structure of such algebras in more detail and show that a free graded commutative Banach algebra $\bigwedge_B E$ over a (purely odd) Banach space E is a Banach-Grassmann algebra in the sense of [14] if and only if $\dim E = \infty$.

Since $\bigwedge_B l_1 = B_\infty$ then our result can be viewed as an extension of a theorem from [19]. However, the method of proof used in [19] cannot be extended beyond the particular case $E = l_1$. For unseparable E's, the result has been stated by us earlier [13]; its generalization to a separable case is not quite trivial.

2. Preliminaries

(2.1). A graded-commutative algebra [2-10] Λ is an associative algebra over the basic field $\boldsymbol{K}$ with a fixed vector space decomposition $\Lambda \cong \Lambda^0 \oplus \Lambda^1$, where Λ^0 is called the *even* and Λ^1 the *odd part (sector)* of Λ, in such a way that the *parity* $\tilde{x}$ of any element $x \in \Lambda^0 \cup \Lambda^1$ defined by letting $x \in \Lambda^{\tilde{x}}$, $\tilde{x} \in \{0, 1\} = \boldsymbol{Z}_2$, meets the following restrictions:

(2.1.1) $$\widetilde{xy} = \tilde{x} + \tilde{y},\ x,\ y \in \Lambda^0 \cup \Lambda^1$$

(2.1.2) $$xy = (-1)^{\tilde{x}\tilde{y}} yx,\ x,\ y \in \Lambda^0 \cup \Lambda^1$$

(2.2) By a *normed algebra* we mean an algebra Λ together with a fixed *submultiplicative* norm on it, $\|\cdot\|_\Lambda$; the *submultiplicativity* of the norm [22] means that $\|xy\|_\Lambda \leq \|x\|_\Lambda \|y\|_\Lambda$ for all $x,\ y \in \Lambda$. For a unital algebra Λ this condition implies $\|1_\Lambda\| = 1$.

(2.3) Recall that an l_1 (resp. l_∞, or c_0) type sum [20] of a family of normed spaces $\{E_\alpha, \alpha \in A\}$ is the Banach space completion of the linear space $\bigoplus_{\alpha \in A} E_\alpha$ endowed with the norm $\|x\| := \sum_{\alpha \in A} \|x_\alpha\|_{E_\alpha}$ (respectively, $\|x\| := \sup_{\alpha \in A} \|x_\alpha\|_{E_\alpha}$).

(2.4) A *Banach-Grassmann algebra* [14] is a complete normed associative unital graded commutative algebra Λ satisfying the following two conditions.

BG_1 (*Jadczyk-Pilch self-duality*). For any $r, s \in \boldsymbol{Z}_2 = \{0, 1\}$ and any bounded Λ^0-linear operator $T\colon \Lambda^r \to \Lambda^s$ there exists a unique element $a \in \Lambda^{r+s}$ such that $Tx = ax$ whenever $x \in \Lambda^r$. In addition, $\|a\|$ equals the operator norm $\|T\|_{op}$ of T.

BG_2. The algebra Λ decomposes into an l_1 type sum $\Lambda \simeq \boldsymbol{K} \oplus J_\Lambda^0 \oplus \Lambda^1$ where $\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{C}$ and J_Λ^0 is the even part of the closed ideal J_Λ topologically generated by the odd part Λ^1. In other words, for an arbitrary $x \in \Lambda$ there exist elements $x_B \in \boldsymbol{K}$, $x_S^0 \in J_\Lambda^0$, and $x^1 \in \Lambda^1$ such that $x = x_B + x_S^0 + x^1$ and $\|x\| = \|x_B\| + \|x_S^0\| + \|x^1\|$.

(2.5) As it was noted in [13], it suffices to verify the condition BG_1 in the case $r = 1$ only. Furthermore, denote by $L_{\Lambda^0}(\Lambda^1, \Lambda)$ the totality of all bounded Λ^0-linear operators from Λ^1 to Λ [14]. It is convenient to split the Jadczyk-Pilch self-duality condition BG_1 into the two ingredients. Denote by $\rho_\Lambda\colon \Lambda \to L_{\Lambda^0}(\Lambda^1, \Lambda)$ the left regular representation of Λ defined by letting $\rho_\Lambda(x)(\xi) = x\xi$. The condition BG_1 is equivalent to the following:

JP) ρ_Λ is an isometric isomorphism of Λ onto $L_{\Lambda^0}(\Lambda^1, \Lambda)$,

or, in more detail:

JP_1) ρ_Λ is an isometric embedding,

JP_2) ρ_Λ is onto.

(2.6) We say that a Hausdorff topological associative unital graded commutative algebra Λ is a *supernumber algebra (SN algebra; SNA)* if it admits a decomposition into a topological direct sum $\Lambda \simeq \boldsymbol{K} \oplus J_\Lambda$ where J_Λ, as above, is a closed ideal topologically generated by the odd part Λ^1. In other terms, Λ is a local graded-commutative topological algebra such that the maximal ideal J_Λ is topologically generated by the odd part. Such an algebra admits a unique (continuous) character $\beta_\Lambda : \Lambda \to \boldsymbol{K}$ called the *body map*. See [13] for a more detailed treatment of *SN* algebras.

(2.7) Assertion. [13] *A Banach graded commutative algebra Λ admits a norm satisfying the condition BG_2 iff Λ is an SN algebra.*

(2.8) Corollary. *A Banach graded commutative algebra Λ admits a norm making it into a Banach-Grassmann algebra iff Λ is an SN algebra meeting the condition JP.*

(2.9) Let E and F be any two normed spaces. Their *weak tensor product* [21] is the completion $E \check{\otimes} F$ of the algebraic $\boldsymbol{K}$-tensor product $E \otimes_{\boldsymbol{K}} F$ endowed with the *uniform cross norm* defined as follows:

$$\|u\|_{E \check{\otimes} F} := \sup\{|(f \otimes g)(u)| : f \in E', \|f\|_{op} \leq 1, g \in F', \|g\|_{op} \leq 1\}.$$

Clearly, for each $x \in E$, $y \in F$ the following holds: $\|x \otimes y\|_{E \check{\otimes} F} = \|x\|_E \cdot \|y\|_F$.

(2.10). If A and B are normed algebras then so is $A \check{\otimes} B$ [22]. If, moreover, both A and B are graded commutative unital complete normed algebras then $A \check{\otimes} B$ is so as well.

(2.11). Remark that if A and B are normed unital algebras then $A \check{\otimes} B$ contains their l_∞ type sum $A \oplus_{l_\infty} B$ as a normed linear subspace under an isometric embedding $a \oplus b \mapsto a \oplus 1_B + 1_A \oplus b$. As a corollary of this really obvious remark, for any pair of normed subspaces $E \hookrightarrow A$, $F \hookrightarrow B$ the l_∞ type sum $E \oplus_{l_\infty} F$ embeds into $A \check{\otimes} B$ isometrically in a canonical way.

(2.12). For an element a of an algebra Λ we denote by ${}^\perp a$ the *left annihilator* of a; this is the set $\{x \in \Lambda : xa = 0\}$. This is an ideal in Λ. If Λ is a topological graded commutative algebra then ${}^\perp a$ is a closed graded ideal. See [3, 6].

We say that a graded commutative algebra Λ is *effective* if $\bigcap\{{}^\perp a : a \in \Lambda^1\} = (0)$. Clearly, this is precisely the case where the representation ρ_Λ (2.5) is effective.

(2.13). Assertion. ([13]; cf. also [19, 6]). *Let Λ be an effective graded commutative Banach algebra and let $T \in L_{\Lambda^0}(\Lambda^1, \Lambda)$. Then for any $a \in \Lambda^1$, $T(a) \in {}^\perp a$.*

(2.14). ASSERTION. [13] *Let Λ and T be as above and let $a, b \in \Lambda^1$. Then $aT(b) = \tilde{T}T(a)b$.*

3. FREE GRADED COMMUTATIVE BANACH ALGEBRAS

(3.1). THEOREM. (Announced in [13] without a proof.) *Let E be a normed space. There exists a complete normed associative unital graded commutative algebra $\bigwedge_B E$ with the following properties.*

1) *$\bigwedge_B(E)$ contains B as a normed subspace of the odd part $(\bigwedge_B E)^1$ in such a way that $E \cup \{1\}$ topologically generates $\bigwedge_B E$.*

2) *Every linear operator f from E to the odd part Λ^1 of a complete normed associative unital graded commutative algebra Λ with a norm $\|f\|_{op} \leq 1$ extends to an even homomorphism $\hat{f}\colon \bigwedge_B E \to \Lambda$ with a norm $\|\hat{f}\|_{op} \leq 1$.*

Such an algebra $\bigwedge_B E$ is unique up to an even isometric isomorphism. Moreover, $\bigwedge_B E$ is a supernumber algebra.

PROOF. Let $\wedge E = \bigoplus_{n=0}^{\infty} \bigwedge^n E$ be the exterior algebra over E (to be more pedantic, what we need is rather a symmetric algebra over a purely odd linear space $(0) \oplus E$, see [23, Ch. 3], but such ideological subtleties do not affect the reasoning that much). Endow each n-th exterior power $\bigwedge^n E$, $n \in \mathbf{N}$ with the maximal norm making it into a normed space in such a way that for every $i = 0, 1, \ldots, n-1$ and every $x \in \bigwedge^i E$, $y \in \bigwedge^{n-i} E$ the following holds: $\|x \wedge y\|_{\bigwedge^n E} \leq \|x\|_{\bigwedge^i E} \|y\|_{\bigwedge^{n-i} E}$. To convince oneself that there is indeed at least one norm with such a property, consider the canonical anti-symmetrization map from the n-th tensor power $E^{\otimes n}$ onto $\bigwedge^n E$ and endow the latter space with the quotient norm of the cross norm $\|\cdot\|_E \hat{\otimes} \|\cdot\|_E \hat{\otimes} \ldots \hat{\otimes} \|\cdot\|_E$ (n times). By the way, the norm one comes to is precisely the desired maximal norm on $\bigwedge^n E$. Similar constructions have been performed, say, in [24], where the uniform cross norm $\|\cdot\|_E \check{\otimes} \ldots \check{\otimes} \|\cdot\|_E$ is used, and in [25] for Hilbert spaces E only.

The completion of $\bigwedge^n E$ relative the norm defined above will be denoted by $\bigwedge^n_B E$.

Now denote by $\bigwedge_B E$ the l_1 type sum of all the $\bigwedge^n_B E$'s, $n \in \mathbf{N}$. A little effort is needed to observe that the norm on that l_1 type sum is the maximal one making $\bigwedge_B E$ into a (complete) normed algebra in such a way that E is a normed subspace of $\bigwedge_B E$. The desired universality property follows from this latter observation more or less directly. (Hint: no submultiplicative *prenorm* on $\bigwedge_B E$, whose restriction to E is less than or equal to $\|\cdot\|_E$, exceeds $\|\cdot\|_{\bigwedge_B E}$ at some point). Both properties of essential uniqueness and of being an *SN* algebra are obvious.

(3.2). EXAMPLE. The algebra $\bigwedge_B l_1$ is just the Rogers algebra B_∞ [17]. Its nonseparable analogues of the type $\bigwedge_B l_1(\Gamma)$ have been considered in [18, 26]. For a finite dimensional E the algebra $\bigwedge_B E$ is an ordinary Grassmann algebra, $\bigwedge_B \boldsymbol{K}^n \simeq \bigwedge(n)$.

(3.3). ASSERTION. *For a normed space E, the condition* $\dim E = \infty$ *is equivalent to the fact that the left regular representation* $\rho_{\bigwedge_B E}: \bigwedge_B E \to L_{(\bigwedge_B E)^0}((\bigwedge_B E)^1, \bigwedge_B E)$ *is an isometric embedding.*

PROOF. The «if» part stems from the observation that for a Grassmann algebra $\bigwedge(q)$ the map $\rho_{\bigwedge(q)}$ merely is not an injection. To prove the «only if» part, we establish the following somewhat stronger result. Let $\dim E = \infty$. Then for an arbitrary $x \in \bigwedge_B E$ and each $\varepsilon > 0$, there is an $y \in E$ with $\|xy\|_{\bigwedge_B E} \geq \|x\|_{\bigwedge_B E}\|y\|_E - \varepsilon$.

Let $x \in \bigwedge_B E$ and $\varepsilon > 0$. Assume without loss of generality that $\|x\|_{\bigwedge_B E} \leq 1$. There is a unique representation of x as a sum $x = \sum_{n=0}^{\infty} x_n$ where $x_n \in \overset{n}{\bigwedge_B} E$ and $\|x\| = \sum_{n=0}^{\infty} \|x_n\|$ (see the proof of 3.1 above). Fix N with $\left\| \sum_{n=N+1}^{\infty} x_n \right\| < \varepsilon/3$. For every $n = 0, 1, \ldots, N$ there are elements $x_n' \in \overset{n}{\bigwedge} E$ such that $\|x_n - x_n'\| < \varepsilon/(3N+3)$. Fix a finite subset $z_1, \ldots, z_k \in E$ with the property: all the elements x_n', $n = 1, \ldots, N$ are in a subalgebra generated by $\{z_1, \ldots, z_k\}$. Without loss of generality, one may assume that $\|x_i\| = 1$, $i = 1, \ldots, k$. Put $x' = \sum_{n=0}^{N} x_n'$.

Thanks to the infinite dimensionality of E, there exists a nontrivial continuous linear functional $f \in E'$ with $\|f\|_{op} = 1$ and $f(z_i) = 0$, $i = 1, \ldots, k$. Fix an $y \in E$ such that $\|y\| \leq 1 + \varepsilon/3$ and $\|f(y)\| = 1$. The map F sending each $a \in E$ to the pair $(a - f(a)y) \oplus \oplus f(a)$ is a contracting linear operator from E to the l_∞ type sum $H \bigoplus_{l_\infty} \boldsymbol{K}^1$ where $H = \ker f$.

The contracting linear operator F from E to the space $H \bigoplus_{l_\infty} \boldsymbol{K}^1$ canonically embedded (2.11) into the (odd part of) the weak tensor product algebra $\bigwedge_B H \,\check{\otimes}\, \bigwedge_B \boldsymbol{K}^1 \simeq \simeq \bigwedge_B H \,\check{\otimes}\, \bigwedge(\xi)$ (here ξ stands for the element $1_{\boldsymbol{K}}$ of the one-dimensional linear space $\boldsymbol{K}^1$) extends to a contracting even homomorphism $\widehat{F}: \bigwedge_B E \to \bigwedge_B H \,\check{\otimes}\, \bigwedge(\xi)$ (3.1). Now one has a chain of simple majorations: $\|xy\|_{\bigwedge_B E} \geq \|\widehat{F}(x'y)\|_{\bigwedge_B H \check{\otimes} \bigwedge(\xi)} = \|\widehat{F}(x')\|_{\bigwedge_B H} \times \times \|\xi\|_{\bigwedge(\xi)} \geq \sum_{n=0}^{N} \|x_n'\|_{\overset{n}{\bigwedge_B} E} = \|x'\|_{\bigwedge_B E} \geq \|x\|_{\bigwedge_B E}\|y\|_E - \varepsilon$.

(3.4). ASSERTION. [13] *A free graded commutative Banach algebra* $\bigwedge_B E$ *is effective iff* $\dim E = \infty$.

(3.5). Remark that *an algebra* $\bigwedge_B E$ *is separable if and only if so is E.* The proof is very similar to the demonstration valid in the case of free topological groups [27].

4. Berezin topology

(4.1). Let E be a normed space. Denote by π_E^n the canonical projection map from the exterior algebra $\wedge E$ onto the n-th exterior power $\overset{n}{\wedge} E$. By the *Berezin topology* on $\wedge E$ (resp. $\wedge_B E$) we mean the projective topology (see [21, Ch. 1]) with respect to the family of maps $\pi_E^n\colon \wedge E \to \overset{n}{\wedge}_B E$ (resp. $\pi_E^n\colon \wedge_B E \to \overset{n}{\wedge}_B E$). In other words, sets of the form $(\pi_E^n)^{-1} U$ where $n \in \mathbf{N}$ and U is open in $\overset{n}{\wedge}_B E$ form a base for the Berezin topology.

(4.2). The completion of $\wedge E$ w.r.t. the Berezin topology is denoted by $\wedge_{\mathrm{Ber}} E$. There is a canonical continuous even monomorphism $i_E\colon \wedge_B E \hookrightarrow \wedge_{\mathrm{Ber}} E$, whose restrictions to the n-th exterior powers $i_E^n\colon \overset{n}{\wedge}_B E \to \overset{n}{\wedge}_{\mathrm{Ber}} E$ are homeomorphisms. Actually, the algebra $\wedge_{\mathrm{Ber}} E$ is isomorphic, as a locally convex space, to the Tychonoff product of all the Banach exterior powers of E, namely, $\wedge_{\mathrm{Ber}} E \simeq \mathop{\times}\limits_{n=0}^{\infty} \overset{n}{\wedge}_B E$; thus, elements of $\wedge_{\mathrm{Ber}} E$ are just arbitrary formal series of the type $\sum_{n=0}^{\infty} x_n$, $x_n \in \overset{n}{\wedge}_B E$.

(4.3). Here is still another description of the Berezin topology. A sequence $(x_k)_{k \in \mathbf{N}}$ of elements of the algebra $\wedge_{\mathrm{Ber}} E$ converges in $\wedge_{\mathrm{Ber}} E$ (to an element x) if and only if for each $n \in \mathbf{N}$ the sequence $(\pi_E^n x_k)_{k \in \mathbf{N}}$ converges in $\overset{n}{\wedge}_B E$ (to an element $\pi_E^n x$).

(4.4). Assertion. *Let E be a normed space. Suppose an element $x \in \wedge_{\mathrm{Ber}} E$ is such that the operator of the left multiplication by x maps $(\wedge_B E)^1$ to $\wedge_B E$ and it is continuous w.r.t. the norm topology on $\wedge_B E$. Then $x \in \wedge_B E$.*

Proof. Let $x \in \wedge_{\mathrm{Ber}} E \setminus \wedge_B E$, that is, $x = \sum_{n=0}^{\infty} x_n$ where $x_n \in \overset{n}{\wedge}_B E$ and $\sum_{n=0}^{\infty} \|x_n\| = +\infty$. Using 3.4, pick for every $k \in \mathbf{N}$ an element $y_k \in E$ such that $\|y_k\|_E \le 1$ and

$$\left\| \left(\sum_{n=0}^{k} x_n \right) y_k \right\|_{\overset{k+1}{\wedge}_B E} \ge \left\| \sum_{n=0}^{k} x_n \right\|_{\overset{k}{\wedge}_B E} - 1/k.$$

It is easy to see that $\|xy\| \ge \left\| \left(\sum_{n=0}^{k} x_n \right) y_k \right\| \to \infty$ as $k \to \infty$. Thus, the operator of the left multiplication by x sends a subset $\{y_k : k \in \mathbf{N}\}$ of the unit ball in $(\wedge_B E)^1$ to an unbounded subset $\{xy_k : k \in \mathbf{N}\}$ of the space $\wedge_B E$ and hence is discontinuous, in contradiction to the conditions of Assertion.

(4.5). Comment. A topology on an exterior algebra called by us the Berezin topology was considered originally in a more general context by F. A. Berezin [28, 1.3.3]. The algebra $\wedge_{\mathrm{Ber}} E$ makes sense for an arbitrary locally convex space E (see [11, Sec. 2]). The most widely known example of a graded commutative algebra endowed with the Berezin topology is the De Witt supernumber algebra Λ_∞ [6, 7] isomorphic to the algebra $\bigwedge_{\mathrm{Ber}} \boldsymbol{K}^\omega$.

5. Self-duality

(5.1). Assertion. (cf. [6, 13, 19]) *Let E be an infinite-dimensional normed space and let $a \in E$. Then the annihilator ${}^\perp a$ in the algebra $\bigwedge_B E$ coincides with $a \bigwedge_B E$.*

Proof. Let $x \in {}^\perp a$; it may be assumed that $x \in \overset{n}{\underset{B}{\bigwedge}} E$ for an $n \in \boldsymbol{N}$. We represent x as $\sum x_{i_1} x_{i_2} \ldots x_{i_n}$ where $x_{i_j} \in E$. Now it remains to note that thanks to the infinite dimensionality of E, for arbitrary linearly independent $y_1, \ldots, y_k \in E$ their (wedge) product does not vanish.

(5.2). Corollary. *Let E be an infinite dimensional normed space. Let $a_1, \ldots, a_n \in E$ Then ${}^\perp a_1 \cap {}^\perp a_2 \cap \ldots \cap {}^\perp a_n = a_1 a_2 \ldots a_n \bigwedge_B E$ in $\bigwedge_B E$.*

(5.3). Lemma. *Let E be an infinite dimensional normed space and let $T \in$ $\in L_{(\bigwedge_B E)^0}((\bigwedge_B E)^1, \bigwedge_B E)$. Then there exists $x \in \bigwedge_{\mathrm{Ber}} E$ such that $xa = T(a)$ for all $a \in$ $\in (\bigwedge_B E)^1$.*

Proof. Choose a sequence of linearly independent elements $a_1, a_2, \ldots, a_n, \ldots$ in E.

Assertions 2.13 and 3.4 imply that $T(a_1) \in {}^\perp a_1$; by virtue of 5.1, there is $b_1 \in \bigwedge_B E$ with $T(a_1) = b_1 a_1$.

Suppose that for an $n \in \boldsymbol{N}$ elements $b_1, \ldots, b_n \in \bigwedge_B E$ have been chosen in such a way that for every $i = 1, \ldots, n$ one has

$$(b_1 + a_1 b_2 + a_1 a_2 b_3 + \ldots + a_1 a_2 \ldots a_{n-1} b_n) a_i = T(a_i).$$

Consider an operator T_{n+1} defined by letting $T_{n+1}(x) := T(x) - (b_1 + a_1 b_2 + \ldots +$ $+ a_1 \ldots a_{n-1} b_n) x$. It is easy to see that $T_{n+1}(a_i) = 0$ for all $i = 1, \ldots, n$ and $T_{n+1}(a_{n+1}) \in$ $\in {}^\perp a_{n+1}$ (use 2.13 and 3.5 together with the boundedness and $(\bigwedge_B E)^0$-linearity of T_{n+1}). This implies that $T_{n+1}(a_{n+1}) \in {}^\perp a_i$ for all $i = 1, \ldots, n+1$ (use 2.14) and hence there is $b_{n+1} \in \bigwedge_B E$ such that $T_{n+1}(a_{n+1}) = a_1 a_2 \ldots a_n b_{n+1} a_{n+1}$ (use 5.2). Now it is obvious that $(b_1 + a_1 b_2 + \ldots + a_1 a_2 \ldots a_n b_{n+1}) a_i = T(a_i)$ for all $i = 1, \ldots, n+1$. The recursion step thus is performed.

Denote $x_n := b_1 + a_1 b_2 + \ldots + a_1 \ldots a_n b_{n+1}$ for every $n \in \boldsymbol{N}$. Since for every $n \in \boldsymbol{N}$ one has $\pi_E^n (a_1 \ldots a_{n+1} b_{n+2}) = 0$ then the sequence $(\pi_E^n x_k)_{k \in \boldsymbol{N}}$ stabilizes in $\overset{n}{\underset{B}{\bigwedge}} E$ for every fixed $n \in \boldsymbol{N}$, that is, all the elements of it coincide pairwise for $k > n$. By force of 4.3, the sequence $(x_n)_{n \in \boldsymbol{N}}$ converges to some $x \in \bigwedge_{\mathrm{Ber}} E$. It is clear that for every $n \in \boldsymbol{N}$, $T(a_n) = x a_n$. Finally, taking into account that $\bigcap_{n \in \boldsymbol{N}} {}^\perp a_n = (0)$ and arguing as in [13, Sect. 7], with the help of 2.14, one deduces that $T(a) = xa$ for an arbitrary $a \in \bigwedge_B E$.

(5.4). MAIN THEOREM. *Let E be a normed space. Then the free graded commutative Banach algebra* $\bigwedge_B E$ *is a Banach-Grassmann algebra in the sense of Jadczyk and Pilch if and only if* $\dim E = \infty$.

PROOF. Combine 2.4, 2.6, 3.2, 3.4, 4.4, and 5.3.

CONCLUSION

In our view, it should be interesting now to extend the concept of Jadczyk-Pilch self-duality beyond the Banach case (for example, in order to make it applicable to any GLAM algebra in the sense of [11]). Some aspects of an extension are discussed in [5, 6]. However, while the properties BG_2, JP_2 and partly JP_1 are readily amenable to such a generalization, it is not quite clear how to generalize the property of ρ_Λ being an isometry, and hence there is still some way to go.

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