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Function spaces of Nikolskii type on compact manifold

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Analisi matematica. — *Function spaces of Nikolskii type on compact manifolds.*
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ABSTRACT. — Nikolskii spaces were defined by way of translations on $\boldsymbol{R}^n$ and by way of coordinate maps on a differentiable manifold. In this paper we prove that, for functions with compact support in $\boldsymbol{R}^n$, we get an equivalent definition if we replace translations by all isometries of $\boldsymbol{R}^n$. This result seems to justify a definition of Nikolskii type function spaces on riemannian manifolds by means of a transitive group of isometries (provided that one exists). By approximation theorems, we prove that – for homogeneous spaces of compact connected Lie groups – our definition is equivalent to the classical one.

KEY WORDS: Nikolskii spaces; Isometry groups; Compact homogeneous spaces.

RIASSUNTO. — *Spazi funzionali del tipo di Nikolskii su varietà compatte.* Gli spazi di Nikolskii sono stati definiti in $\boldsymbol{R}^n$ tramite le traslazioni e su varietà differenziabili mediante carte locali. In questa *Nota* si dimostra che, per funzioni a supporto compatto in $\boldsymbol{R}^n$, si ottiene una definizione equivalente sostituendo le traslazioni con tutte le isometrie dirette. Ciò giustifica la definizione tramite isometrie, che viene qui proposta successivamente per spazi del tipo di Nikolskii su spazi omogenei di gruppi di Lie compatti e connessi. Una caratterizzazione mediante approssimazione permette infine di dimostrare che, per tali varietà, la definizione qui proposta è equivalente a quella usuale.

INTRODUCTION

Several classes of function spaces on $\boldsymbol{R}^n$ can be defined by means of translations. Then the question might arise: if we replace the translations by all the isometries of $\boldsymbol{R}^n$, do we get an equivalent definition?

In this paper we will consider a family of Nikolskii spaces on $\boldsymbol{R}^n$ and give (in n. 1) an affirmative answer to the previous question, for functions of compact support.

This result suggests to define (as we do in n. 2) Nikolskii spaces on a riemannian manifold by way of the isometries of the manifold, provided that they are – shall we say – enough. Nikolskii spaces on a C^∞ manifold are usually defined by coordinate maps. Here we propose the definition by isometries only for a more restricted class of manifolds, namely the homogeneous spaces of compact connected Lie groups (among which there are, for example, spheres and tori). However, for these spaces, our definition is equivalent to the classical one (as we prove in n. 3); moreover, it allows us to use the invariance property of the Laplace-Beltrami operator of the manifold with respect to the isometries. This fact could prove useful in some applications to evolution problems on riemannian manifolds (see the last remark of the paper).

The result of the equivalence quoted in n. 1 is based on a characterization of Nikolskii functions in $\boldsymbol{R}^n$, which we report as Theorem 1. A similar characterization for the function spaces which we have introduced on manifolds is given in Theorem 3. Both results are used in n. 3 to show that – for compact homogeneous spaces – our definition and the classical one are in fact equivalent.

(*) Nella seduta dell'8 febbraio 1992.

1. – NOTATION. Let $x=(x_1,\dots,x_n)\in \boldsymbol{R}^n$, $A=[a_{ij}]$ be a real $n\times n$ matrix, r be a real positive number and $f\in L^1(\boldsymbol{R}^n)$. Then we define:

$$|x|=\left(\sum_{i=1}^n x_i^2\right)^{1/2},\qquad |A|=\left(\sum_{i,j=1}^n a_{ij}\right)^{1/2},\qquad B(0,r)=\{x\in \boldsymbol{R}^n\mid |x|<r\},$$

$$\int f(x)\,dx=\int_{\boldsymbol{R}^n} f(x)\,dx,\qquad \|f\|=\int |f(x)|\,dx.$$

In this Section we consider the following Nikolskii spaces (see, *e.g.*, [4, § 8.2] and [9, § 8.4]).

DEFINITION 1. *For every* $\lambda\in]0,1[$ *let*

$$N^\lambda(\boldsymbol{R}^n)=N^\lambda=\Big\{f\in L^1(\boldsymbol{R}^n)\mid \sup_{h\in \boldsymbol{R}^n,h\neq 0}|h|^{-\lambda}\|f(x+h)-f(x)\|<+\infty\Big\}.$$

The space N^λ can be characterized by the following Theorem 1. The proof we expose here is a discretisation of the one given by [12, Ch. III, Th. 4]; see, also, [11, V. 4.2] and [10, Lemma 2.2]. For a more general statement, see, for example, [2, 6.2] and [8, Ch. II, 9].

THEOREM 1. $f\in N^\lambda$ *if and only if there is a sequence* $(f_j)_{j=0,1,\dots}$ *of functions in* $C^\infty\cap W^{1,1}(\boldsymbol{R}^n)$ *satisfying the conditions*:

(*i*) $$f=\sum_{j=0}^{+\infty} f_j \quad \text{in } L^1(\boldsymbol{R}^n);$$

(*ii*) *there exists* $c>0$ *such that, for every* j,
$$\|f_j\|\leqslant c2^{-\lambda j},\qquad \|D_k f_j\|\leqslant c2^{(1-\lambda)j}\qquad k=1,\dots,n.$$

PROOF. Assume that $f\in N^\lambda$. Let φ be a smooth function, supported on $B(0,1)$, such that $\varphi\geqslant 0$ and $\int\varphi(x)\,dx=1$.

Define $\varphi_j(x)=2^{jn}\varphi(2^jx)$ $(j=0,1,2,\dots)$; $\psi_j=\varphi_j-\varphi_{j-1}$ $(j=1,2,\dots)$; $\psi_0=\varphi_0=\varphi$. In this way we have:

$$\text{for every } j>0,\quad \int\psi_j(x)\,dx=0,\qquad \operatorname{supp}\psi_j\subset B(0,2^{-j+1}),\qquad \int D_k\psi_j(x)\,dx=0;$$

there exists $C>0$ such that for every $j\geqslant 0$, for every $k=1,\dots,n$, $\|D_k\psi_j\|\leqslant C2^j$.

Let $f_j=f*\psi_j$ $(j=0,1,\dots)$.

Then $f_j\in C^\infty\cap W^{1,1}(\boldsymbol{R}^n)$ and $f=\sum_j f_j$ in $L^1(\boldsymbol{R}^n)$.

Moreover, since $f\in N^\lambda$, there is a constant $M>0$ such that, for every $h\in \boldsymbol{R}^n$, $\|f(x+h)-f(x)\|\leqslant M|h|^\lambda$; hence, for every $j\neq 0$,

$$\|f_j\|=\int\left|\int f(x-y)\,\psi_j(y)\,dy\right|dx=\int\left|\int (f(x-y)-f(x))\,\psi_j(y)\,dy\right|dx\leqslant$$

$$\leqslant\int|\psi_j(y)|\int|f(x-y)-f(x)|\,dx\,dy\leqslant\int|\psi_j(y)|\,M|y|^\lambda\,dy\leqslant 2M2^{(-j+1)\lambda}=c_1 2^{-j\lambda}.$$

Similarly, we estimate the norms $\|D_k f_j\|$ for every $k = 1, 2, \ldots, n$, and we obtain

$$\|D_k f_j\| = \|f * D_k \psi_j\| = \int \left| \int (f(x-y) - f(x))\, D_k \psi_j(y)\, dy \right| dx \leqslant$$

$$\leqslant \int |D_k \psi_j(y)|\, M|y|^\lambda\, dy \leqslant C 2^j M 2^{(-j+1)\lambda} = c_2 2^{(1-\lambda)j}.$$

Conversely, let

$$f = \sum_{j=0}^{+\infty} f_j,$$

where the f_j satisfy the conditions of Theorem 1.

For every $j > 0$, for every $h = (h_1, \ldots, h_n)$ we have:

$$\|f_j(x+h) - f_j(x)\| = \int \left| \sum_{k=1}^{n} \int_0^1 D_k f_j(x + th)\, h_k\, dt \right| dx \leqslant$$

$$\leqslant |h| \int_0^1 \left(\sum_{k=1}^{n} \int |D_k f_j(x+th)|\, dx \right) dt = |h| \sum_{k=1}^{n} \int |D_k f_j(x)|\, dx.$$

So we obtain

$$\|f_j(x+h) - f_j(x)\| \leqslant \begin{cases} 2c2^{-\lambda j} \\ cn|h|\, 2^{(1-\lambda)j} \end{cases} \quad \forall j.$$

Let j_0 be such that $2^{j_0} \leqslant 2/(n|h|) < 2^{j_0+1}$. (Thus $cn|h|\, 2^{(1-\lambda)j} \leqslant 2c2^{-\lambda j}$ for $j \leqslant j_0$; $cn|h|\, 2^{(1-\lambda)j} > 2c2^{-\lambda j}$ for $j > j_0$).

This implies that, for every $h \in \boldsymbol{R}^n$,

$$\|f(x+h) - f(x)\| \leqslant \sum_{j=0}^{+\infty} \|f_j(x+h) - f_j(x)\| \leqslant \sum_{j=0}^{j_0} cn\,|h|\, 2^{(1-\lambda)j} + \sum_{j=j_0+1}^{+\infty} 2c2^{-\lambda j} \leqslant$$

$$\leqslant cn|h| \sum_{j=-\infty}^{j_0} 2^{(1-\lambda)j} + 2c \sum_{j=j_0+1}^{+\infty} 2^{-\lambda j} = cn|h|\,(2^{-j_0})^{\lambda-1} \left(\sum_{j=0}^{+\infty} 2^{(\lambda-1)j} \right) +$$

$$+ 2c2^{-\lambda(j_0+1)} \left(\sum_{j=0}^{+\infty} 2^{-\lambda j} \right) = c_1\,|h|\,(2^{j_0})^{1-\lambda} + c_2\,(2^{j_0+1})^{-\lambda} \leqslant$$

$$\leqslant c_1\,|h| \left(\frac{2}{n\,|h|} \right)^{1-\lambda} + c_2 \left(\frac{n|h|}{2} \right)^{\lambda} \leqslant M|h|^\lambda.$$

Therefore $f \in N^\lambda$.

Now let us introduce new function spaces, which are closely related to the Nikolskii spaces N^λ. They are defined by replacing, in Definition 1, the group of translations with the identity component $SO(n) \rtimes \boldsymbol{R}^n$ in the group of all isometries of the euclidean space $\boldsymbol{R}^n$.

Let G be the group $SO(n) \rtimes \boldsymbol{R}^n$ and V be the Lie algebra of $SO(n)$.

We define $d(A) = \min\{|X| \mid X \in V \text{ and } \exp X = A\}$ for every $A \in SO(n)$; $|(A, b)| = d(A) + |b|$ for every $(A, b) \in G$.

DEFINITION 2. *For every* $\lambda \in]0,1[$, *let*

$$E^\lambda = \left\{ f \in L^1(\boldsymbol{R}^n) \mid \sup_{(A,b) \in G, (A,b) \neq (I,0)} |(A,b)|^{-\lambda} \| f(Ax+b) - f(x) \| < +\infty \right\}.$$

Evidently, $E^\lambda \subseteq N^\lambda$.

Conversely, we have

THEOREM 2. *Let supp f be compact. Then $f \in E^\lambda$ if and only $f \in N^\lambda$.*

PROOF. Let us suppose that $f \in N^\lambda$ and supp f is compact. It is sufficient to show that

$$\sup_{A \in SO(n), A \neq I} |(A,0)|^{-\lambda} \| f(Ax) - f(x) \| < +\infty .$$

Let us write $f = \sum_j f_j$ as in Theorem 1.

Since supp f is compact, we may choose $r > 0$ so that supp $f_j \subset B(0,r)$ for every j. Let $A \in SO(n)$ and $x \in \boldsymbol{R}^n$; choose $X \in V$ so that $\exp X = A$ and $|X| = d(A)$. Consider the curve in $\boldsymbol{R}^n$ given by $\alpha(t) = \exp tX \cdot x$, $t \in [0,1]$. Remember that $\exp tX \in SO(n)$ and $\alpha'(t) = X \cdot \exp tX \cdot x$.

Therefore

$$\| f_j(Ax) - f_j(x) \| = \int \left| \int_0^1 \nabla f_j(\exp tX \cdot x) \cdot \alpha'(t)\, dt \right| dx \leqslant$$

$$\leqslant \int_0^1 \int |\nabla f_j(\exp tX \cdot x)| \, |\alpha'(t)| \, dx\, dt \leqslant \int_0^1 \int \sum_{k=1}^n |D_k f_j(\exp tX \cdot x)| \, |X \cdot \exp tX \cdot x| \, dx\, dt \leqslant$$

$$\leqslant n^{5/2} \int_0^1 \int \sum_{k=1}^n |D_k f_j(\exp tX \cdot x)| \, |X| \, |\exp tX| \, |x| \, dx\, dt \leqslant$$

$$\leqslant n^3 |X| \int_0^1 \int_{B(0,r)} \sum_{k=1}^n |D_k f_j(\exp tX \cdot x)| \, |x| \, dx\, dt \leqslant n^3 r |X| \, 2^{(1-\lambda)j} = c_1 |X| \, 2^{(1-\lambda)j} .$$

Thus we have both the following estimates

$$\| f_j(Ax) - f_j(x) \| \leqslant \begin{cases} 2c2^{-\lambda j} \\ c_1 |X| \, 2^{(1-\lambda)j} \end{cases} \quad \forall j .$$

Following the same reasoning as in the second part of Theorem 1 we get

$$\sup_{A \in SO(n), A \neq I} |(A,0)|^{-\lambda} \| f(Ax) - f(x) \| < +\infty .$$

We therefore have $f \in E^\lambda$.

REMARK 1. If supp f is not compact, it is not true that $f \in N^\lambda$ implies $f \in E^\lambda$. We outline a counterexample; details will appear in [1].

Suppose that $n = 2$. Fix $\lambda \in]0,1[$; choose $\rho > 0$ such that $\rho^\lambda > 2$. Denote, for every $k \in \boldsymbol{N}$, $a_k = (\tan(\rho^{-k}\pi/4))^{-1}$; Q_k the square with vertices in $(a_k, 0)$, $(a_k + 1, 0)$, $(a_k +$

$+1, 1)$, $(a_k, 1)$; χ_k the characteristic function of Q_k. Let

$$f(x) = \sum_{k=1}^{+\infty} 2^{-k} \chi_k(x).$$

Then we have $f \in N^\lambda$ and $f \notin E^\lambda$.

2. – In this Section we consider compact homogeneous spaces and on them we define and characterize function spaces of the Nikolskii type. Our main references here are the books [3, Ch. II, Ch. X; 7, Ch. IV, Ch. V].

Let G be a compact connected Lie group, K a closed Lie subgroup of G, M the space of left cosets xK, $x \in G$. Fix on M, once and for all, a G-invariant riemannian metric (by G-invariant we mean that each transformation $\tau(x): M \to M$, $\tau(x)p = x \cdot p$ $(x \in G)$ is an isometry). Let μ denote the invariant measure on M corresponding to the riemannian metric. Let $\mathcal{G}$ be the Lie algebra of G and $\exp: \mathcal{G} \to G$ the exponential mapping; since G is compact, exp is onto G.

For every $X \in \mathcal{G}$, we define a vector field on M by the formula

$$X^* f(p) = \lim_{s \to 0} \frac{f(\exp sX \cdot p) - f(p)}{s} \qquad \text{for } f \in C^\infty(M),\ p \in M.$$

Finally, let $|\ |$ denote a norm in $\mathcal{G}$, induced by an inner product.

The previous Theorem 2 seems to justify the following

Definition 3. *For every* $\lambda \in]0, 1[$ *let*

$$E^\lambda(M) = \left\{ f \in L^1(M) \Big| \sup_{X \in \mathcal{G}, X \neq 0} |X|^{-\lambda} \|f(\exp X \cdot p) - f(p)\|_{L^1(M)} < +\infty \right\}.$$

As before, we have

Theorem 3. $f \in E^\lambda(M)$ *if and only if there is a sequence* $(f_j)_{j=0,1,\dots}$ *of functions in* $C^\infty(M)$, *satisfying the conditions:*

(*i*)
$$f = \sum_{j=0}^{+\infty} f_j \quad \text{in } L^1(M);$$

(*ii*) *there exists* $c > 0$ *such that, for every* j,

$$\|f_j\|_{L^1(M)} \leq c 2^{-\lambda j}; \qquad \|X^* f_j\|_{L^1(M)} \leq c 2^{(1-\lambda)j}, \qquad \forall X \in \mathcal{G} \quad \text{with } |X| = 1.$$

Proof. Suppose that $f \in E^\lambda(M)$.

Let the measures on M, G, K be normalized so that $\int_G F(x)\,dx = \int_M \left(\int_K F(xk)\,dk \right) d\mu$ for every $F \in C(G)$.

For $j = 0, 1, \dots$, let B_j be the ball of radius 2^{-j} in $\mathcal{G}$; we may suppose that B_0 is a normal neighborhood of 0 in $\mathcal{G}$. Let φ_j be a smooth function on G, supported on $\exp B_j$, such that $\varphi_j \geq 0$, $\int_G \varphi_j(x)\,dx = 1$ and $\|\tilde{X}\varphi_j\|_{L^1(G)} \leq C 2^j |X|$ for every $X \in \mathcal{G}$, where $\tilde{X}$ is the right invariant vector field on G, defined by X.

Let $\psi_0 = \varphi_0$, $\psi_j = \varphi_j - \varphi_{j-1}$ for $j \geq 1$. For every $x \in G$, let $\tilde{f}(x) = f(xK)$. The function $\tilde{f}$ is right K-invariant; since $f \in L^1(M)$, $\tilde{f} \in L^1(G)$ and $F_j = \psi_j * \tilde{f}$ is a smooth, right K-invariant function on G. Since (φ_j) is an approximate identity, $\tilde{f} = \sum_{j=0}^{+\infty} F_j$ in $L^1(G)$.

Define on $M = G/K$ $f_j(xK) = F_j(x)$; then $f_j \in C^\infty(M)$ and $f = \sum_{j=0}^{+\infty} f_j$ in $L^1(M)$. We have to estimate the norms $\|f_j\|_{L^1(M)}$ and $\|X^* f_j\|_{L^1(M)}$.

For every $j \geq 1$ we obtain

$$\|f_j\|_{L^1(M)} = \|F_j\|_{L^1(G)} = \int_G |\psi_j * \tilde{f}(x)|\, dx =$$

$$= \int_G \left| \int_G \psi_j(y) \tilde{f}(y^{-1}x)\, dy - \int_G \psi_j(y) \tilde{f}(x)\, dy \right| dx \leq \int_G \int_G |\psi_j(y)|\, |\tilde{f}(y^{-1}x) - \tilde{f}(x)|\, dx\, dy .$$

Since $\operatorname{supp} \psi_j \subseteq \exp B_{j-1}$, for every y in $\operatorname{supp} \psi_j$ there exists exactly one $Y \in B_{j-1}$ such that $y = \exp Y$.

Then it follows that

$$\int_G |\tilde{f}(y^{-1}x) - \tilde{f}(x)|\, dx = \|\tilde{f}(\exp(-Y)\, x) - \tilde{f}(x)\|_{L^1(G)} =$$

$$= \|f(\exp(-Y)\cdot p) - f(p)\|_{L^1(M)} \leq L|Y|^\lambda \leq L 2^{(1-j)\lambda} .$$

This implies

$$\|f_j\|_{L^1(M)} \leq L 2^{(1-j)\lambda} \int_G |\psi_j(y)|\, dy \leq c_1 2^{-\lambda j} .$$

In a similar way, we estimate $\|X^* f_j\|_{L^1(M)}$. Let $X \in \mathcal{G}$ with $|X| = 1$. For every $x \in G$ we have

$$X^* f_j(xK) = \tilde{X} F_j(x) = \tilde{X}(\psi_j * \tilde{f})(x) = (\tilde{X}\psi_j) * \tilde{f}(x) .$$

Therefore

$$\|X^* f_j\|_{L^1(M)} = \|(\tilde{X}\psi_j) * \tilde{f}\|_{L^1(G)} .$$

Since $\tilde{X}\psi_j$ has mean value zero (see [13, p. 387]) we obtain

$$\|X^* f_j\|_{L^1(M)} \leq \int_G \int_G |\tilde{X}\psi_j(y)|\, |\tilde{f}(y^{-1}x) - \tilde{f}(x)|\, dx\, dy \leq L 2^{(1-j)\lambda} 2C 2^j |X| = c_2 2^{(1-\lambda)j} .$$

Conversely, let $f = \sum_{j=0}^{+\infty} f_j$, where the f_j satisfy the conditions (i) and (ii).

Let $p \in M$ and $Y \in \mathcal{G}$; write $Y = sX$ with $|X| = 1$ and $s \geq 0$. For every $j = 0, 1, \ldots$ we define

$$h_j : [0,1] \to \boldsymbol{R}, \qquad h_j(t) = f_j(\exp tY \cdot p) .$$

Since

$$h_j'(t) = \lim_{u \to 0} \frac{f_j(\exp(u+t)\, Y \cdot p) - f_j(\exp tY \cdot p)}{u} = Y^* f_j(\exp tY \cdot p)$$

we have

$$|f_j(\exp Y\cdot p) - f_j(p)| = \left|\int_0^1 h_j'(t)\,dt\right| \leqslant s\int_0^1 |X^* f_j(\exp tsX\cdot p)|\,d\mu .$$

Moreover, the G-invariance of the measure μ implies that

$$\|f_j(\exp Y\cdot p) - f_j(p)\|_{L^1(M)} = \int_M |f_j(\exp Y\cdot p) - f_j(p)|\,d\mu(p) \leqslant$$

$$\leqslant s\int_0^1\int_M |X^* f_j(\exp tsX\cdot p)|\,d\mu(p)\,dt = s\int_0^1\int_M |X^* f_j(p)|\,d\mu(p)\,dt \leqslant cs2^{(1-\lambda)j} = c|Y|\,2^{(1-\lambda)j}.$$

Therefore

$$\|f_j(\exp Y\cdot p) - f_j(p)\|_{L^1(M)} \leqslant \begin{cases} 2c2^{-\lambda j} \\ c|Y|\,2^{(1-\lambda)j} \end{cases} \quad \forall j.$$

The conclusion follows from the same argument as in the second part of Theorem 1.

Corollary 1. *Let f be a smooth function. Then $f\in E^\lambda(M)$.*

Proof. Let $f_0 = f$, $f_j = 0$ for every $j > 0$. It suffices to prove that there exists $C > 0$ such that, for every $X \in \mathcal{G}$ with $|X| = 1$, $\|X^* f\|_{L^1(M)} \leqslant C$. Choose an orthonormal basis $X_1, \ldots, X_q$ in $\mathcal{G}$. Let $X \in \mathcal{G}$ with unit norm; then it follows that $b_1, \ldots b_q \in \boldsymbol{R}$ exist such that $X = \sum_{i=1}^{q} b_i X_i$, with $|b_i| \leqslant 1$. Since $X^* = \sum_{i=1}^{q} b_i X_i^*$, the result is easily proven.

3. – In this Section we shall compare the spaces $E^\lambda(M)$ introduced in n. 2, with the Nikolskii spaces, defined on a riemannian manifold by coordinate maps.

Let M be a riemannian manifold; let $\{(U_\alpha, \varphi_\alpha) | \alpha\in A\}$ be an atlas on M.

Definition 4. *For every $\lambda \in]0,1[$, let*

$$N^\lambda(M) = \{f\in L^1(M) | \forall\alpha\in A, \forall\xi\in C_c^\infty(U_\alpha)\,(\xi f)\circ\varphi_\alpha^{-1} \in N^\lambda(\boldsymbol{R}^n)\}.$$

It is easy to see that this definition is independent of the various choices of the atlas; in fact one need only check $F\circ\Phi\in N^\lambda(\boldsymbol{R}^n)$, if F is a function in $N^\lambda(\boldsymbol{R}^n)$ with compact support and Φ is a diffeomorphism (and this result follows, for example, from Theorem 1).

Now let M be a compact homogeneous space, as in the previous Section; let n be its dimension.

Theorem 4.

$$N^\lambda(M) = E^\lambda(M)$$

Proof. Let us assume that $f\in N^\lambda(M)$. Choose an atlas $\{(U_l, \varphi_l) | l = 1, \ldots, r\}$ of M.

Select two open coverings $\{V_l\}$ and $\{W_l\}$ of M and a partition of unity $\{\psi_l\}$ such that $\operatorname{supp}\psi_l \subseteq \overline{W}_l \subseteq V_l \subseteq \overline{V}_l \subseteq U_l$.

For every l, let $f_l = \psi_l \cdot f$ and $F_l = f_l \circ \varphi_l^{-1}$ in $\varphi_l(U_l)$ (and $F_l = 0$ elsewhere). Since $f \in N^\lambda(M)$, $F_l \in N^\lambda(\boldsymbol{R}^n)$; therefore there exists a sequence $(F_{lj})_{j=0,1,\dots}$ of functions satisfying the conditions of Theorem 1. Moreover, since $\operatorname{supp} F_l \subseteq \varphi_l(\overline{W}_l)$, we may suppose that $\operatorname{supp} F_{lj} \subseteq \varphi_l(\overline{V}_l)$.

If we define $f_{lj} = F_{lj} \circ \varphi_l$ in U_l (and $f_{lj} = 0$ elsewhere), then $f_{lj} \in C^\infty(M)$. Furthermore it is easily seen that $f_l = \sum_{j=0}^{+\infty} f_{lj}$ in $L^1(M)$ and that there is a constant $c_1 > 0$ such that, for every j, $\|f_{lj}\|_{L^1(M)} \leq c_1 2^{-\lambda j}$.

Now we have to estimate $\|X^* f_{lj}\|_{L^1(M)}$, for $X \in \mathcal{G}$ with $|X| = 1$. To do this, take an orthonormal basis $X_1, \dots, X_{n+k}$ of $\mathcal{G}$. Let $X_1^*, \dots, X_{n+k}^*$ be the corresponding vector fields on M. Let $y_1, \dots, y_n$ be the system of coordinates on U_l; then, for every $s = 1, \dots, n+k$, there exist functions $a_1^s, \dots, a_n^s$ in $C^\infty(U_l)$ such that

$$X_s^*(p) = \sum_{i=1}^{n} a_i^s(p) \frac{\partial}{\partial y_i}(p) \qquad \forall p \in U_l .$$

Let $A = \max_{1 \leq s \leq n+k} \max_{1 \leq i \leq n} \max_{p \in \overline{V}_l} |a_i^s(p)|$. Then

$$\|X_s^* f_{lj}\|_{L^1(M)} \leq \sum_{i=1}^{n} \int_{\overline{V}_l} |a_i^s(p)| \left| \frac{\partial}{\partial y_i}(p) f_{lj} \right| d\mu(p) \leq$$

$$\leq A \sum_{i=1}^{n} \int_{\overline{V}_l} \left| \frac{\partial}{\partial y_i}(p) f_{lj} \right| d\mu(p) \leq c_2 A \sum_{i=1}^{n} \int_{\varphi_l(\overline{V}_l)} |D_i F_{lj}(x)| dx \leq c_3 2^{(1-\lambda)j} .$$

Now let $X \in \mathcal{G}$ and $|X| = 1$; then $X = \sum_{s=1}^{n+k} b_s X_s$ with $|b_s| \leq 1$; therefore

$$\|X^* f_{lj}\|_{L^1(M)} = \left\| \sum_{s=1}^{n+k} b_s X_s^* f_{lj} \right\| \leq (n+k)\, c_3 2^{(1-\lambda)j} .$$

It follows from Theorem 3 that $f_l \in E^\lambda(M)$. Since $f = \left(\sum_{l=1}^{r} \psi_l \right) f = \sum_{l=1}^{r} f_l$, we obtain $f \in E^\lambda(M)$.

To prove the other half of the Theorem, take an atlas $\{(U_l, \varphi_l) \mid l = 1, \dots, r\}$ with the property that for every coordinate neighborhood (U_l, φ_l) there are n unitary vectors $X_{l1}, \dots, X_{ln} \in \mathcal{G}$ such that the corresponding vector fields $X_{l1}^*, \dots, X_{ln}^*$ generate the module of all vector fields on U_l.

Now suppose that $f \in E^\lambda(M)$. Let $\xi \in C_c^\infty(U_l)$; we have to prove that $\xi f \circ \varphi_l^{-1} \in N^\lambda(\boldsymbol{R}^n)$. Choose W_l and V_l open sets of M such that $\operatorname{supp}\xi \subseteq \overline{W}_l \subseteq V_l \subseteq \overline{V}_l \subseteq U_l$. Since $\xi f \in E^\lambda(M)$ (the proof is similar to that of Corollary 1), there exists a sequence (f_j) of functions of $C^\infty(M)$ satisfying the following conditions:

$$(i) \qquad \operatorname{supp} f_j \subseteq \overline{V}_l \quad \textit{and} \quad \xi f = \sum_{j=0}^{+\infty} f_j \quad \textit{in } L^1(M);$$

(ii) *there exists* $c>0$ *such that,* $\forall j$

$$\|f_j\|_{L^1(M)} \leqslant c2^{-\lambda j}; \quad \|X^* f_j\|_{L^1(M)} \leqslant c 2^{(1-\lambda)j}, \quad \forall X \in \mathfrak{G} \text{ with } |X| = 1.$$

We define $F = \xi f \circ \varphi_l^{-1}$ in $\varphi_l(U_l)$ (and $F=0$ elsewhere); $F_j = f_j \circ \varphi_l^{-1}$ in $\varphi_l(U_l)$ (and $F_j = 0$ elsewhere). Then $F_j \in C_c^\infty(\boldsymbol{R}^n)$, $\sum_j F_j = F$ in $L^1(\boldsymbol{R}^n)$ and there exists $c_1 > 0$ such that, for every j, $\|F_j\|_{L^1(\boldsymbol{R}^n)} \leqslant c_1 2^{-\lambda j}$.

It only remains to estimate the norms $\|D_h F_j\|_{L^1(\boldsymbol{R}^n)}$ for $h = 1, \dots, n$.

Let $y_1, \dots, y_n$ be local coordinates with respect to (U_l, φ_l). Then

$$\|D_h F_j\|_{L^1(\boldsymbol{R}^n)} = \int_{\varphi_l(\overline{V}_l)} |D_h F_j(x)|\, dx \leqslant c_l \int_{\overline{U}_l} \left| \frac{\partial}{\partial y_h} f_j(p) \right| d\mu(p) = c_l \left\| \frac{\partial}{\partial y_h} f_j \right\|_{L^1(U_l)}.$$

From our hypothesis on the atlas, it follows that there are n functions $a_1^h, \dots, a_n^h$ in $C^\infty(U_l)$ such that

$$\frac{\partial}{\partial y_h}(p) = \sum_{i=1}^n a_i^h(p)\, X_{li}^*(p) \qquad \text{for every } p \in U_l.$$

Let $A = \max\limits_{1 \leqslant h \leqslant n} \max\limits_{1 \leqslant i \leqslant n} \max\limits_{p \in \overline{V}_l} |a_i^h(p)|$.

We obtain

$$\left\| \frac{\partial}{\partial y_h} f_j \right\|_{L^1(U_l)} \leqslant \sum_{i=1}^n \int_{\overline{V}_l} |a_i^h(p)|\, |X_{li}^* f_j(p)|\, d\mu(p) \leqslant$$

$$\leqslant A \sum_{i=1}^n \|X_{li}^* f_j\|_{L^1(U_l)} \leqslant Ac \sum_{i=1}^n 2^{(1-\lambda)j} \leqslant c_2 2^{(1-\lambda)j}.$$

Hence, by Theorem 1, $F \in N^\lambda(\boldsymbol{R}^n)$.

REMARK 2. Nikolskii spaces on open sets of $\boldsymbol{R}^n$ naturally occur in the study of partial differential equations. By analogy, the function spaces $E^\lambda(M)$, that we have introduced here, could prove interesting in problems concerning differential equations on manifolds, particularly if we have to consider differential operators, which are invariant under global transformations of the manifold.

An example of such a situation is the following.

In [6] a regularity result is proven for the solution of the two-phase Stefan problem in a bounded domain Ω of $\boldsymbol{R}^n$, under the assumption that the initial data are assigned in a suitable Nikolskii space on Ω. The demonstration makes use, among other things, of the invariance property of the Laplacian with respect to the translations of $\boldsymbol{R}^n$: $\Delta(f(x+h)) = (\Delta f)(x+h)$.

Also for applications, it is interesting to study a similar problem no longer in Ω, but rather on a riemannian manifold M (for instance, on the boundary of Ω) and – also in this case – to take the initial data in $N^\lambda(M)$ (this problem is proposed in [5, 5.c]). The definition of $N^\lambda(M)$ utilizes coordinate maps; therefore, it is not clear how to use the invariance property of the Laplace-Beltrami operator of M under all isometries of M:

$$(*) \qquad \Delta(f(g \cdot x)) = (\Delta f)(g \cdot x) \qquad \text{for every isometry } g \text{ of } M.$$

If however M is a homogeneous space of a compact Lie group, the definition of $E^\lambda(M)$

(which makes use of a transitive group of isometries of M) and Theorem 4 allow us to avail ourselves of the property (*). In this case therefore it seems possible to extend the techniques of [6] and consequently to obtain a similar regularity result.

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