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Infinite locally soluble $k$-Engel groups


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Teoria dei gruppi. — *Infinite locally soluble k-Engel groups.* Nota di Lucia Serena Spiezia, presentata (*) dal Socio G. Zappa.

**Abstract.** — In this paper we deal with the class $\mathcal{S}_k^*$ of groups $G$ for which whenever we choose two infinite subsets $X, Y$ there exist two elements $x \in X, y \in Y$ such that $[x, y, ..., y] = 1$. We prove that an infinite finitely generated soluble group in the class $\mathcal{S}_k^*$ is in the class $\mathcal{S}_k$ of $k$-Engel groups. Furthermore, with $k = 2$, we show that if $G \in \mathcal{S}_2^*$ is infinite locally soluble or hyperabelian group then $G \in \mathcal{S}_2$.

**Key words:** Groups; Engel; Varieties.

**Riassunto.** — $k$-Engel gruppi infiniti localmente risolubili. Si definisce la classe $\mathcal{S}_k^*$ dei gruppi $G$ per i quali comunque si prendano due sottoinsiemi $X, Y$, esistono due elementi $x \in X, y \in Y$ tali che $[x, y, ..., y] = 1$. Si prova che un gruppo infinito risolubile finitamente generabile nella classe $\mathcal{S}_k^*$ è nella classe $\mathcal{S}_k$ dei gruppi $k$-Engel. Inoltre, per $k = 2$ si è provato che se $G \in \mathcal{S}_2^*$ è infinito e localmente risolubile od iperabeliano allora $G \in \mathcal{S}_2$.

**1. Introduction**

Let $\mathcal{V}$ be a variety of groups defined by the law $w(x_1, ..., x_n) = 1$. Following [3] we denote by $\mathcal{V}^*$ the class of groups $G$ such that, whenever $X_1, ..., X_n$ are infinite subsets of $G$, there exist $x_i \in X_i, i = 1, ..., n$, such that $w(x_1, ..., x_n) = 1$.

In [2] P. Longobardi, M. Maj and A. H. Rhemtulla proved that $\mathcal{V}^* = \mathcal{V} \cup \mathcal{F}$, where $\mathcal{F}$ is the class of finite groups and $\mathcal{V}$ is the class of nilpotent groups of class $\leq n - 1$, while in [3] P. S. Kim, A. H. Rhemtulla and H. Smith studied $\mathcal{V}^*$, where $\mathcal{V}$ is the class $\mathcal{C}_2$ of metabelian groups. They proved, among other things, that an infinite locally soluble $\mathcal{C}_2^*$-group is actually a metabelian group. In this paper we study $\mathcal{S}_k^*$, where $\mathcal{S}_k$ is the class of $k$-Engel groups and $k \geq 2$ is an integer.

We show the following results:

**Theorem A.** Let $G$ be an infinite locally soluble group. If $G \in \mathcal{S}_2^*$, then $G \in \mathcal{S}_2$.

**Theorem B.** Let $G$ be an infinite hyperabelian group. If $G \in \mathcal{S}_2^*$, then $G \in \mathcal{S}_2$.

**Theorem C.** Let $G$ be an infinite finitely generated soluble group. If $G \in \mathcal{S}_k^*$, then $G \in \mathcal{S}_k$.

Finally we show that if $G \in \mathcal{S}_k^*$ is infinite, then the centralizer $C_G(x)$ is infinite for every element $x$ of $G$.

Notation and terminology are the usual ones (see for instance [4]). We recall that a group is said to be a $k$-Engel group if $[x, y, ..., y] = 1$ for all $x, y \in G$.

2. Soluble $\mathfrak{M}_2$-groups

We start with two useful Lemmas:

**Lemma 2.1.** Let $G$ be an infinite nilpotent $\mathfrak{M}_2$-group. Then $G$ is a 2-Engel group.

**Proof.** Let $i \geq 1$ be the greatest integer such that $Z_{i-1}(G)$ is finite while $Z_i(G)$ is infinite. Put $C = Z_i(G)$ and $D = Z_{i-1}(G)$. Let $x, y$ be arbitrary elements of $G$. Then $[x, c]$ and $[y, c]$ are in $D$, for every $c \in C$, and the identity $x^c = x[x, c]$ implies that $|x^C| \leq |D|$, i.e., $|x^C|$ is finite. Thus $|C: C_c(x)|$ is finite. Similarly $|C: C_c(y)|$ is finite and so also $|C: C_c(x) \cap C_c(y)|$ is finite. Hence $C_c(x) \cap C_c(y)$ is an infinite nilpotent subgroup of $G$. Then there exists an infinite abelian subgroup $A$ of $C_c(x) \cap C_c(y)$. Now the two subsets $xA$ and $yA$ are infinite, and, by the hypothesis there exist $a_1$ and $a_2$ in $A$ such that: $1 = [xa_1, ya_1, ya_2] = [x, y, y]$.

Therefore $G$ is a 2-Engel group, as required. 

**Lemma 2.2.** Let $G$ be an infinite $\mathfrak{M}_2$-group. If $G$ has an infinite normal abelian subgroup $A$, then $G$ is a 2-Engel group.

**Proof.** First of all we remark that, if $N$ is an infinite normal subgroup of an $\mathfrak{M}_2$-group, then $G/N \in \mathfrak{M}_2$. For, if $x, y$ are arbitrary elements of $G$, then $xN, yN$ are infinite, so that there exist $n_1, n_2 \in N$ such that $[xn_1, yn_2, yn_2] = 1$. Thus we have $[xN, yN, yN] = N$, and $G/N \in \mathfrak{M}_2$.

Now let $A$ be an infinite normal abelian subgroup of $G$. Then $A$ consists of right 2-Engel elements of $G$. In fact, fix $x$ in $G$ and let $B$ be an infinite subset of $A$. Then the subsets $xB$ and $B$ of $G$ are infinite and there exist $a, b \in B$ such that: $1 = [a, xb, xb] = [a, x, x]$.

Hence for any infinite subset $B$ of $A$ there is an element $a$ in $B$ such that $[a, x, x] = 1$. Thus the set $B(x) = \{a \in A/[a, x, x] = 1\}$ is an infinite subset of $A$.

Now if there is an element $b \in A$ such that $[b, x, x] \neq 1$, then the set $\{ab/a \in B(x)\}$ is also infinite, and there exists $b' \in B(x)$ such that $1 = [bb', x, x] = [b, x, x]$ which is a contradiction. Finally we can get the conclusion $G \in \mathfrak{M}_2$. Obviously it is enough to show that for each $x, y \in G$, the subgroup $H = A(x, y)$ is a 2-Engel group. First we remark that $A \leq Z_1(H)$. In fact, let $a$ be in $A$, then $a$ is a right 2-Engel element of $G$, so that $[a, x, y] = [a, y, x]^{-1}$. Furthermore $[a, x, y, y] = [a^{-1}, y, y][a, y, y] = [a, x, y, y] = 1$, because $a^G$ consists of 2-Engel elements of $G$. In the same way $[a, y, x, x] = 1$.

Then $[a, x, y, y]^{-1} \in C_C(x) \cap C_C(y) \cap A$ and $[a, x, y, y] \in Z(H)$. Analogously $[a, y, x] \in Z(H)$, so that $[a, x] \in Z_2(H)$ and $[a, y] \in Z_2(H)$, and $a \in Z_3(H)$. Now we have $H/A \leq G/A$ nilpotent, and $A \leq Z_3(H)$, so that $H$ is nilpotent. By Lemma 2.1, $H$ is a 2-Engel group, as required. 

From Lemma 2.2 it follows easily:
**Corollary 2.3.** Let $G$ be an infinite metabelian group, $G \in \mathfrak{S}_2$. Then $G$ is a 2-Engel group.

**Proof.** If the derived subgroup $G'$ is infinite, then the result follows from Lemma 2.2. Assume that $G'$ is finite. Then $G$ is a FC-group and $|G: C_G(x)|$ is finite for any $x \in G$. Thus, if $x, y$ are in $G$, $|G: C_G(x) \cap C_G(y)|$ is also finite, and $C_G(x) \cap C_G(y)$ is infinite. Then there exists an infinite abelian subgroup $A \leq C_G(x) \cap C_G(y)$ and the subsets $xA, yA$ are infinite, and there are $a_1, a_2 \in A$ such that:

$$1 = [xa_1, ya_2, ya_2] = [x, y, y].$$

Hence $G$ is a 2-Engel group as required. □

Now we can prove the following:

**Theorem 2.4.** Let $G$ be an infinite soluble group in $\mathfrak{S}_2$. Then $G$ is a 2-Engel group.

**Proof.** We show that $G$ is metabelian, i.e., $G^{(2)} = \{1\}$. Then the result will follow from Corollary 2.3. Assume that $G^{(2)} \neq \{1\}$. If $G^{(3)}$ is infinite, then $G/G^{(3)}$ is a 2-Engel group, arguing as in Lemma 2.2, and $G/G^{(3)}$ is metabelian, a contradiction. Thus $G^{(3)}$ is finite.

Furthermore $G'$ is infinite, otherwise $G$ is an infinite FC-group in $\mathfrak{S}_2$, and $G$ is a 2-Engel group arguing as in Corollary 2.3, a contradiction. If $G^{(2)}$ is infinite, then $G/G^{(3)} \leq \mathfrak{S}_2$, by Lemma 2.2, and $G/G^{(3)}$ is metabelian, again a contradiction. Then $G^{(2)}$ is finite, and $G'$ is an infinite FC-group in $\mathfrak{S}_2$, so that $G'$ is nilpotent. Furthermore $G/G^{(2)}$ is an infinite metabelian group in $\mathfrak{S}_2$, and so it is nilpotent by Corollary 2.3. Thus, by a result of P. Hall (see [4, vol. I, Th. 2.27, p. 56]), $G$ is nilpotent and $G$ is a 2-Engel group by Lemma 2.1, which is the final contradiction. □

3. **Proofs of Theorems A and B**

Now we can prove our first statement in the introduction:

**Proof of Theorem A.** Assume that $G$ is an infinite locally soluble group in $\mathfrak{S}_2$. If there exists in $G$ an element $g$ of infinite order, then for any two elements $x, y \in G$, we can consider the subgroup $N = \langle x, y, g \rangle$. $N$ is an infinite and soluble $\mathfrak{S}_2$-group, so by Theorem 2.4, it is a 2-Engel group. Thus $[x, y, y] = 1$ for any $x, y \in G$, and $G \in \mathfrak{S}_2$.

Now we can assume that $G$ is a periodic group. If $G$ is a Černikov group, then it has a normal abelian subgroup $A$ of finite index and, by Lemma 2.2, $G \in \mathfrak{S}_2$.

Then it remains to consider the case of a periodic locally soluble group which is not Černikov. In this case $H = \langle x, y \rangle$ is finite, for any $x, y \in G$. Then by a result of D. I. Zaicëv (see [5, Th. 1, p. 342]) this $H$ normalizes some infinite abelian subgroup $B$ of $G$. Therefore the group $HB$ is an infinite soluble group in $\mathfrak{S}_2$ and by Theorem 2.4 it is a 2-Engel group, hence $[x, y, y] = 1$ for any $x, y \in G$, i.e., $G \in \mathfrak{S}_2$ as required. □
In order to prove Theorem B in the introduction we need the following useful result:

**Lemma 3.1.** Let \( G \) be in \( \mathfrak{S}_2 \), \( G \) infinite. Then \( C_G(x) \) is infinite for any \( x \in G \).

**Proof.** Suppose that there exists \( y \in G \) such that \( C_G(y) \) is finite, and look for a contradiction. There exists an infinite sequence \( x_1, x_2, \ldots, x_n, \ldots \) of elements of \( G \) such that:

i) \( x_i \notin C_G(y) \) for any \( i \in N \),

ii) \( x_i x_j^{-1} \notin C_G(y) \) for any \( i \neq j, i, j \in N \),

iii) \( x_i x_j^{-1} x_b x_k^{-1} \notin C_G(y) \) for any \( i, j, b, k \in N \), pairwise different.

For, suppose that \( x_1, x_2, \ldots, x_n \) satisfy the properties, then there exists an element \( x_{n+1} \) such that:

\[
x_{n+1} \notin \bigcup_{i=1}^{n} C_G(y) x_i \cup C_G(y) \cup \bigcup_{i, b, k = 1}^{n} C_G(y) x_b x_k^{-1} x_i \cup \bigcup_{i, j, k = 1}^{n} x_i x_j^{-1} x_k C_G(y)^{x_b},
\]

since this union is finite. Thus the elements \( x_1, x_2, \ldots, x_n, x_{n+1} \) satisfy the properties.

Now, let \( N = \bigcup_{n \in N} I_n \cup \bigcup_{n \in N} J_n \), where each \( I_n \) and \( J_n \) is infinite and the unions are disjoint unions. The sets \( A_n = \{ y^{x_n}/i \in I_n \} \) and \( B_n = \{ y^{x_n}/j \in J_n \} \) are infinite, so that, for any \( n \in N \), there exist \( i_n \in I_n, j_n \in J_n \) such that:

\[
1 = [y^{x_n}, y^{x_n}, y^{x_n}] = [y^{x_n} x_n^{-1}, y, y], \quad i.e. \quad [y^{x_n} x_n^{-1}, y] \in C_G(y),
\]

that implies

\[
y^{x_n} x_n^{-1} \in C_G(y).
\]

But \( C_G(y) \) is finite, so there exists \( n \in N \) such that, for infinitely many \( m \), we have:

\[
y^{x_n} x_n^{-1} = y^{x_n} x_n^{-1}, \quad i.e. \quad y^{x_n} x_n^{-1} (y^{x_n} x_n^{-1})^{-1} \in C_G(y).
\]

Therefore there are \( i \neq s \) such that:

\[
y^{x_s} x_s^{-1} (y^{x_i} x_i^{-1})^{-1} = y^{x_s} x_s^{-1} (y^{x_i} x_i^{-1})^{-1},
\]

that implies \( y^{x_s} x_s^{-1} = y^{x_i} x_i^{-1} \) and \( x_i x_j^{-1} (x_i x_j^{-1})^{-1} \in C_G(y) \), a contradiction. \( \square \)

From Lemma 3.1 it follows, with straightforward arguments:

**Corollary 3.2.** Let \( G \in \mathfrak{S}_2 \), \( G \) infinite. Then there exists an infinite abelian subgroup \( A \) of \( G \).

Now we can prove Theorem B.

**Proof of Theorem B.** We show that \( G \) is metabelian, the result will follow from Theorem 2.4. Assume that \( G \) is not metabelian and let \( N \triangleleft G \) be a maximal normal metabelian subgroup of \( G \). There exists a non-trivial normal abelian subgroup \( M/N \) of
$G/N$, since $G$ is hyperabelian. Then $M$ is not metabelian. Furthermore, by Corollary 3.2 there exists an infinite abelian subgroup $A$ of $G$.

Now we consider the group $H = MA$. Then $A \leq H$ so that $H$ is infinite, and $H$ is not metabelian since $M \leq H$. But $H$ is soluble and that contradicts Theorem 2.4. □

4. FINITELY GENERATED $\mathfrak{S}_k$-GROUPS

Arguing exactly as in Lemma 2.1 and in Corollary 2.3, we can show:

**Lemma 4.1.** Let $G$ be either a nilpotent or FC infinite $\mathfrak{S}_k$-group. Then $G$ is a $k$-Engel group.

If $G$ is a finitely generated soluble group, we have also an analogue of Lemma 2.2:

**Lemma 4.2.** Let $G \in \mathfrak{S}_k$ be an infinite finitely generated soluble group. If $G$ has an infinite normal abelian subgroup $A$, then $G$ is a $k$-Engel group.

**Proof.** Arguing as in the proof of Lemma 2.2 we can easily show that if $N$ is an infinite normal subgroup of $G$, then $G/N \in \mathfrak{S}_k$, and also that $A$ consists of right $k$-Engel elements. But in a finitely generated soluble group the set of the right Engel elements is contained in some $Z_l(G)$, by a theorem of Grunenberg (see [1]). Hence $G/A$ is nilpotent and $A \leq Z_l(G)$, so that $G$ is nilpotent too. The result follows from Lemma 4.1. □

We are now able to prove Theorem C:

**Proof of Theorem C.** We show that $G$ is nilpotent by induction on the derived length, $l = l(G)$ of $G$ and the Theorem will follow from Lemma 4.2. If $l = 1$, the result is obviously true. Assume that $l > 1$, and let $G^{(l-1)}$ be the last non-trivial term of the derived series of $G$. If $G^{(l-1)}$ is infinite, then $G$ is nilpotent by Lemma 4.2. Now assume that $G^{(l-1)}$ is finite, then $G/G^{(l-1)}$ is infinite and, by induction $G/G^{(l-1)}$ is nilpotent.

Let $0 \leq i < l - 1$ be maximum such that $G^{(i)}$ is infinite. Then $G^{(i+1)}$ is finite and $G^{(i)}$ is an infinite FC-group in $\mathfrak{S}_k$. Thus $G^{(i)}$ is a $k$-Engel group by Lemma 4.1. Furthermore $G^{(i)}$ is finitely generated since $G/G^{(l-1)}$ is polycyclic and $G^{(l-1)} \leq G^{(i)}$. Hence $G^{(i)}$ is nilpotent by a theorem of Grunenberg (see [1]). The group $G/G^{(i+1)}$ is also nilpotent, because $i + 1 \leq l - 1$ and $G^{(i+1)} \geq G^{(l-1)}$. Then $G$ is nilpotent, by a theorem of P. Hall (see [4], vol.I, Th.2.27, p.56), as required. □

Finally we remark that there is an analogue of Lemma 3.1:

**Lemma 4.3.** Let $G \in \mathfrak{S}_k$, $G$ infinite. Then $C_G(x)$ is infinite, for every $x \in G$.

**Proof.** Assume, by the way of contradiction, that $C_G(y)$ is finite, for some $y \in G$. 

Then by arguing as in Lemma 3.1, there exists an infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of $G$ such that:

i) $x_i \notin C_G(y)$ for any $i \in \mathbb{N}$,

ii) $x_i x_j^{-1} \notin C_G(y)$ for any $i \neq j, i, j \in \mathbb{N}$,

iii) $x_i x_j^{-1} (x_b x_k^{-1})^{-1} \notin C_G(y)$ for any $i, j, b, k \in \mathbb{N}$, pairwise different.

Again as in Lemma 3.1, we can find, for any $n \in \mathbb{N}$, some integers $i_n, j_n \in \mathbb{N}$, such that:

$$1 = [y_{x_n}, k y_{x_n}] = [y_{x_n}, k y_{x_n}] = 1 \quad (1).$$

Then $[y_{x_n}, k y_{x_n}] \notin C_G(y)$, so that $y_{[y_{x_n}, k y_{x_n}]} \notin C_G(y)$. But $C_G(y)$ is finite, thus, for some $n \in \mathbb{N}$ there exist infinitely many $m$ such that:

$$y_{[y_{x_n}, k y_{x_n}]} = y_{[y_{x_n}, k y_{x_n}]}.$$

Therefore, for infinitely many $m$,

$$[y_{x_n}, k y_{x_n}] = [y_{x_n}, k y_{x_n}]^{-1} \in C_G(y).$$

But this subgroup is finite, thus, for some $s$, there exist infinitely many $t$ such that:

$$[y_{x_n}, k y_{x_n}] = [y_{x_n}, k y_{x_n}]^{-1} \in C_G(y).$$

That holds for infinitely many $t$. Continuing in this way, after $k - 3$ steps, we get:

$$y_{x_n} (y_{x_n} x_b^{-1})^{-1} \in C_G(y), \quad \text{for some } b \in \mathbb{N}, \text{ and infinitely many } k.$$

Therefore:

$$y_{x_n} (y_{x_n} x_b^{-1})^{-1} = y_{x_n} (y_{x_n} x_b^{-1})^{-1}, \quad \text{for some } l \neq r,$$

since $C_G(y)$ is finite and so:

$$y_{x_n} x_b^{-1} = y_{x_n} x_b^{-1}, \quad \text{i.e.,} \quad x_{x_n} x_b^{-1} (x_b x_k^{-1})^{-1} \in C_G(y),$$

contradicting iii). \qed

An easy consequence of Theorem C is:

**Corollary 4.4.** Let $G \in S_k$ be a non periodic locally soluble group. Then $G$ is a $k$-Engel group.

(1) Following [4, part II, p. 40], if $n \in \mathbb{N}$, $G$ is a group and $a, b \in G$, we put $[a, b] = [a, b, \ldots, b]$. 

INFINITE LOCALLY SOLUBLE $k$-ENGL GroupS

References


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