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Infinite locally soluble k-Engel groups

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Teoria dei gruppi. — *Infinite locally soluble* k-*Engel groups.* Nota di LUCIA SERENA SPIEZIA, presentata (*) dal Socio G. Zappa.

ABSTRACT. — In this paper we deal with the class $\mathcal{E}_k^*$ of groups G for which whenever we choose two infinite subsets X, Y there exist two elements $x \in X$, $y \in Y$ such that $[x, \underbrace{y, \ldots, y}_{k}] = 1$. We prove that an infinite finitely generated soluble group in the class $\mathcal{E}_k^*$ is in the class $\mathcal{E}_k$ of k-Engel groups. Furthermore, with $k = 2$, we show that if $G \in \mathcal{E}_2^*$ is infinite locally soluble or hyperabelian group then $G \in \mathcal{E}_2$.

KEY WORDS: Groups; Engel; Varieties.

RIASSUNTO. — k-*Engel gruppi infiniti localmente risolubili.* Si definisce la classe $\mathcal{E}_k^*$ dei gruppi G per i quali comunque si prendano due sottoinsiemi X, Y, esistono due elementi $x \in X$, $y \in Y$ tali che $[x, \underbrace{y, \ldots, y}_{k}] = 1$. Si prova che un gruppo infinito risolubile finitamente generabile nella classe $\mathcal{E}_k^*$ è nella classe $\mathcal{E}_k$ dei gruppi k-Engel. Inoltre, per $k = 2$ si è provato che se $G \in \mathcal{E}_2^*$ è infinito e localmente risolubile od iperabeliano allora $G \in \mathcal{E}_2$.

1. INTRODUCTION

Let $\mathfrak{V}$ be a variety of groups defined by the law $w(x_1, \ldots, x_n) = 1$. Following [3] we denote by $\mathfrak{V}^*$ the class of groups G such that, whenever $X_1, \ldots, X_n$ are infinite subsets of G, there exist $x_i \in X_i$, $i = 1, \ldots, n$, such that $w(x_1, \ldots, x_n) = 1$.

In [2] P. Longobardi, M. Maj and A. H. Rhemtulla proved that $\mathfrak{V}^* = \mathfrak{V} \cup \mathcal{F}$, where $\mathcal{F}$ is the class of finite groups and $\mathfrak{V}$ is the class of nilpotent groups of class $\leq$ $\leq n - 1$, while in [3] P. S. Kim, A. H. Rhemtulla and H. Smith studied $\mathfrak{V}^*$, where $\mathfrak{V}$ is the class $\mathfrak{A}_2$ of metabelian groups. They proved, among other things, that an infinite locally soluble $\mathfrak{A}_2^*$-group is actually a metabelian group. In this paper we study $\mathcal{E}_k^*$, where $\mathcal{E}_k$ is the class of k-Engel groups and $k \geq 2$ is an integer.

We show the following results:

THEOREM A. *Let G be an infinite locally soluble group. If $G \in \mathcal{E}_2^*$, then $G \in \mathcal{E}_2$.*

THEOREM B. *Let G be an infinite hyperabelian group. If $G \in \mathcal{E}_2^*$, then $G \in \mathcal{E}_2$.*

THEOREM C. *Let G be an infinite finitely generated soluble group. If $G \in \mathcal{E}_k^*$, then $G \in \mathcal{E}_k$.*

Finally we show that if $G \in \mathcal{E}_k^*$ is infinite, then the centralizer $C_G(x)$ is infinite for every element x of G.

Notation and terminology are the usual ones (see for instance [4]). We recall that a group is said to be a k-Engel group if $[x, \underbrace{y, \ldots, y}_{k}] = 1$ for all $x, y \in G$.

(*) Nella seduta del 24 aprile 1992.

2. Soluble $\mathcal{E}_2^*$-groups

We start with two useful Lemmas:

Lemma 2.1. *Let G be an infinite nilpotent $\mathcal{E}_2^*$-group. Then G is a 2-Engel group.*

Proof. Let $i \geqslant 1$ be the greatest integer such that $Z_{i-1}(G)$ is finite while $Z_i(G)$ is infinite. Put $C = Z_i(G)$ and $D = Z_{i-1}(G)$. Let x, y be arbitrary elements of G. Then $[x, c]$ and $[y, c]$ are in D, for every $c \in C$, and the identity $x^c = x[x, c]$ implies that $|x^C| \leq |D|$, *i.e.*, $|x^C|$ is finite. Thus $|C: C_C(x)|$ is finite. Similarly $|C: C_C(y)|$ is finite and so also $|C: C_C(x) \cap C_C(y)|$ is finite. Hence $C_G(x) \cap C_G(y)$ is an infinite nilpotent subgroup of G. Then there exists an infinite abelian subgroup A of $C_G(x) \cap \cap C_G(y)$. Now the two subsets xA and yA are infinite, and, by the hypothesis there exist a_1 and a_2 in A such that: $1 = [xa_1, ya_2, ya_2] = [x, y, y]$.

Therefore G is a 2-Engel group, as required. □

Lemma 2.2. *Let G be an infinite $\mathcal{E}_2^*$-group. If G has an infinite normal abelian subgroup A, then G is a 2-Engel group.*

Proof. First of all we remark that, if N is an infinite normal subgroup of an $\mathcal{E}_2^*$-group, then $G/N \in \mathcal{E}_2$. For, if x, y are arbitrary elements of G, then xN, yN are infinite, so that there exist $n_1, n_2 \in N$ such that $[xn_1, yn_2, yn_2] = 1$. Thus we have $[xN, yN, yN] = N$, and $G/N \in \mathcal{E}_2$.

Now let A be an infinite normal abelian subgroup of G. Then A consists of right 2-Engel elements of G. In fact, fix x in G and let B be an infinite subset of A. Then the subsets xB and B of G are infinite and there exist $a, b \in B$ such that: $1 = [a, xb, xb] = = [a, x, x]$.

Hence for any infinite subset B of A there is an element a in B such that $[a, x, x] = = 1$. Thus the set $B(x) = \{a \in A/[a, x, x] = 1\}$ is an infinite subset of A.

Now if there is an element $b \in A$ such that $[b, x, x] \neq 1$, then the set $\{ab/a \in B(x)\}$ is also infinite, and there exists b' in $B(x)$ such that $1 = [bb', x, x] = [b, x, x]$ which is a contradiction. Finally we can get the conclusion $G \in \mathcal{E}_2$. Obviously it is enough to show that for each $x, y \in G$, the subgroup $H = A\langle x, y\rangle$ is a 2-Engel group. First we remark that $A \leq Z_3(H)$. In fact, let a be in A, then a is a right 2-Engel element of G, so that $[a, x, y] = [a, y, x]^{-1}$. Furthermore $[a, x, y, y] = [a^{-1}, y, y]^{[a^x, y]} [a^x, y, y] = 1$, because a^G consists of 2-Engel elements of G. In the same way $[a, y, x, x] = 1$

Then $[a, x, y] = [a, y, x]^{-1} \in C_G(x) \cap C_G(y) \cap A$ and $[a, x, y] \in Z(H)$. Analogously $[a, y, x] \in Z(H)$, so that $[a, x]$ and $[a, y] \in Z_2(H)$ and $a \in Z_3(H)$. Now we have $H/A \leq G/A$ nilpotent, and $A \leq Z_3(H)$, so that H is nilpotent. By Lemma 2.1, H is a 2-Engel group, as required. □

From Lemma 2.2 it follows easily:

COROLLARY 2.3. *Let G be an infinite metabelian group,* $G \in \mathcal{E}_2^*$. *Then G is a 2-Engel group.*

PROOF. If the derived subgroup G' is infinite, then the result follows from Lemma 2.2. Assume that G' is finite. Then G is a FC-group and $|G: C_G(x)|$ is finite for any $x \in G$. Thus, if x, y are in G, $|G: C_G(x) \cap C_G(y)|$ is also finite, and $C_G(x) \cap C_G(y)$ is infinite. Then there exists an infinite abelian subgroup $A \leq C_G(x) \cap C_G(y)$ and the subsets xA, yA are infinite, and there are $a_1, a_2 \in A$ such that: $1 = [xa_1, ya_2, ya_2] = [x, y, y]$.

Hence G is a 2-Engel group as required. □

Now we can prove the following:

THEOREM 2.4. *Let G be an infinite soluble group in* $\mathcal{E}_2^*$. *Then G is a 2-Engel group.*

PROOF. We show that G is metabelian, *i.e.,* $G^{(2)} = \{1\}$. Then the result will follow from Corollary 2.3. Assume that $G^{(2)} \neq \{1\}$. If $G^{(3)}$ is infinite, then $G/G^{(3)}$ is a 2-Engel group, arguing as in Lemma 2.2, and $G/G^{(3)}$ is metabelian, a contradiction. Thus $G^{(3)}$ is finite.

Furthermore G' is infinite, otherwise G is an infinite FC-group in $\mathcal{E}_2^*$, and G is a 2-Engel group arguing as in Corollary 2.3, a contradiction. If $G^{(2)}$ is infinite, then $G/G^{(3)} \in \mathcal{E}_2$, by Lemma 2.2, and $G/G^{(3)}$ is metabelian, again a contradiction. Then $G^{(2)}$ is finite, and G' is an infinite FC-group in $\mathcal{E}_2^*$, so that G' is nilpotent. Furthermore $G/G^{(2)}$ is an infinite metabelian group in $\mathcal{E}_2^*$, and so it is nilpotent by Corollary 2.3. Thus, by a result of P. Hall (see [4, vol. I, Th. 2.27, p. 56]), G is nilpotent and G is a 2-Engel group by Lemma 2.1, which is the final contradiction. □

3. PROOFS OF THEOREMS A AND B

Now we can prove our first statement in the introduction:

PROOF OF THEOREM A. Assume that G is an infinite locally soluble group in $\mathcal{E}_2^*$. If there exists in G an element g of infinite order, then for any two elements $x, y \in G$, we can consider the subgroup $N = \langle x, y, g \rangle$. N is an infinite and soluble $\mathcal{E}_2^*$-group, so by Theorem 2.4, it is a 2-Engel group. Thus $[x, y, y] = 1$ for any $x, y \in G$, and $G \in \mathcal{E}_2$. Now we can assume that G is a periodic group. If G is a Černikov group, then it has a normal abelian subgroup A of finite index and, by Lemma 2.2, $G \in \mathcal{E}_2$.

Then it remains to consider the case of a periodic locally soluble group which is not Černikov. In this case $H = \langle x, y \rangle$ is finite, for any $x, y \in G$. Then by a result of D. I. Zaičev (see [5, Th. 1, p. 342]) this H normalizes some infinite abelian subgroup B of G. Therefore the group HB is an infinite soluble group in $\mathcal{E}_2^*$ and by Theorem 2.4 it is a 2-Engel group, hence $[x, y, y] = 1$ for any $x, y \in G$, *i.e.*, $G \in \mathcal{E}_2$ as required. □

In order to prove Theorem B in the introduction we need the following useful result:

LEMMA 3.1. *Let G be in $\mathcal{E}_2^*$, G infinite. Then $C_G(x)$ is infinite for any $x \in G$.*

PROOF. Suppose that there exists $y \in G$ such that $C_G(y)$ is finite, and look for a contradiction. There exists an infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of G such that:

i) $x_i \notin C_G(y)$ for any $i \in \mathbf{N}$,

ii) $x_i x_j^{-1} \notin C_G(y)$ for any $i \neq j$, $i, j \in \mathbf{N}$,

iii) $x_i x_j^{-1}(x_h x_k^{-1})^{-1} \notin C_G(y)$ for any $i, j, h, k \in \mathbf{N}$, pairwise different.

For, suppose that $x_1, x_2, \ldots, x_n$ satisfy the properties, then there exists an element x_{n+1} such that:

$$x_{n+1} \notin \bigcup_{i=1}^{n} C_G(y) x_i \cup C_G(y) \cup \bigcup_{i,h,k=1}^{n} C_G(y) x_h x_k^{-1} x_i \cup \bigcup_{i,j,k=1}^{n} x_i x_j^{-1} x_k C_G(y)^{x_k},$$

since this union is finite. Thus the elements $x_1, x_2, \ldots, x_n, x_{n+1}$ satisfy the properties. Now, let $\mathbf{N} = \left(\dot{\bigcup}_{n \in \mathbf{N}} I_n\right) \dot{\cup} \left(\dot{\bigcup}_{n \in \mathbf{N}} J_n\right)$, where each I_n and J_n is infinite and the unions are disjoint unions. The sets $A_n = \{y^{x_i} / i \in I_n\}$ and $B_n = \{y^{x_j} / j \in J_n\}$ are infinite, so that, for any $n \in \mathbf{N}$, there exist $i_n \in I_n$, $j_n \in J_n$ such that:

$$1 = [y^{x_{i_n}}, y^{x_{j_n}}, y^{x_{j_n}}] = [y^{x_{i_n} x_{j_n}^{-1}}, y, y], \quad i.e. \quad [y^{x_{i_n} x_{j_n}^{-1}}, y] \in C_G(y),$$

that implies

$$y^{y^{x_{i_n} x_{j_n}^{-1}}} \in C_G(y).$$

But $C_G(y)$ is finite, so there exists $n \in \mathbf{N}$ such that, for infinitely many m, we have:

$$y^{y^{x_{i_n} x_{j_n}^{-1}}} = y^{y^{x_{i_m} x_{j_m}^{-1}}}, \quad i.e. \quad y^{x_{i_n} x_{j_n}^{-1}} (y^{x_{i_m} x_{j_m}^{-1}})^{-1} \in C_G(y).$$

Therefore there are $t \neq s$ such that:

$$y^{x_{i_n} x_{j_n}^{-1}} (y^{x_{i_s} x_{j_s}^{-1}})^{-1} = y^{x_{i_n} x_{j_n}^{-1}} (y^{x_{i_t} x_{j_t}^{-1}})^{-1},$$

that implies $y^{x_{i_s} x_{j_s}^{-1}} = y^{x_{i_t} x_{j_t}^{-1}}$ and $x_{i_s} x_{j_s}^{-1} (x_{i_t} x_{j_t}^{-1})^{-1} \in C_G(y)$, a contradiction. □

From Lemma 3.1 it follows, with straightforward arguments:

COROLLARY 3.2. *Let $G \in \mathcal{E}_2^*$, G infinite. Then there exists an infinite abelian subgroup A of G.*

Now we can prove Theorem B.

PROOF OF THEOREM B. We show that G is metabelian, the result will follow from Theorem 2.4. Assume that G is not metabelian and let $N \triangleleft G$ be a maximal normal metabelian subgroup of G. There exists a non-trivial normal abelian subgroup M/N of

G/N, since G is hyperabelian. Then M is not metabelian. Furthermore, by Corollary 3.2 there exists an infinite abelian subgroup A of G.

Now we consider the group $H = MA$. Then $A \leq H$ so that H is infinite, and H is not metabelian since $M \leq H$. But H is soluble and that contradicts Theorem 2.4. □

4. Finitely generated $\mathcal{E}_k^*$-groups

Arguing exactly as in Lemma 2.1 and in Corollary 2.3, we can show:

Lemma 4.1. *Let G be either a nilpotent or FC infinite $\mathcal{E}_k^*$-group. Then G is a k-Engel group.*

If G is a finitely generated soluble group, we have also an analogue of Lemma 2.2:

Lemma 4.2. *Let $G \in \mathcal{E}_k^*$ be an infinite finitely generated soluble group. If G has an infinite normal abelian subgroup A, then G is a k-Engel group.*

Proof. Arguing as in the proof of Lemma 2.2 we can easily show that if N is an infinite normal subgroup of G, then $G/N \in \mathcal{E}_k$, and also that A consists of right k-Engel elements. But in a finitely generated soluble group the set of the right Engel elements is contained in some $Z_i(G)$, by a theorem of Gruenberg (see [1]). Hence G/A is nilpotent and $A \leq Z_i(G)$, so that G is nilpotent too. The result follows from Lemma 4.1. □

We are now able to prove Theorem C:

Proof of Theorem C. We show that G is nilpotent by induction on the derived length, $l = l\,(G)$ of G and the Theorem will follow from Lemma 4.2. If $l = 1$, the result is obviously true. Assume that $l > 1$, and let $G^{(l-1)}$ be the last non-trivial term of the derived series of G. If $G^{(l-1)}$ is infinite, then G is nilpotent by Lemma 4.2. Now assume that $G^{(l-1)}$ is finite, then $G/G^{(l-1)}$ is infinite and, by induction $G/G^{(l-1)}$ is nilpotent.

Let $0 \leq i < l - 1$ be maximum such that $G^{(i)}$ is infinite. Then $G^{(i+1)}$ is finite and $G^{(i)}$ is an infinite FC-group in $\mathcal{E}_k^*$. Thus $G^{(i)}$ is a k-Engel group by Lemma 4.1. Furthermore $G^{(i)}$ is finitely generated since $G/G^{(l-1)}$ is polycyclic and $G^{(l-1)} \leq G^{(i)}$. Hence $G^{(i)}$ is nilpotent by a theorem of Gruenberg (see [1]). The group $G/G^{(i+1)}$ is also nilpotent, because $i + 1 \leq l - 1$ and $G^{(i+1)} \geq G^{(l-1)}$. Then G is nilpotent, by a theorem of P. Hall (see [4], vol. I, Th. 2.27, p. 56), as required. □

Finally we remark that there is an analogue of Lemma 3.1:

Lemma 4.3. *Let $G \in \mathcal{E}_k^*$, G infinite. Then $C_G(x)$ is infinite, for every $x \in G$.*

Proof. Assume, by the way of contradiction, that $C_G(y)$ is finite, for some $y \in G$.

Then by arguing as in Lemma 3.1, there exists an infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ of elements of G such that:

i) $x_i \notin C_G(y)$ for any $i \in \mathbf{N}$,

ii) $x_i x_j^{-1} \notin C_G(y)$ for any $i \neq j$, $i, j \in \mathbf{N}$,

iii) $x_i x_j^{-1}(x_h x_k^{-1})^{-1} \notin C_G(y)$ for any $i, j, h, k \in \mathbf{N}$, pairwise different.

Again as in Lemma 3.1, we can find, for any $n \in \mathbf{N}$, some integers $i_n, j_n \in \mathbf{N}$, such that:

$$1 = [y^{x_{i_n}}, {}_k y^{x_{j_n}}] = [y^{x_{i_n} x_{j_n}^{-1}}, {}_k y] = 1\ (^1).$$

Then $[y^{x_{i_n} x_{j_n}^{-1}}, {}_{k-1} y] \in C_G(y)$, so that $y^{[y^{x_{i_n} x_{j_n}^{-1}}, {}_{k-2} y]} \in C_G(y)$. But $C_G(y)$ is finite, thus, for some $n \in \mathbf{N}$ there exist infinitely many m such that:

$$y^{[y^{x_{i_n} x_{j_n}^{-1}}, {}_{k-2} y]} = y^{[y^{x_{i_m} x_{j_m}^{-1}}, {}_{k-2} y]}.$$

Therefore, for infinitely many m,

$$[y^{x_{i_n} x_{j_n}^{-1}}, {}_{k-2} y][y^{x_{i_m} x_{j_m}^{-1}}, {}_{k-2} y]^{-1} \in C_G(y).$$

But this subgroup is finite, thus, for some s, there exist infinitely many t such that:

$$[y^{x_{i_n} x_{j_n}^{-1}}, {}_{k-2} y][y^{x_{i_s} x_{j_s}^{-1}}, {}_{k-2} y]^{-1} = [y^{x_{i_n} x_{j_n}^{-1}}, {}_{k-2} y][y^{x_{i_t} x_{j_t}^{-1}}, {}_{k-2} y]^{-1}$$

and

$$[y^{x_{i_s} x_{j_s}^{-1}}, {}_{k-2} y] = [y^{x_{i_t} x_{j_t}^{-1}}, {}_{k-2} y] \quad \text{i.e.,}$$

$$[y^{x_{i_s} x_{j_s}^{-1}}, {}_{k-3} y][y^{x_{i_t} x_{j_t}^{-1}}, {}_{k-3} y]^{-1} \in C_G(y).$$

That holds for infinitely many t. Continuing in this way, after $k-3$ steps, we get:

$$y^{x_{i_h} x_{j_h}^{-1}}(y^{x_{i_k} x_{j_k}^{-1}})^{-1} \in C_G(y), \qquad \text{for some } h \in \mathbf{N}, \text{ and infinitely many } k.$$

Therefore:

$$y^{x_{i_h} x_{j_h}^{-1}}(y^{x_{i_l} x_{j_l}^{-1}})^{-1} = y^{x_{i_h} x_{j_h}^{-1}}(y^{x_{i_r} x_{j_r}^{-1}})^{-1}, \qquad \text{for some } l \neq r,$$

since $C_G(y)$ is finite and so:

$$y^{x_{i_l} x_{j_l}^{-1}} = y^{x_{i_r} x_{j_r}^{-1}}, \qquad \text{i.e.,} \qquad x_{i_l} x_{j_l}^{-1}(x_{i_r} x_{j_r}^{-1})^{-1} \in C_G(y),$$

contradicting iii). □

An easy consequence of Theorem C is:

COROLLARY 4.4. *Let $G \in \mathcal{E}_k^*$ be a non periodic locally soluble group. Then G is a k-Engel group.*

(^{1}) Following [4, part II, p. 40], if $n \in \mathbf{N}$, G is a group and $a, b \in G$, we put $[a, {}_n b] = [a, \underbrace{b, \ldots, b}_{n}]$.

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