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Dependence of fractional powers of elliptic operators on boundary conditions

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Matematica. — *Dependence of fractional powers of elliptic operators on boundary conditions.* Nota di PAVEL E. SOBOLEVSKII, presentata (*) dal Socio E. Vesentini.

ABSTRACT. — The realization of an elliptic operator A under suitable boundary conditions is considered and the dependence of the square-root of A from the various conditions is studied.

KEY WORDS: Elliptic operators; Fractional powers of operators; Resolvent function.

RIASSUNTO. — *Dipendenza di potenze frazionarie di operatori dalle condizioni al contorno.* Viene considerata la realizzazione di un operatore ellittico A sotto opportune condizioni al contorno e viene studiata la dipendenza della radice quadrata di A dalle diverse condizioni.

1. The character of the results obtained is demonstrated by the following example. Let Ω be a bounded domain in $\mathbf{R}^n$ with boundary $\partial\Omega$ of class $\mathbb{C}^2$ and let $\sigma(x)$ be a (smooth) nonnegative function defined on $\partial\Omega$. We define the operator A_σ which acts in $L_p(\Omega)$ $(1<p<\infty)$ by the formula

$$A_\sigma v(x) = -\Delta v(x) + v(x), \qquad (x \in \Omega), \tag{1}$$

for the function $v(x) \in W_p^2(\Omega)$ which satisfy the boundary condition

$$\partial v(x)/\partial N_x + \sigma(x)\, v(x) = 0, \qquad (x \in \partial\Omega). \tag{2}$$

Here Δ is the Laplace's operator, N_x is an outward normal vector to the boundary $\partial\Omega$ in the point x. This set is called the domain of the operator A_σ in the space $L_p(\Omega)$, and it is denoted by $\mathfrak{D}(A_\sigma, p) = \mathfrak{D}(A_\sigma)$. Evidently $\mathfrak{D}(A_{\sigma_1}) = \mathfrak{D}(A_{\sigma_2})$ iff $\sigma_1(x) \equiv \sigma_2(x)$.

The operator A_σ defined in (1)-(2) is positive and therefore any integral or fractional A_σ^α is defined [1]. It is known (Seeley [2]) that $\mathfrak{D}(A_\sigma^{1/2})$ doesn't depend on σ.

THEOREM 1. *The operator* $K = A_1^{1/2} - A_2^{1/2}$, $A_i = A_{\sigma_i}$, *has a closure* $\overline{K}$ *which is a bounded operator, and the following estimate*

$$\|\overline{K}\|_{L_p \to L_p} \leq Mp^2 (p-1)^{-1} \max_{x \in \partial\Omega} |\sigma_1(x) - \sigma_2(x)| \tag{3}$$

holds.

Here M is a positive constant depending on p.

The case $p = 2$ of this statement was earlier proved in [3].

2. We shall prove Theorem 1 for the case when $\Omega = \mathbf{R}_+^n = \{x = (x_1, \ldots, x_n),\ -\infty < x_i < +\infty,\ i = 1, \ldots, n-1,\ x_n > 0\}$ and $\sigma_i(x)$ are uniformly continuous functions defined on $\partial\Omega$. We can investigate the general case by means of the localization principle [4].

(*) Nella seduta dell'11 maggio 1991.

By formula [1] for fractional powers of positive operators given in [1], we obtain the identity

$$G \equiv (A_2^{1/2}\phi_1, \phi_2) - (\phi_1, A_2^{1/2}\phi_2) = \frac{1}{\pi}\int_0^\infty \lambda^{1/2}\{(A_1[\lambda + A_1]^{-1}\phi_1, [\lambda + A_2]^{-1}\phi_2) -$$

$$-([\lambda + A_1]^{-1}\phi_1, A_2[\lambda + A_2]^{-1}\phi_2)\}\, d\lambda, \quad \phi_1 \in \mathfrak{D}(A_1, p),\ \phi_2 \in \mathfrak{D}(A_2, q),\ 1/p + 1/q = 1$$

and

$$(\psi_1, \psi_2) = \int_\Omega \psi_1(x) \cdot \psi_2(x)\, dx.$$

Integrating by parts and using the boundary conditions (2) for $\sigma = \sigma_i$ we obtain

$$G = \frac{1}{\pi}\int_0^\infty \lambda^{1/2} \int_{\partial\Omega_x} [\sigma_1(x) - \sigma_2(x)][\lambda + A_1]^{-1}\phi_1(x) \cdot [\lambda + A_2]^{-1}\phi_2(x)\, dx\, d\lambda.$$

Let $G_i(x, y; \lambda)$ be a Green-function of the resolvent equation for the operator A_i. The integral representation given in [1]

$$[\lambda + A_i]^{-1}\phi_i(x) = \int_{\Omega_y} G_i(x, y; \lambda)\, \phi_i(y)\, dy$$

yields

$$G = \frac{1}{\pi}\int_0^\infty \lambda^{1/2}\, d\lambda \int_{\partial\Omega_x} [\sigma_1(x) - \sigma_2(x)]\, dx \int_{\Omega_y} G_1(x, y; \lambda)\, \phi_1(y)\, dy \int_{\Omega_z} G_2(x, z; \lambda)\, \phi_2(z)\, dz.$$

It follows from the maximum principle that $0 \leqslant G_i(x, y; \lambda) \leqslant G_0(x - y; \lambda)$. Here $G_0(x; \lambda)$ is the fundamental solution of the resolvent equation for the operator $-\Delta$. Hence the inequality

$$|G| \leqslant \frac{1}{\pi}\int_0^\infty \lambda^{1/2}\, d\lambda \int_{\partial\Omega_x} |\sigma_1(x) - \sigma_2(x)|\, dx \int_{\mathbf{R}_y^n} G_0(x - y; \lambda)\, |\tilde{\phi}_1(y)|\, dy \int_{\mathbf{R}_z^n} G_0(x - y; \lambda)\, |\tilde{\phi}_2(z)|\, dz$$

holds, where

$$\tilde{\phi}_i(x) = \begin{cases} \phi_i(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$$

Further we represent $G_0(x; \lambda)$ by

$$G_0(x; \lambda) = \int_0^\infty \left(2\sqrt{\pi t}\right)^{-n} \exp\left(-\frac{|x|^2}{4t} - \lambda t\right) dt, \qquad |x|^2 = \sum_{i=1}^n x_i^2 \tag{4}$$

with the help of the fundamental solution of the heat equation [1]. That implies the identity $G_0(x;\lambda)=\lambda^{n/2-1}G_0\left(x\sqrt{\lambda};1\right)$, and therefore the inequality

$$|G|\leqslant\frac{1}{\pi}\int\limits_0^\infty\lambda^{-3/2-n}\,d\lambda\int\limits_{\partial\Omega_x}|\sigma_1(x)-\sigma_2(x)|\,dx\int\limits_{\mathbf{R}_y^n}G_0\left(y\sqrt{\lambda};1\right)|\tilde{\varphi}_1(x+y)|dy\,\times$$

$$\times\int\limits_{\mathbf{R}_z^n}G_0\left(z\sqrt{\lambda};1\right)|\tilde{\varphi}_2(x+z)|dz$$

holds. The replacements $y\sqrt{\lambda}\to y$, $z\sqrt{\lambda}\to z$, $\lambda^{-1/2}\to\lambda$ lead then to the inequality

$$|G|\leqslant\frac{2}{\pi}\int\limits_0^\infty d\lambda\int\limits_{\partial\Omega_x}|\sigma_1(x)-\sigma_2(x)|\,dx\int\limits_{\mathbf{R}_y^n}G_0(y;1)|\tilde{\varphi}_1(x+\lambda y)|dy\int\limits_{\mathbf{R}_z^n}G_0(z;1)|\tilde{\varphi}_2(x+\lambda z)|dz.$$

By means of the Hölder inequality we obtain the inequality

$$|G|\leqslant\frac{2}{\pi}\max_{x\in\partial\Omega}|\sigma_1(x)-\sigma_2(x)|\int\limits_{\mathbf{R}_y^n}G_0(y;1)\,dy\int\limits_{\mathbf{R}_z^n}G_0(z;1)dz\left[\int\limits_0^\infty d\lambda\int\limits_{\partial\Omega_x}|\tilde{\varphi}_1(x+\lambda y)|^p\,dx\right]^{1/p}\times$$

$$\times\left[\int\limits_0^\infty d\lambda\int\limits_{\partial\Omega_x}|\tilde{\varphi}_2(x+\lambda z)|^q\,dx\right]^{1/q},\qquad\frac{1}{p}+\frac{1}{q}=1.$$

As $\Omega=\mathbf{R}_+^n$, then

$$\left[\int\limits_0^\infty d\lambda\int\limits_{\partial\Omega_x}|\tilde{\varphi}_1(x+\lambda y)|^p\,dx\right]^{1/p}=\left[\int\limits_0^\infty d\lambda\int\limits_{\partial\Omega_x}|\tilde{\varphi}_1[x+\lambda(0,\ldots,0,y_n)]|^p\,dx\right]^{1/p}=|y_n|^{-1/p}\|\varphi_1\|_{L_p(\Omega)}.$$

The identity

$$\left[\left(\int\limits_0^\infty d\lambda\int\limits_{\partial\Omega_x}|\tilde{\varphi}_2(x+\lambda z)|^q\,dx\right)\right]^{1/q}=|z_n|^{-1/q}\|\varphi_2\|_{L_q(\Omega)}$$

is established in a similar way, and therefore the inequality

$$|G|\leqslant(2/\pi)\max_{x\in\partial\Omega}|\sigma_1(x)-\sigma_2(x)|\,\|\varphi_1\|_{L_p(\Omega)}\,\|\varphi_2\|_{L_q(\Omega)}\int\limits_{\mathbf{R}_y^n}G_0(y;1)\,|y_n|^{-1/p}\,dy\int\limits_{\mathbf{R}_z^n}G_0(z;1)\,|z_n|^{-1/q}\,dz$$

holds.

By (4) we obtain the identity

$$\int\limits_{\mathbf{R}_y^n}G_0(y;1)\,|y_n|^{-1/p}\,dy=\int\limits_0^\infty\exp(-t)\,dt\int\limits_{\mathbf{R}_{y'}^{n-1}}\left(2\sqrt{\pi t}\right)^{-(n-1)}\times$$

$$\times\exp\left(-\frac{|y'|^2}{4t}\right)dy'=\int\limits_0^\infty\left(2\sqrt{\pi t}\right)^{-1}\exp\left(-\frac{|y_n|^2}{4t}\right)dy_n,$$

$$y' = (y_1, \ldots, y_{n-1}), \quad |y'|^2 = \sum_{i=1}^{n-1} y_i^2,$$

and, by direct computations,

$$\int_{\mathbf{R}_y^n} G_0(y;1)\,|y_n|^{-1/p}\,dy = 2^{-1/p}\,\pi^{-1/2}\,\Gamma\left(1-\frac{1}{2p}\right)\Gamma\left[\frac{1}{2}\left(1-\frac{1}{p}\right)\right]$$

and

$$\int_{\mathbf{R}_z^n} G_0(z;1)\,|z_n|^{-1/q}\,dz = 2^{-1/q}\,\pi^{-1/2}\,\Gamma\left(1-\frac{1}{2q}\right)\Gamma\left[\frac{1}{2}\left(1-\frac{1}{q}\right)\right].$$

In conclusion the inequality

$$|G| \leqslant (\pi)^{-2}\,\Gamma(1-1/(2p))\,\Gamma(1/2+1/(2p)) \times$$
$$\times \Gamma[(1-1/p)/2]\,\Gamma(1/(2p)) \max_{x\in\partial\Omega} |\sigma_1(x)-\sigma_2(x)| \times \|\phi_1\|_{L_p(\Omega)}\,\|\phi_2\|_{L_q(\Omega)}$$

holds.

As the sets $\mathfrak{D}(A_1)$ and $\mathfrak{D}(A_2)$ are dense in $L_p(\Omega)$ and $L_q(\Omega)$ respectively, then the last inequality leads to inequality (3).

References

[1] M. A. Krasnoselskii *et al., Integral operators in spaces of summable functions.* Noordhoff International Publishing, Groningen 1976.

[2] R. Seeley, *Fraction powers of boundary problems.* Actes Cong. Math., Nice, vol. 2, 1970, 795-801.

[3] P. E. Sobolevskii, *Comparison theorems for fractional powers of operators.* Soviet Math. Dokl., vol. 8, n. 3, 1967, 25-28.

[4] C. Miranda, *Equazioni alle derivate parziali di tipo ellittico.* Springer Verlag, 1955.

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