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Analisi matematica. — *On the eigen-values of an elliptic operator* $a(x, H(u))$. Nota (*) del Corrisp. Sergio Campanato.

Abstract. — Let Ω be a bounded open convex set of class C^2. Let $a(x, H(u))$ be a non linear operator satisfying the condition (A) (elliptic) with constants α, γ, δ. We prove that a number $\lambda \geq 0$ is an eigen-value for the operator $a(x, H(u))$ if and only if the number $\alpha\lambda$ is an eigen-value for the operator Δu. If $\lambda \geq 0$, the two systems $a(x, H(u)) = \lambda u$ and $\Delta u = \alpha\lambda u$ have the same solutions. In particular, also the eventual eigen-values of the operator $a(x, H(u))$ should all be negative. Finally, we obtain a sufficient condition for the existence of solutions $u \in H^2 \cap H_0^1(\Omega)$ of the system $a(x, H(u)) = b(x, u, Du)$ where $b(x, u, p)$ is a vector in $\boldsymbol{R}^N$ with a controlled growth.

Key words: Non linear elliptic systems; Eigen-values; Conditions for existence.

Riassunto. — *Sugli autovalori di un operatore ellittico* $a(x, H(u))$. Sia Ω un aperto limitato di classe C^2 e convesso. Sia $a(x, H(u))$ un operatore non lineare che verifica la condizione (A) (è ellittico) con costanti α, γ, δ. Si dimostra che il numero $\lambda \geq 0$ è un autovalore per l'operatore $a(x, H(u))$ se e solo se il numero $\alpha\lambda$ è un autovalore per l'operatore Δu. Se $\lambda \geq 0$, i due sistemi $a(x, H(u)) = \lambda u$ e $\Delta u = \alpha\lambda u$ hanno le stesse soluzioni. In particolare anche gli eventuali autovalori dell'operatore $a(x, H(u))$ sono tutti negativi. Si ottiene infine una condizione sufficiente per l'esistenza di soluzioni $u \in H^2 \cap H_0^1(\Omega)$ del sistema $a(x, H(u)) = b(x, u, Du)$ dove $b(x, u, p)$ è un vettore di $\boldsymbol{R}^N$ ad andamento controllato.

1. Introduction

In the present *Note*, Ω is a bounded convex open set of $\boldsymbol{R}^n$ of class C^2 and of generic point x. N is an integer ≥ 1.

Let $u(x): \Omega \to \boldsymbol{R}^N$ be a vector; we shall set $Du = (D_1 u, \ldots, D_n u)$ and $H(u) = \{D_{ij} u\}$, $i, j = 1, \ldots, n$. $H(u)$ is an element of $\boldsymbol{R}^{n^2 N}$, that is, an $n \times n$ matrix of vectors of $\boldsymbol{R}^N$.

Let $a(x, \xi): \Omega \times \boldsymbol{R}^{n^2 N} \to \boldsymbol{R}^N$ be a vector which is measurable in x and continuous in ξ and satisfying the condition that $a(x, 0) = 0$ in Ω and the condition (A): there exist three positive constants α, γ, δ with $\gamma + \delta < 1$, such that $\forall \xi, \tau \in \boldsymbol{R}^{n^2 N}$ we have

$$(1) \qquad \Big\| \sum_i \xi_{ii} - \alpha[a(x, \xi + \tau) - a(x, \tau)] \Big\|_N \leq \gamma \|\xi\| + \delta \Big\| \sum_i \xi_{ii} \Big\|_N .$$

We shall also say that the vector $a(x, \xi)$ is elliptic. It is known that, if the vector $a(x, \xi)$ satisfies the condition (A), then $\forall u \in H^2 \cap H_0^1(\Omega)$ we have $\|a(x, H(u))\|_N \leq \alpha^{-1} c(n) \|H(u)\|$ and hence the operator $a(x, H(u))$ maps $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$.

Moreover, it is also known that, if $a(x, \xi)$ satisfies the condition (A) and Ω is of class C^2 and is convex, then the operator $a(x, H(u))$ is near the operator Δu in the sense that there exist two positive constants α and K, $K \in (0, 1)$ such that, $\forall u, v \in H^2 \cap H_0^1(\Omega)$, we have

$$(2) \qquad \|\Delta u - \alpha[a(x, H(u + v)) - a(x, H(v))]\|_{L^2(\Omega)} \leq K \|\Delta u\|_{L^2(\Omega)}$$

(*) Presentata nella seduta dell'8 febbraio 1992.

(see [1] and [2]). It follows that, $\forall f \in L^2(\Omega)$, the problem

$$a(x, H(u))) = f \quad \text{in } \Omega, \qquad u \in H^2 \cap H_0^1(\Omega)$$

has a solution (has a unique solution) if the problem

$$\Delta u = f \quad \text{in } \Omega, \qquad u \in H^2 \cap H_0^1(\Omega)$$

has a solution (has a unique solution) (see [3, Theorem 2.1]).

2. The problem of eigenvalues

We observe that $\forall u \in H^2 \cap H_0^1(\Omega)$ we have

$$(3) \qquad (\Delta u | u)_{L^2(\Omega)} \leq 0$$

infact $(\Delta u | u)_{L^2(\Omega)} = -\|Du\|^2_{L^2(\Omega)}$. It now follows that

Lemma 1. *If $a(x, H(u))$ is an operator near Δu with constants α and K then, $\forall \lambda \geq 0$ the operator $[a(x, H(u)) - \lambda u]$ is near operator $[\Delta u - \alpha\lambda u]$, considered as operators $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$.*

Proof. From the hypothesis that $a(x, H(u))$ is near Δu it follows that, $\forall u, v \in H^2 \cap H_0^1(\Omega)$, we have

$$(4) \qquad \|\Delta u - \alpha\lambda u - \alpha[a(x, H(u+v)) - a(x, H(v)) - \lambda u]\|_{L^2(\Omega)} =$$

$$= \|\Delta u - \alpha[a(x, H(u+v)) - a(x, H(v))]\|_{L^2(\Omega)} \leq K\|\Delta u\|_{L^2(\Omega)}.$$

On the other hand, in view of the observation (3) and the fact that $\lambda \geq 0$, it follows that

$$(5) \qquad \|\Delta u\|_{L^2(\Omega)} \leq \|\Delta u - \alpha\lambda u\|_{L^2(\Omega)}$$

$\forall u \in H^2 \cap H_0^1(\Omega)$. The assertion follows from (4) and (5).

It follows from the preceding Lemma and Theorem 2.1 of [3] that

Theorem 1. *Under the hypothesis made on Ω and on the vector $a(x, \xi)$ we have that $\forall \lambda \geq 0$ the vector $u \in H^2 \cap H_0^1(\Omega)$ is a solution of the system*

$$(6) \qquad a(x, H(u)) = \lambda u \quad \text{in } \Omega$$

if and only if u is a solution of the system

$$(7) \qquad \Delta u = \alpha\lambda u \quad \text{in } \Omega$$

and hence $u = 0$.

Proof. From the nearness relations (4) and (5), assuming $v = 0$, it follows that

$$(8) \qquad \|\Delta u - \alpha\lambda u - \alpha[a(x, H(u)) - \lambda u]\|_{L^2(\Omega)} \leq K\|\Delta u - \alpha\lambda u\|_{L^2(\Omega)}.$$

Since $K \in (0, 1)$, the assertion follows from the relation (8).

We can conclude that the operator $a(x, H(u)) \colon H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$, has possibly eigenvalues which are all negative.

3. A GENERALIZATION OF THE METHOD

The method used to study the existence of eigenvalues can be generalized to obtain a sufficient condition for the existence of solutions $u \in H^2 \cap H_0^1(\Omega)$ of the system

$$a(x, H(u)) = b(x, u, Du) = B(u). \tag{9}$$

Suppose that $b(x, u, p): \Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{n^2 N} \to \boldsymbol{R}^N$, is a vector which is measurable in x and continuous in the other variables, with respect to which it has a controlled growth, which means that

$$u \in H^2(\Omega) \Rightarrow B(u) \in L^2(\Omega). \tag{10}$$

Then the operators $a(x, H(u)) - B(u)$ and $\Delta u - \alpha B(u)$ map $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$.

DEFINITION 1. *We say that the operator $B(u)$ is monotone with respect to the operator Δu if $\forall u,\ v \in H^2 \cap H_0^1(\Omega)$ we have*

$$(B(u) - B(v) | \Delta(u - v))_{L^2(\Omega)} \leq 0. \tag{11}$$

We have the following

THEOREM 2. *If $B(u)$ is monotone with respect to Δu, then the problem*

$$u \in H^2 \cap H_0^1(\Omega), \qquad a(x, H(u)) = B(u) \quad \text{in } \Omega \tag{12}$$

has a solution (it has exactly one) if the problem

$$u \in H^2 \cap H_0^1(\Omega), \qquad \Delta u = \alpha B(u) \quad \text{in } \Omega \tag{13}$$

has a solution (it has exactly one).

PROOF. In view of Theorem 2.1 of [3] it would be sufficient to prove that the operator $a(x, H(u)) - B(u)$ is near the operator $\Delta u - \alpha B(u)$. Since, by the hypothesis made on Ω and the vector $a(x, \xi)$, the operator $a(x, H(u))$ is near the operator Δu with constants α and K, we have $u, v \in H^2 \cap H_0^1(\Omega)$

$$\|\Delta(u - v) - \alpha[B(u) - B(v)] - \alpha[a(x, H(u)) - a(x, H(v))] - [B(u) - B(v)]\|_{L^2(\Omega)} = \tag{14}$$
$$= \|\Delta(u - v) - \alpha[a(x, H(u)) - a(x, H(v))]\|_{L^2(\Omega)} \leq K \|\Delta(u - v)\|_{L^2(\Omega)}.$$

On the other hand, by the assumption (11), we have $\forall u,\ v \in H^2 \cap H_0^1(\Omega)$

$$\|\Delta(u - v)\|_{L^2(\Omega)} \leq \|\Delta(u - v) - \alpha[B(u) - B(v)]\|_{L^2(\Omega)}. \tag{15}$$

It now follows from (14) and (15) that the operator $[a(x, H(u)) - B(u)]$ is near the operator $[\Delta u - \alpha B(u)]$. The Theorem is thus proved.

REMARK. We observe that a condition weaker than the monotonicity (11), but still sufficient in order that the Theorem 2 continues to hold, is that, $\forall u,\ v \in H^2 \cap H_0^1(\Omega)$, we have

$$(B(u) - B(v) | \Delta(u - v))_{L^2(\Omega)} \leq 2^{-1} \alpha \|B(u) - B(v)\|^2_{L^2(\Omega)}. \tag{16}$$

References

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