Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

Gérard A. Maugin, Carmine Trimarco

Note on a mixed variational principle in finite elasticity

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **3** (1992), n.1, p. 69–74.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1992_9_3_1_69_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1992.

Rend. Mat. Acc. Lincei
s. 9, v. 3:69-74 (1992)

Fisica matematica. — *Note on a mixed variational principle in finite elasticity.* Nota (*) di Gérard A. Maugin e Carmine Trimarco, presentata dal Corrisp. T. Manacorda.

Abstract. — In the present context the variation is performed keeping the deformed configuration fixed while a suitable material stress tensor $\mathcal{S}$ and the material coordinates are required to vary independently. The variational principle turns out to be equivalent to an equilibrium problem of placements and tractions prescribed at the boundary of a body of finite extent.

Key words: Elastostatics; Conservation laws; Fracture.

Riassunto. — *Su di un principio variazionale misto in elasticità finita.* Si fissa la configurazione deformata di un solido elastico mentre si richiede che le coordinate materiali e che $\mathcal{S}$, un opportuno tensore materiale degli sforzi, possano variare in modo indipendente. Si tova che il principio variazionale proposto corrisponde ad un problema di equilibrio meccanico.

1. Introduction

The variational equation here proposed is equivalent to an equilibrium problem of placements and tractions expressed in a material frame work. The introduced functional depends explicitly on the material position $\boldsymbol{X}$ in the reference configuration through the energy density. Such a dependence accounts for material inhomogeneities of the body. The forces related to the inhomogeneities turn out to be balanced inside the body by the so called «configurational forces» through the derived Euler-Lagrange equations. At the boundary an extra force has to be added to the configurational force in order to balance the external tractions and attain equilibrium.

The functional may possibly depend explicitly on $\boldsymbol{x}$ both through the energy density and through the external tractions. Since the variation is performed at $\boldsymbol{x}$ fixed, these Eulerian fields are unaffected by such a variation. Thus, the tractions behave as dead-loading though they are not «dead» in the common understanding of the term [1, p. 310]. This class of Eulerian fields and tractions is of some interest in mechanics and in electromagnetism.

The functional may also depend on a suitably chosen stress tensor $\mathcal{S}$ as well as on $\boldsymbol{x}$ (through $\boldsymbol{C}^{-1}$, $\boldsymbol{C}$ being the right Cauchy strain tensor). In this case the variational principle is a mixed one in the sense that the fields $\mathcal{S}$ and $\boldsymbol{X}$ may vary independently. We will be concerned with this class of functionals and the corresponding E. L. equations will be interpreted as material equilibrium equations. These equations turn out to be equivalent to the Eshelby's conservation law provided that the quantities involved are properly transposed in the reference configuration [2-5]. Since the divergence of the Eshelby energy-momentum tensor is interpreted as the configurational force in the deformed configuration we are led to interpret the tensor's divergence

(*) Pervenuta all'Accademia il 25 settembre 1991.

term, which appears in the derived conservation law, as the configurational force in the deformed configuration. The important role of this force has been pointed out by many authors who have been concerned with the description of microscopic defects, craks or even liquid crystals in the continuum theory [3-6]. A comparison with a classical variational problem shows that an equivalence holds between the latter and the present principle provided that the identity of the material point be preserved.

The present variational principle seems to be of interest in the study of brittle fracture. In fact there is a possibility of establishing path-independent integrals in finite elasticity. These integrals represent an extension of the so called Rice integral [7] or of the Bui integral [8]. The latter is concerned with the complementary stress energy.

2. The variational principle

Consider a hyperelastic body of finite extent whose deformed configuration is denoted by $\mathcal{V}$. $\partial\mathcal{V}$ and $\boldsymbol{n}$ denote the boundary and the external unit normal to it, respectively. The corresponding reference configuration is denoted by V, while ∂V and $\boldsymbol{N}$ represent the corresponding boundary and external unit normal, respectively. V and $\mathcal{V}$ are bounded open sets in $\boldsymbol{R}^3$ and their boundaries are assumed to be regular enough to apply Gauss Theorem. Consider the mapping $\boldsymbol{\chi}: \boldsymbol{X} \to \boldsymbol{x}$, $\boldsymbol{X} \in V$ and $\boldsymbol{\chi} \in \mathcal{C}^3(V)$. The gradient of deformation is denoted by $\boldsymbol{F}$ and the usual assumption $\det \boldsymbol{F} \equiv J_F > 0$ is assumed to hold.

As we wish to perform the variation of the reference configuration while the deformed one is kept fixed, a natural choice of the field describing the deformation is the inverse mapping $\boldsymbol{\chi}^{-1}(\boldsymbol{x})$. Then, having introduced the elastic energy density $\tilde{w}$ (per unit volume of the deformed configuration), we assume it to depend on the deformation through $\hat{\boldsymbol{E}} = 1/2(\boldsymbol{I} - \boldsymbol{C}^{-1}(\boldsymbol{x}))$ We also introduce the following Lagrangian stress tensors

$$\mathcal{S}^* = \partial(\tilde{W})/\partial\hat{\boldsymbol{E}} \quad \text{and} \quad \mathcal{S} = J_F^{-1}\,\mathcal{S}^* \tag{2.1}$$

where $\tilde{W} = J_F \tilde{w}$ represents the energy density per unit volume of the reference configuration. In addition, we introduce the complementary energy density $W_c(\mathcal{S}^*, X)$, the quantity W which is defined as

$$W(\mathcal{S}^*, \hat{E}, X) = \mathcal{S}^* \cdot \hat{E} - W_c(\mathcal{S}^*, X) \tag{2.2}$$

as well as the functional

$$E(\boldsymbol{X}, \mathcal{S}, \boldsymbol{x}) = \int_{\mathcal{V}} [w + \phi(\boldsymbol{x})]\, dv + \int_{\partial\mathcal{V}_1} \mathcal{S}\boldsymbol{F}^{-1}\boldsymbol{n} \cdot (\boldsymbol{X} - \boldsymbol{X}_0)\, ds + \int_{\partial\mathcal{V}_2} \boldsymbol{T}^*(\boldsymbol{x}) \cdot \boldsymbol{X}\, ds \tag{2.3}$$

where $\partial\mathcal{V}_1 \cup \partial\mathcal{V}_2 = \partial\mathcal{V}$; $\partial\mathcal{V}_1 \cap \partial\mathcal{V}_2 = \emptyset$ and $w = J_F^{-1} W$.

Notice that the relation

$$\partial\tilde{W}/\partial\hat{E} \equiv \partial W/\partial\hat{E} \equiv \mathcal{S}^* \tag{2.4}$$

holds true. We are now able to state the

THEOREM. The fields $X(x)$ and $\mathcal{S}$ are solutions of the equilibrium problem of placements and tractions

$$
(2.5)\qquad \begin{cases} \operatorname{div}(\mathcal{S}F^{-1}) - J_F^{-1}\nabla_R(W) + J_F^{-1}\left(\dfrac{\partial W}{\partial X}\right)_{\text{expl.}} = 0 & \text{in } \mathcal{V}, \\ \mathcal{S}F^{-1}n = T^* & \text{on } \partial\mathcal{V}_2 \\ X - X_0 = 0 & \text{on } \partial\mathcal{V}_1 \end{cases}
$$

if and only if they satisfy the following variational equation

$$
(2.6)\qquad \delta_x E - \int_{\partial V} WN\cdot\delta_x X\, ds = 0 .
$$

REMARK. The mixed stress tensor $\mathcal{T} \equiv \mathcal{S}F^{-1} \equiv -J_F^{-1}\,\partial W/\partial F^{-1}$, plays the dual role of the first Piola-Kirchhoff stress tensor.

Performing the two-fold variation with respect to X and $\mathcal{S}$ we derive the following expression

$$
(2.7)\qquad \int_{\mathcal{V}}\left\{\left[-\operatorname{div}\frac{\partial w}{\partial F^{-1}} + \left(\frac{\partial w}{\partial X}\right)_{\text{expl}}\right]\cdot\delta X + \frac{\partial w}{\partial \mathcal{S}}\cdot\delta\mathcal{S}\right\} dv +
$$

$$
+\int_{\partial\mathcal{V}_2}\left[\left(\frac{\partial w}{\partial F^{-1}} - wF^T\right)n + T^*\right]\cdot\delta X\, ds + \int_{\partial\mathcal{V}_1}(X - X_0)\cdot\delta(\mathcal{S}F^{-1}n)\, ds = 0 .
$$

Since δX and $\delta\mathcal{S}$ are independent and arbitrary in $\mathcal{V}$ and on $\partial\mathcal{V}$, the equations

$$
(2.8)\qquad \begin{cases} \operatorname{div}\dfrac{\partial w}{\partial F^{-1}} - \left(\dfrac{\partial w}{\partial X}\right)_{\text{expl.}} = 0 & \text{in } \mathcal{V}, \\ \dfrac{\partial w}{\partial \mathcal{S}} = 0 & \text{in } \mathcal{V}, \\ \left(\dfrac{\partial w}{\partial F^{-1}} - w\,F^T\right)n + T^* = 0 & \text{on } \partial\mathcal{V}_2 , \\ X - X_0 = 0 & \text{on } \partial\mathcal{V}_1 , \end{cases}
$$

hold. Equation $(2.8)_1$ establishes the equilibrium among the «configurational forces» $\operatorname{div}(\partial w/\partial F^{-1})$ and the forces $(\partial w/\partial X)$ due to the inhomogeneities. Equation $(2.8)_2$ explicitly reads

$$
(2.9)\qquad \hat{E} = \frac{\partial W_c}{\partial \mathcal{S}^*} \equiv J_F^{-1}\,\frac{\partial W_c}{\partial \mathcal{S}}
$$

and the remaining equations (2.8) are equivalent to eqs. (2.5). This equivalence can be proved taking into account the following identities:

$$(2.10)\qquad \begin{cases} i) \ \dfrac{\partial W}{\partial \boldsymbol{F}^{-1}} = -\boldsymbol{F}^T \dfrac{\partial W}{\partial \boldsymbol{F}} \boldsymbol{F}^T, \\ ii) \ \dfrac{\partial W}{\partial \boldsymbol{F}^{-1}} = \dfrac{\partial w}{\partial \boldsymbol{F}^{-1}} J_F - w J_F \boldsymbol{F}^T, \\ iii) \ \operatorname{div}(J_F^{-1} \boldsymbol{F}^T) = 0. \end{cases}$$

Once converted to the Eulerian form, the Lagrangian equation $(2.8)_3$ is required to be equivalent to the well known Eulerian condition of equilibrium at the boundary $\partial \mathcal{V}_2$:

$$(2.11)\qquad \boldsymbol{tn} = \boldsymbol{T}^d .$$

Then one finds the Cauchy's stress tensor $\boldsymbol{t}$ under the following form of a stress-energy tensor

$$(2.12)\qquad -\boldsymbol{F}^{T-1} \frac{\partial w}{\partial \boldsymbol{F}^{-1}} + w\boldsymbol{I} = \boldsymbol{t}$$

and

$$(2.13)\qquad \boldsymbol{F}^{T-1} \boldsymbol{T}^* = \boldsymbol{T}^d .$$

Consequently, $\boldsymbol{T}^*$ corresponds to the Lagrangian form of the external forces at equilibrium in the deformed configuration. With reference to equation $(2.8)_3$ notice that the equilibrium surface force acting at the internal boundary cannot be recovered by the «configurational force» $(\partial w/\partial \boldsymbol{F}^{-1})\boldsymbol{n}$ alone. Notice also that since $\boldsymbol{T}^*$ is assumed to depend only on $\boldsymbol{x}$, $\boldsymbol{T}^*$ does not change under the variational operator δ_x while the material points in $\boldsymbol{x}$ may change with the variation and are acted upon different forces. This may be the case of a body immersed in a fluid or in an electromagnetic field.

Remark. Equation $(2.8)_1$ is also equivalent to Cauchy's equilibrium equation in the absence of body forces. This equivalence may be proved by multiplying eq. $(2.8)_1$ to the left by $\boldsymbol{F}^{T-1}$ and taking into account the explicit form of $\nabla w(\boldsymbol{F}^{-1}(\boldsymbol{x}), \boldsymbol{X}(\boldsymbol{x}))$.

The final result is

$$(2.14)\qquad \operatorname{div}\left(\boldsymbol{F}^{T-1} \frac{\partial w}{\partial \boldsymbol{F}^{-1}} - w\boldsymbol{I}\right) \equiv -\operatorname{div} \boldsymbol{t} = 0$$

which is consistent with the previous remark.

3. A classical variational problem

Several authors [9-12], have proposed a principle of stationary complementary energy for a body of finite extent following Hellinger's original idea (1914) [12]. We follow Reissner version of the principle.

The equilibrium problem reads

$$(3.1)\qquad \begin{cases} \mathrm{div}_R \boldsymbol{F}\boldsymbol{S} - \nabla\Phi = 0 & \text{in } V, \\ \boldsymbol{F}\boldsymbol{S}\boldsymbol{N} = \boldsymbol{t}_R & \text{on } \partial V_2, \\ \boldsymbol{x} - \boldsymbol{x}_0 = 0 & \text{on } \partial V_1, \end{cases}$$

where $\Phi(\boldsymbol{x})$ represents the potential energy of the body forces and $\boldsymbol{S}$ the II° Piola-Kirchhoff stress tensor. Equations (3.1) as well as the following

$$(3.2)\qquad \boldsymbol{E} \equiv 1/2\,(\boldsymbol{C} - \boldsymbol{I}) = \partial \overline{W}_c / \partial \boldsymbol{S}$$

may be derived from the variational principle

$$(3.3)\qquad \delta_X F_c = 0$$

where

$$(3.4)\quad F_c = \int_V \{\boldsymbol{S}\cdot\boldsymbol{E} - \overline{W}_c(\boldsymbol{S},\boldsymbol{X}) + \Phi(\boldsymbol{x})\}\, dV - \int_{\partial V_1} \{\boldsymbol{x} - \boldsymbol{x}_0\}\cdot \boldsymbol{F}\boldsymbol{S}\boldsymbol{N}\, ds_0 - \int_{\partial V_2} \boldsymbol{t}_R\{\boldsymbol{X}\}\cdot \boldsymbol{x}\, ds.$$

Once again the variation is mixed since $\boldsymbol{x}$ and $\boldsymbol{S}$ may vary independently. As the tractions $\boldsymbol{t}_R$ are assumed to depend only on $\boldsymbol{X}$, they behave like dead-loading, *i.e.* are unaffected by the variational operator δ_X.

4. A theorem of equivalence

We require now that the *identity of the material point has to be preserved in the variation* δ_X. Such a requirement is formulated in the following form:

$$(4.1)\qquad \frac{\partial}{\partial \varepsilon}\, \chi^{-1}(\boldsymbol{x}(\boldsymbol{X}_\varepsilon), \varepsilon)\Big|_{\boldsymbol{X}\text{ fixed}} = 0.$$

Whence we derive the

Lemma. The identity

$$(4.2)\qquad \delta_x \boldsymbol{X} + \boldsymbol{F}^{-1}\delta_X \boldsymbol{x} = 0$$

holds true.

We are able now to state the

Theorem. The variational equations (2.6) and (3.3) are equivalent to one another provided Φ is constant.

The proof of the Theorem follows straightforwardly by substituting the identity (4.2) into (2.6) or (3.3).

Comments.

(i) The correspondence among $\mathcal{S}$, $\boldsymbol{S}$ and $\boldsymbol{t}$ is the following

$$\mathcal{S}^* = \boldsymbol{C}\boldsymbol{S}\boldsymbol{C} = J_F\, \boldsymbol{F}^T \boldsymbol{t}\boldsymbol{F}.$$

The stress tensor $\mathcal{S}^*$ has no specific denomination in the proper literature while $\mathcal{S}$ is known as the *convected* stress tensor. The latter has been introduced by Green and Rivlin [13].

(ii) The body forces do not appear in the conservation law expressed by eq. $(2.8)_1$ though they have been taken into account in the functional (2.3). On the other hand, they appear explicitly in eq. (3.1), while the material force density $-\partial w/\partial \boldsymbol{X}$, which appears in eq. $(2.8)_1$, is absent in eq. $(3.1)_1$. Since the explicit dependence of W on $\boldsymbol{X}$ takes into account the inhomogeneities of the material space (true material inhomogeneities) we may refer to $\phi(\boldsymbol{x})$ as expressing the presence of inhomogeneities in Eulerian space. Gravitation and interaction with Maxwellian electromagnetic fields belong to this last class.

Acknowledgements

Work supported by G.N.F.M. of C.N.R. and by M.U.R.S.T. Part of the present work was completed while G.A.M. was a guest of the Istituto di Matematiche Applicate «U. Dini» Univ. Pisa, as a Visiting Professor of C.N.R.

References

[1] R. W. Ogden, *Nonlinear Elastic Deformations.* J. Wiley & S., Ellis Horwood Series, Chichester 1984.

[2] J. D. Eshelby, *The Elastic Energy Momentum Tensor.* J. of Elasticity, 5, 1975, 321-335.

[3] R. C. Batra, *The force on a Lattice Defect in an Elastic Body.* J. of Elasticity, 17, 1981, 3-8.

[4] M. Epstein - G. A. Maugin, *The Energy Momentum Tensor and Material Uniformity in Finite Elasticity.* Acta Mechanica, 83, 1990, 127-133.

[5] A. G. Herrmann, *Material Momentum Tensor and Path independent Integrals of Fracture Mechanics.* Int. J. Solids and Structures, 18, 1982, 319-324.

[6] J. D. Eshelby, *The force on a Disclination in a Liquid Crystal.* Philosophical Magazine, 42A, 1980, 359-367.

[7] J. R. Rice, *A path independent Integral and the approximate Analysis of strain Concentrations by notches and cracks.* J. Appl. Mechs., 35, 1968, 379-386.

[8] D. Bui, *Dualité entre les intégrales indépendantes du contour dans la théorie des solides fissurés.* C. R. Acad. Sc. Paris, Série A, t. 276, 1973, 1425-1428.

[9] E. Reissner, *On a Variational Theorem for Finite Elastic Deformations.* J. Math. Physics, 32, 1953, 129-135.

[10] T. Manacorda, *Sopra un Principio Variazionale di E. Reissner per la Statica dei Mezzi Continui.* Boll. U.M.I., 9 (232A), 1954, 154-159.

[11] C. A. Truesdell - R. A. Toupin. In: *Encyclopedia of Physics*, vol. III/1, Springer, Berlin 1960.

[12] E. Hellinger, *Die allgemeinen Ansätze der Mechanik der Kontinua.* Enz. Math. Wiss., 4, 1914, 602-694.

[13] A. E. Green - R. S. Rivlin, *The Mechanics of Non-Linear Materials with Memory.* Arch. Rat. Mech. An., 1, 1957, 1-21; 470.

G. A. Maugin: Laboratoire de Modélisation en Mecanique
Université Pierre-et-Marie Curie
Place Jussieu, 4 - 75252 Paris Cedex 05 (Francia)

C. Trimarco: Istituto di Matematiche Applicate «U. Dini»
Università degli Studi di Pisa
Via Bonanno, 25B - 56126 Pisa