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Cohomology of tensor product of quantum planes

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Algebra. — *Cohomology of tensor product of quantum planes.* Nota (*) di Paolo Papi, presentata dal Corrisp. C. Procesi.

Abstract. — We consider the Lie algebra of inner derivations of the n-fold tensor product of Manin quantum planes and compute its second cohomology group with trivial coefficients.

Key words: Lie algebra; Cohomology; Non-commutative space.

Riassunto. — *Coomologia del prodotto tensoriale di piani quantistici.* Viene considerata l'algebra di Lie delle derivazioni interne del prodotto tensoriale di n piani quantistici alla Manin; di quest'algebra viene calcolato il secondo gruppo di coomologia a coefficienti banali.

Introduction

The skew polynomial ring $\mathbf{C}_q[x,y]$, where $yx = qxy$, $q \in \mathbf{C} - \{0\}$ a fixed non-root of unity, has been called quantum plane in [3]; this is a primitive ring which can be localized at the Öre set of powers of x and y to give a ring of Laurent polynomials $\mathbf{C}_q[x,y,x^{-1},y^{-1}]$. This ring was studied in [2], where its derivations and its automorphism group were determined; in the same paper it was considered the Lie algebra $\mathcal{E}_q^{(1)}$ of inner derivations of $\mathbf{C}_q[x,y,x^{-1},y^{-1}]$ which is an infinite dimensional simple Lie algebra. Its specialization $\mathcal{E}_1^{(1)}$, obtained taking the limit as $q \to 1$, can be regarded as a generalization of a centerless Virasoro algebra and it can be realized as the algebra of periodic function on the torus under usual Poisson bracket.

In the present paper we consider the tensor product of n quantum planes; we show that the corresponding Laurent polynomial ring $S_q^{(n)}$ is simple and then we examine the Lie algebra of inner derivations $\mathcal{E}_q^{(n)}$ in order to show that $\dim H^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) = 2n$, where $2n$ is the number of generators of $S_q^{(n)}$.

The idea of the proof is very straightforward: first we show that we can select a well determined representative for each cohomology class through suitable normalizations; so the computation of $H^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q))$ is reduced to that of normalized cocycles. Then we prove through a series of technical Lemmas that a normalized cocycle must necessarily be of a very special form; this allows to say that $2n$ is an upper bound for $\dim H^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q))$. Finally we get the result exhibiting $2n$ linearly independent normalized cocycles.

1. Tensor product of quantum planes and inner derivations

The object of our study will be the Lie algebra of inner derivations of the n-fold tensor product of a quantum plane by itself. Following Manin [3],

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we call quantum plane the non-commutative space given by the skew polynomial ring $\mathbf{C}_q[x,y]$ where $yx=qxy$ and $q\in\mathbf{C}-\{0\}$ is a fixed non-root of unity.

We consider then $\bigotimes_{i=1}^{n}\mathbf{C}_q[x_i,y_i]$, which can be clearly regarded as the quasipolynomial ring $R_q^{(n)}:=\mathbf{C}[x_1,\dots,x_n,y_1,\dots,y_n]$ with the commutation relations:

$$x_i\,x_j=x_j\,x_i,\ \ y_i\,y_j=y_j\,y_i,\ \ x_i\,y_j=q^{-\delta_{ij}}y_j\,x_i,\qquad \forall i,j=1,\dots,n.$$

Let's denote by $S_q^{(n)}$ the Laurent polynomial ring corresponding to $R_q^{(n)}$.

Proposition 1.1. $S_q^{(n)}$ is a simple ring.

Proof. We refer to the following criterion, due to McConnell and Pettit: If S is the Laurent polynomial ring corresponding to a quasipolynomial ring R, S is simple if and only if its center reduces to the scalars; equivalently, if and only if the following condition is verified:

there does not exist $\underline{m}=(m_1,\dots,m_n)\in\mathbf{Z}^n$, $\underline{m}\neq\underline{0}$, such that for all j, $1\leqslant j\leqslant n$

$$\prod_{i=1}^{n}\lambda_{ij}^{m_i}=1$$

(here, as in [1], (λ_{ij}) is the matrix which expresses the quasi-commutation relations in R; for a proof of the criterion, see [4]).

In our situation, for fixed j, $1\leqslant j\leqslant n$, we have

$$\lambda_{ij}=\begin{cases}q & \text{if } i=j+n,\\ 1 & \text{otherwise.}\end{cases}$$

So

$$\prod_{i=1}^{2n}\lambda_{ij}^{m_i}=1$$

iff $q^{m_{j+n}}=1$, that is $m_{j+n}=0$, $\forall j$, $1\leq j\leq n$.

If instead $n+1\leq j\leq 2n$, one has

$$\lambda_{ij}=\begin{cases}q^{-1} & \text{if } i=j-n,\\ 1 & \text{otherwise}\end{cases}$$

and the above argument works, so that $m_{j-n}=0$, $\forall j$, $n+1\leq j\leq 2n$.

So $m_j=0$, $\forall j$, $1\leq j\leq 2n$: we are in the hypothesis of the criterion, and all follows. □

We now introduce the Lie algebra of inner derivations of $S_q^{(n)}$, $\mathcal{E}_q^{(n)}:=ad\,S_q^{(n)}$.

It is clear that a vector space basis for $\mathcal{E}_q^{(n)}$ is given by $\mathcal{B}=\{E_\alpha\,|\,\alpha\in\mathbf{Z}^n\times\mathbf{Z}^n,\ \alpha\neq(0,0)\}$ where $E_\alpha=ad(x_1^{a_1}\dots x_n^{a_n}\,y_1^{b_1}\dots y_n^{b_n})(\alpha=(\underline{a},\underline{b})\in\mathbf{Z}^n\times\mathbf{Z}^n$, $\underline{a}=(a_1,\dots,a_n)$, $\underline{b}=(b_1,\dots b_n))$.

An easy calculation shows that

$$[E_\alpha,E_{\alpha'}]=\lambda(\alpha,\alpha')\,E_{\alpha+\alpha'}$$

where

$$\lambda(\alpha,\alpha')=q^{\underline{b}\cdot\underline{a}'}-q^{\underline{a}\cdot\underline{b}'}\,(\alpha=(\underline{a},\underline{b}),\ \alpha'=(\underline{a}',\underline{b}'))$$

and $\cdot$ denotes the usual inner product in $\boldsymbol{R}^n$

From Prop. 1.1 we can now deduce

COROLLARY 1.1. $\mathcal{E}_q^{(n)}$ is a simple Lie algebra.

For the proof see [2].

2. COHOMOLOGY OF $\mathcal{E}_q^{(n)}$

We will consider the second cohomology group of the Lie algebra $\mathcal{E}_q^{(n)}$ with coefficients in $\boldsymbol{C}(q)$ (field of rational functions in the variable q).

We denote by $C^r(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))$ the space of r-linear alternating maps from $\mathcal{E}_q^{(n)}\times\ldots\times\mathcal{E}_q^{(n)}$ (r times) to $\boldsymbol{C}(q)$.

To simplify notation, when $f\in C^r(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))$ we write $f(\alpha_1,\ldots,\alpha_r)$ for $f(E_{\alpha_1},\ldots,E_{\alpha_r})$ where $\alpha_i=(\underline{a}_i,\underline{b}_i)$, $\underline{a}_i,\underline{b}_i\in\boldsymbol{Z}^n$.

So, recalling the usual definitions for the complex and the coboundary operator ∂, we have

$$Z^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q)):=\{f\in C^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q)):\partial f=0\}=$$

$$=\{f\in C^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q)):\lambda(\alpha,\beta)f(\gamma,\alpha+\beta)+\lambda(\beta,\gamma)f(\alpha,\beta+\gamma)+\lambda(\gamma,\alpha)f(\beta,\gamma+\alpha)=0\},$$

$$B^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q)):=\{f\in C^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q)):f=\partial g,\ g\in C^1(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))\}=$$

$$=\{f\in C^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))\,|\,f(\alpha,\beta)=\lambda(\alpha,\beta)\,g(\alpha+\beta),\ g\in C^1(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))\}.$$

The basic remark which allows to calculate $H^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))$ is the following: in the cocycle condition we may impose without restriction that the sum $\alpha+\beta+\gamma$ is fixed – and equal, say, to u. So we have a natural way to separate variables; for fixed $u\in\boldsymbol{Z}^{2n}$ we set $Z_u^2=\{\psi_u\,|\,\psi_u(\alpha)=f(\alpha,u-\alpha),\ f\in Z^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))\}$.

Now a cocycle f can be regarded as a collection (ψ_u) where each ψ_u verifies

(A) $\psi_u(\alpha)=-\psi_u(u-\alpha)$

(C) $\lambda(\alpha,\beta)\,\psi_u(\gamma)+\lambda(\beta,\gamma)\,\psi_u(\alpha)+\lambda(\gamma,\alpha)\,\psi_u(\beta)=0,\ \forall\alpha,\beta,\gamma:\alpha+\beta+\gamma=u.$

A direct product decomposition

$$Z^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))=\prod_{u\in\boldsymbol{Z}^{2n}}Z_u^2$$

is also induced; similarly

$$B^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))=\prod_{u\in\boldsymbol{Z}^{2n}}B_u^2,\qquad H^2(\mathcal{E}_q^{(n)},\boldsymbol{C}(q))=\prod_{u\in\boldsymbol{Z}^{2n}}H_u^2.$$

We introduce on $\boldsymbol{Z}^n\times\boldsymbol{Z}^n$ the following symplectic form $\langle\alpha,\alpha'\rangle:=\underline{b}\cdot\underline{a}'-\underline{a}\cdot\underline{b}'$ $(\alpha=(\underline{a},\underline{b}),\ \alpha'=(\underline{a}',\underline{b}'))$.

REMARK 2.1. $\langle\alpha,\alpha'\rangle=0$ if and only if $\lambda(\alpha,\alpha')=0$.

Now we want to normalize cocycles, associating to each one a canonical coboundary verifying a given property.

More precisely, for $g \in C^1(\mathcal{E}_q^{(n)}, \mathbf{C}(q))$, we regard, with respect to the decomposition $B^2 = \prod B_u^2$, the cocycle ∂g as the collection $(\tilde{g}_u)$, where $g_{\tilde{u}}(\alpha) := \lambda(\alpha, u - \alpha)\, g(u)$.

Since $(\tilde{g}_u)$ is a coboundary, it is clear that a cocycle (ψ_u) is cohomologous to $(\psi_u - \tilde{g}_u)$.

Now, we first note that in case $u = (\underline{0}, \underline{0})$, $B^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) = \{0\}$, and we have no problem; if instead $u \neq (\underline{0}, \underline{0})$, we choose $\bar{\alpha}_u, \bar{\beta}_u$ such that $\bar{\alpha}_u + \bar{\beta}_u = u$ and $\lambda(\bar{\alpha}_u, \bar{\beta}_u) \neq 0$; to see that this is possible, it suffices to show that exists $\bar{\alpha}$ with $\langle \bar{\alpha}, u - \bar{\alpha} \rangle \neq 0$. But $\langle \bar{\alpha}, u - \bar{\alpha} \rangle = \langle \bar{\alpha}, u \rangle$ and such an $\bar{\alpha} \equiv \bar{\alpha}_u$ exists since $u \neq (\underline{0}, \underline{0})$ and the symplectic form is non-degenerate.

Then, given (ψ_u), we can always find a cocycle (φ_u) cohomologous to (ψ_u) and such that $\varphi_u(\bar{\alpha}_u) = 0$, $\forall u \neq (\underline{0}, \underline{0})$.

If in fact we define $h \in C^1(\mathcal{E}_q^{(n)}, \mathbf{C}(q))$ setting, for each $u \neq (\underline{0}, \underline{0})$

$$h(u) := \psi(\bar{\alpha}_u)/\lambda(\bar{\alpha}_u, \bar{\beta}_u), \quad h((\underline{0}, \underline{0})) = 0,$$

then we can consider $\varphi_u := \psi_u - \tilde{h}_u$ obtaining a cocycle (φ_u) which verifies the condition demanded.

So we can define *normalized cocycles* as

$$Z_N^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) := \{(\varphi_u) \in Z^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) \colon \varphi_u(\bar{\alpha}_u) = 0, \ \forall u \neq (\underline{0}, \underline{0})\}$$

PROPOSITION 2.1. $H^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) \cong Z_N^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q))$.

PROOF. We have just shown that we can associate to each 2-cocycle a well determined normalized one in the same cohomology class. So we have only to prove that $Z_N^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) \cap B^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) = \{0\}$. Clearly, it suffices to show that (with obvious notations) $Z_{u,N}^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) \cap B_u^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) = \{0\}$. Let $\varphi \in Z_{u,N}^2$ be such that $\varphi = \partial g$; then $\varphi(\alpha, \beta) = \lambda(\alpha, \beta)\, g(\alpha + \beta)$, $\alpha + \beta = u$, and $\varphi(\bar{\alpha}_u, u - \bar{\alpha}_u) = 0 = \lambda(\bar{\alpha}_u, u - \bar{\alpha}_u)\ g(u)$ implies $g(u) = 0$ (since $\lambda(\bar{\alpha}_u, u - \bar{\alpha}_u) \neq 0$) and therefore $\varphi = 0$. $\square$

Now we fix normalization determining explicitly the elements $\bar{\alpha}_u$ for each u; it is clear from the previous discussion that such choices are consistent.

(N1) $\psi_u(\underline{h}, \underline{0}) = 0$, if $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} \neq 0$;

(N2) $\psi_u(\underline{k}, \underline{0}) = 0$, if $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} = 0$, $\underline{k} \neq \underline{0}$;

(N3) $\psi_u(\underline{0}, \underline{h}) = 0$, if $u = (\underline{h}, \underline{0})$, $\underline{h} \neq \underline{0}$.

REMARK 2.2. Note that the condition $\psi_u(\underline{k}, -\underline{h}) = 0$, if $u \neq (\underline{0}, \underline{0})$ would work in any case; however, we prefer to take different normalizations in the three cases since this will simplify the subsequent proofs.

3. DETERMINATION OF $H^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q))$

We want to show now that $Z_{u,N}^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q)) = 0$, $\forall u \neq (\underline{0}, \underline{0})$. Let then $\psi_u \in Z_{u,N}^2(\mathcal{E}_q^{(n)}, \mathbf{C}(q))$ be a normalized cocycle: we want to show $\psi_u \equiv 0$, if $u \neq (\underline{0}, \underline{0})$.

We introduce the following notations: $S = \mathbf{Z}^n \times \mathbf{Z}^n$; $S_u^0 = \{\alpha \in S \colon \psi_u(\alpha) = 0,$

$\forall \psi_u \in Z^2_{u,N}(\mathcal{E}_q^{(n)}, \mathbf{C}(q))\}$. Our goal is to prove $S_u^0 = S$, $\forall u \neq (\underline{0}, \underline{0})$.

PROPOSITION 3.1.

1) $\alpha \in S_u^0$ implies $u - \alpha \in S_u^0$;
2′) $(\underline{h}, \underline{0}) \in S_u^0$, if $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} \neq 0$;
2″) $(\underline{k}, \underline{0}) \in S_u^0$, if $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} = 0$, $\underline{k} \neq \underline{0}$;
2‴) $(\underline{0}, \underline{h}) \in S_u^0$, if $u = (\underline{h}, \underline{0})$, $\underline{h} \neq \underline{0}$;
3) $\alpha \in S_u^0$, $\beta \in S_u^0$, $\langle \alpha, \beta \rangle \neq 0$ imply $u - \alpha - \beta \in S_u^0$, $\alpha + \beta \in S_u^0$;
4) $\alpha \in S_u^0$, $\langle \alpha, u - \alpha - \beta \rangle = 0$, $\langle \alpha, \beta \rangle \neq 0$ imply $u - \alpha - \beta \in S_u^0$, $\alpha + \beta \in S_u^0$;
5) $\langle \beta, u - \alpha - \beta \rangle = 0$, $\langle \alpha, u - \alpha - \beta \rangle = 0$, $\langle \alpha, \beta \rangle \neq 0$ imply $u - \alpha - \beta \in S_u^0$, $\alpha + \beta \in S_u^0$.

PROOF. All statements are obvious: 1) follows from (A); the first assertion of 3), 4), 5) follows from (C) whereas the second follows from the first applying 1); finally 2′), 2″), 2‴) follow from (N1), (N2), (N3) respectively. □

Now we begin the work in view of the main Theorem.

As a notational convention, we refer to the statements of Prop. 3.1 only with their number in Prop. 3.1.

The strategy is slightly different according to whether we consider the case $\mathcal{E}_q^{(1)}$ or the case $\mathcal{E}_q^{(n)}$, $n \geq 2$; in this last instance we begin showing that

i) $(\underline{0}, \underline{n}) \in S_u^0$, $\forall \underline{n}$;
ii) $(\underline{m}, \underline{0}) \in S_u^0$, $\forall \underline{m}$.

Having decided to fix different normalizations in the three cases $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} \neq 0$, $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} = 0$, $\underline{k} \neq 0$, $u = (\underline{h}, \underline{0})$, $\underline{h} \neq \underline{0}$ we will have to give a different argument in each case to prove *i*) and *ii*); the details will be given in Lemmas 3.1-3.3.

Then, using *i*) and *ii*), we show that

iii) $(\underline{m}, \underline{n}) \in S_u^0$, $\forall \underline{m}, \underline{n}$, $\underline{m} \cdot \underline{n} \neq 0$

and this in turn will serve to prove the remaining case:

iv) $(\underline{m}, \underline{n}) \in S_u^0$, $\forall \underline{m}, \underline{n}$, $\underline{m} \cdot \underline{n} = 0$.

The proof of *iii*) and *iv*) – which, being independent of normalization, does not require distinguishing different cases as before – is given in Prop. 3.2.

The situation in the case $\mathcal{E}_q^{(1)}$ is not very different, so we can adapt the previous arguments; this will be done in Prop. 3.3.

LEMMA 3.1. Statements *i*) and *ii*) hold in $\mathcal{E}_q^{(r)}$, $r \geq 2$, for any $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} \neq 0$.

PROOF.

i) $(\underline{0}, \underline{n}) \in S_u^0$, $\forall \underline{n}$.

Let's consider $\alpha = (\underline{0}, \underline{k})$, $\beta = (\underline{h}, -\underline{n})$, $\gamma = (\underline{0}, \underline{n})$. $\alpha \in S_u^0$ by 1) and 2′); besides

$\langle \alpha, \gamma \rangle = 0$ and $\langle \alpha, \beta \rangle = \underline{h} \cdot \underline{k} \neq 0$. By 4), *i*) follows; but clearly a symmetric argument proves *ii*). $\square$

LEMMA 3.2. Statements *i*) and *ii*) hold in $\mathcal{E}_q^{(r)}$, $r \geq 2$, for any $u = (\underline{h}, \underline{k})$, $\underline{h} \cdot \underline{k} = 0$, $\underline{h} \neq \underline{0}$, $\underline{k} \neq \underline{0}$.

PROOF.

i) $(\underline{0}, \underline{n}) \in S_u^0$, $\forall \underline{n}$.

We first suppose $\underline{n} \cdot \underline{k} = 0$, and consider $\alpha = (\underline{k}, \underline{0})$, $\beta = (\underline{h} - \underline{k}, \underline{k} - \underline{n})$, $\gamma = (\underline{0}, \underline{n})$. By 2'') $\alpha \in S_u^0$; $\langle \alpha, \beta \rangle = - |\underline{k}|^2 \neq 0$, $\langle \alpha, \gamma \rangle = - \underline{k} \cdot \underline{n} = 0$.

Now 4) implies $\gamma \in S_u^0$. To solve the remaining case, we may certainly pick an $\underline{n}'$ such that $\underline{n}' \cdot \underline{k} = 0$, $\underline{n}' \cdot \underline{h} \neq 0$ (it cannot be $\underline{h} = \lambda\, \underline{k}$, $\lambda \in \mathbf{Q} - \{0\}$, since $\underline{h} \cdot \underline{k} = 0$). Consider $\alpha = (\underline{0}, \underline{n}')$, $\beta = (\underline{h}, \underline{k} - \underline{n} - \underline{n}')$, $\gamma = (\underline{0}, \underline{n})$; we have just shown that $\alpha \in S_u^0$, but $\langle \alpha, \gamma \rangle = 0$ and $\langle \alpha, \beta \rangle = \underline{n}' \cdot \underline{h}$ which is non-zero by hypothesis, so we can conclude using 4) again.

ii) $(\underline{m}, \underline{0}) \in S_u^0$, $\forall \underline{m}$.

We consider $\alpha = (\underline{k}, \underline{0})$, $\beta = (\underline{h} - \underline{k} - \underline{m}, \underline{k})$, $\gamma = (\underline{m}, \underline{0})$; $\alpha \in S_u^0$ by 2''), $\langle \alpha, \beta \rangle = - |\underline{k}|^2 \neq 0$, $\langle \alpha, \gamma \rangle = 0$, so by 4) we conclude. $\square$

LEMMA 3.3. Statements *i*) and *ii*) hold in $\mathcal{E}_q^{(r)}$, $r \geq 2$ for any $u = (\underline{h}, \underline{0})$, $\underline{h} \neq \underline{0}$ and for any $u = (\underline{0}, \underline{k})$, $\underline{k} \neq \underline{0}$.

PROOF. We first consider the case $u = (\underline{h}, \underline{0})$, $\underline{h} \neq \underline{0}$.

i) $(\underline{0}, \underline{n}) \in S_u^0$, $\forall \underline{n}$.

Let's consider $\alpha = (\underline{0}, \underline{h})$, $\beta = (\underline{h}, -\underline{n} - \underline{h})$, $\gamma = (\underline{0}, \underline{n})$. By 2''') $\alpha \in S_u^0$ and $\langle \alpha, \gamma \rangle = 0$, $\langle \alpha, \beta \rangle = |\underline{h}|^2 \neq 0$, so that 4) gives the assertion.

ii) $(\underline{m}, \underline{0}) \in S_u^0$, $\forall \underline{m}$.

We first prove the assertion when $\underline{m} = \mu\, \underline{h}$, $\mu \in \mathbf{Q}$, $\mu\, \underline{h} \in Z^n$; note that, by *i*), we can essume $\mu \neq 0$; the proof requires three steps:

a) $(\mu\, \underline{h}, \underline{n}) \in S_u^0$, $\forall \mu \in \mathbf{Q} (\mu\, \underline{h} \in Z^n)$, $\forall \underline{n} \neq \underline{0} : \underline{n} \cdot \underline{h} = 0$.

We can pick $\underline{x}$ and $\underline{y}$ such that $\underline{x} \cdot \underline{n} = \mu\, \underline{h} \cdot \underline{y} \neq 0$; then we consider $\alpha = ((1 - \mu)\, \underline{h} - \underline{x}, -\underline{n} - \underline{y})$, $\beta = (\underline{x}, \underline{y})$, $\gamma = (\mu\, \underline{h}, \underline{n})$, obtaining $\langle \beta, \alpha \rangle = \underline{x} \cdot \underline{n} + (1 - \mu)\, \underline{h} \cdot \underline{y} = \mu\, \underline{h} \cdot \underline{y} + (1 - \mu)\, \underline{h} \cdot \underline{y} = \underline{h} \cdot \underline{y} \neq 0$; $\langle \alpha, \gamma \rangle = \mu\, \underline{h} \cdot (-\underline{n} - \underline{y}) - \underline{n} \cdot [(1 - \mu)\, \underline{h} - \underline{x}] = -\mu\, \underline{h} \cdot \underline{n} - \mu\, \underline{h} \cdot \underline{y} - (1 - \mu)\, \underline{h} \cdot \underline{n} + \underline{n} \cdot \underline{x} = -\, \underline{h} \cdot \underline{n} = 0$; $\langle \gamma, \beta \rangle = -\mu\, \underline{h} \cdot \underline{y} + \underline{x} \cdot \underline{n} = 0$ so that we can use 5) to get the assertion.

b) $(\mu\, \underline{h}, \underline{n}) \in S_u^0$, $\forall \mu \in \mathbf{Q} (\mu\, \underline{h} \in Z^n)$, $\forall \underline{n} : \underline{n} \cdot \underline{h} \neq 0$.

Let $\underline{n}' \neq \underline{0}$ be such that $\underline{h} \cdot \underline{n}' = 0$. We consider $\alpha = (\underline{h}, -\underline{n} - \underline{n}')$, $\beta = (-\mu\, \underline{h}, \underline{n}')$, $\gamma = (\mu\, \underline{h}, \underline{n})$; $\beta \in S_u^0$ by *a*) whereas $\alpha \in S_u^0$ by virtue of 1) and *i*). Finally $\langle \beta, \alpha \rangle = \underline{h} \cdot \underline{n}' - \mu\, \underline{h} \cdot \underline{n}' - \mu\, \underline{h} \cdot \underline{n} = -\mu\, \underline{h} \cdot \underline{n} \neq 0$. Now all follows from 3).

c) $(\mu\, \underline{h}, \underline{0}) \in S_u^0$, $\forall \mu \in \mathbf{Q} (\mu\, \underline{h} \in Z^n)$.

We choose $\underline{y}$ such that $\underline{y}\cdot\underline{h}\neq 0$ and set $\alpha=((1-\mu)\,\underline{h},-\underline{y})$, $\beta=(\underline{0},\underline{y})$, $\gamma=(\mu\,\underline{h},\underline{0})$. We note that we can assume $\mu\neq 1$, so that $\langle\alpha,\beta\rangle=(\mu-1)\,\underline{y}\cdot\underline{h}\neq 0$; but α and β belong to S_u^0 respectively by part *b*) and *i*), so we can conclude using 3).

Now we can deal with the general case; we consider $\alpha=(\underline{h},-\underline{n})$, $\beta=(-\underline{m},\underline{0})$, $\gamma=$ $=(\underline{m},\underline{n})$, $\langle\alpha,\beta\rangle=\underline{m}\cdot\underline{n}$, $\langle\gamma,\alpha\rangle=(\underline{m}+\underline{h})\cdot\underline{n}$. So $\gamma\in S_u^0$ if $(\underline{m}+\underline{h})\cdot\underline{n}=0$ and $\underline{m}\cdot\underline{n}\neq 0$ (since $\alpha\in S_u^0$ by *i*) and 1) we can apply 4)).

Given $\underline{m}$, we look for $\tilde{\underline{n}}$ which verifies the above conditions. Such an $\tilde{\underline{n}}$ certainly exists if $\underline{m}\neq\mu\,\underline{h}$ ($\mu\in\mathbf{Q}$) and if we work in dimension ≥ 2 – *i.e.* if we consider $\mathcal{E}_q^{(r)}$ with $r\geq 2$ as we are doing – but the case $\underline{m}=\mu\,\underline{h}$ has been previously examined, whereas we shall deal with $\mathcal{E}_q^{(1)}$ in next Prop. 3.3.

Now we set $\alpha=(\underline{m},\tilde{\underline{n}})$, $\beta=(\underline{0},-\tilde{\underline{n}})$, $\gamma=(\underline{h}-\underline{m},\underline{0})$, and obtain that α and β belong to S_u^0 respectively by the above argument and *i*); since $\langle\alpha,\beta\rangle=\underline{m}\cdot\tilde{\underline{n}}\neq 0$, 3) implies $\alpha+\beta\in S_u^0$, so we get the assertion and the proof of the Lemma in the case $u=(\underline{h},\underline{0})$, $\underline{h}\neq\underline{0}$ is completed.

On the other side, if $u=(\underline{0},\underline{k})$, $\underline{k}\neq\underline{0}$, we can use a similar argument, since normalization (N2) is analogous to normalization (N3) used in the previous case. □

PROPOSITION 3.2. $Z^2_{u,\,N}(\mathcal{E}_q^{(n)},C(q))=0$, $\forall u\neq(\underline{0},\underline{0})$, $\forall n\geq 2$.

PROOF.

iii) $(\underline{m},\underline{n})\in S_u^0$, $\forall\underline{m},\underline{n}$, $\underline{m}\cdot\underline{n}\neq 0$.

We set $\alpha=(\underline{m},\underline{0})$, $\beta=(\underline{0},\underline{n})$, $\gamma=(\underline{h}-\underline{m},\underline{k}-\underline{n})$. By parts *i*) and *ii*) we know that α and β belong to S_u^0; on the other side $\langle\alpha,\beta\rangle=-\,\underline{m}\cdot\underline{n}\neq 0$ so by 3) the statement is proven.

iv) $(\underline{m},\underline{n})\in S_u^0$, $\forall\underline{m},\underline{n}$, $\underline{m}\cdot\underline{n}=0$.

If $(\underline{m},\underline{n})\neq\mu(\underline{h},\underline{k})$, $\mu\in\mathbf{Q}$, we pick $\underline{x}$ and $\underline{y}$ such that $\underline{x}\cdot\underline{y}\neq 0$, $\underline{x}\cdot\underline{k}\neq\underline{y}\cdot\underline{h}$, $\underline{m}\cdot\underline{y}=$ $=\underline{n}\cdot\underline{x}$. Now we consider, in the previous hypothesis $\alpha=(\underline{x},\underline{y})$, $\beta=(\underline{h}-\underline{m}-\underline{x},\underline{k}-$ $-\underline{n}-\underline{y})$, $\gamma=(\underline{m},\underline{n})$.

It results: $\langle\alpha,\gamma\rangle=\underline{m}\cdot\underline{y}-\underline{n}\cdot\underline{x}=0$; $\langle\alpha,\beta\rangle=-(\underline{k}-\underline{n}-\underline{y})\cdot\underline{x}+(\underline{h}-\underline{m}-\underline{x})\cdot\underline{y}=-$ $-\underline{x}\cdot\underline{k}+\underline{y}\cdot\underline{h}\neq 0$.

Since $\alpha\in S_u^0$ – see *iii*) –, 4) applies and therefore $\gamma\in S_u^0$. We have only to deal with the case $(\underline{m},\underline{n})=\mu(\underline{h},\underline{k})$, $\mu\in\mathbf{Q}$; we may also assume $\mu\neq 1$, and choose again $\underline{x}$ and $\underline{y}$ so that $\underline{x}\cdot\underline{y}\neq 0$, $\underline{x}\cdot\underline{k}\neq\underline{y}\cdot\underline{h}$, $(\mu-1)^2\underline{h}\cdot\underline{k}+(\mu-1)\;(\underline{x}\cdot\underline{k}+\underline{y}\cdot\underline{h})+\underline{x}\cdot\underline{y}\neq 0$.

Then, if we set $\alpha=((1-\mu)\;\underline{h}-\underline{x},\;(1-\mu)\;\underline{k}-\underline{y})$, $\beta=(\underline{x},\underline{y})$, $\gamma=(\mu\,\underline{h},\mu\,\underline{k})$, we obtain $\langle\alpha,\beta\rangle=(1-\mu)\;(\underline{x}\cdot\underline{k}-\underline{y}\cdot\underline{h})\neq 0$, and, by virtue of *iii*), $\alpha\in S_u^0$, $\beta\in S_u^0$. Hence $\gamma\in S_u^0$ in this last case too. □

PROPOSITION 3.3. $Z^2_{u,\,N}(\mathcal{E}_q^{(1)},C(q))=0$, $\forall u\neq(\underline{0},\underline{0})$.

PROOF. We first consider the case $u=(h,k)$, $hk\neq 0$, and note that we have nothing to prove since steps *i*), *ii*) in Lemma 3.1 and step *iii*) in Prop. 3.2 work without changes.

So we have only to examine the case $u=(h,0)$, $h\neq 0$ (the case $u=(0,k)$, $k\neq 0$ is treated analogously).

Proceeding exactly as in the proof of Lemma 3.3 *i*), we show that

a) $(0,n)\in S_u^0$, $\forall n\in \mathbf{Z}$.

Then we prove:

b) $(mh,n)\in S_u^0$, $\forall m\in \mathbf{Z}$, $\forall n\neq 0$.

We use the cocycle condition (C) with $\alpha=u$, $\beta=-\gamma=(r,s)$, obtaining at once, if $s\neq 0$

$$(*)\qquad \psi_u(-r,-s)=-q^{-hs}\psi_u(r,s).$$

Now, using (A), $(*)$ and induction on $m\in \mathbf{N}$ – whose first step is given by part *a*) – we get the statement for any $m\geq 0$: $\psi_u(mh,n)=\psi_u((h,0)-((1-m)h,-n))=$ $=-\psi_u(-(m-1)h,-n)=q^{-hn}\psi_u((m-1)h,n)=0$.

If $m\leq 0$ we can use $(*)$ and conclude.

c) $(mh,0)\in S_u^0$, $\forall m\in \mathbf{Z}$.

By *a*), we can assume $m\neq 0$; let's consider $\alpha=(h,-h)$, $\beta=(-mh,h)$, $\gamma=(mh,0)$, α and β belong to S_u^0 respectively by normalization (and antisimmetry) and the previous part; furthermore $\langle\alpha,\beta\rangle=h^2(m-1)\neq 0\Leftrightarrow m\neq 1$.

Now, if $m\neq 1$, we conclude using 3). If instead $m=1$, applying 1) to part *a*) with $n=0$ we get the assertion in this case too. Until now we have proven that

$$(**)\qquad (m,n)\in S_u^0,\ \forall n, \forall m\equiv 0 \bmod h.$$

d) $(t,n)\in S_u^0$, $\forall n\neq 0$, $\forall t$.

Let's consider $\alpha=(-th,-hn)$, $\beta=(h+th-t,hn-n)$, $\gamma=(t,n)$; $\alpha\in S_u^0$ by $(**)$, $\langle\alpha,\gamma\rangle=0$, $\langle\alpha,\beta\rangle=-h^2n\neq 0$ so we conclude by 4).

e) $(rh+t,0)\in S_u^0$, $\forall r\in \mathbf{Z}$, $\forall t=1,\dots,h-1$.

For $n\neq 0$ we set $\alpha=(-t,-n)$, $\beta=((-r+1)h,n)$, $\gamma=(rh+t,0)$. Again $\beta\in S_u^0$ by $(**)$, $\alpha\in S_u^0$ by part *d*) and $\langle\beta,\alpha\rangle\neq 0\Leftrightarrow t\neq h(1-r)$ which is always true. Now we are in the hypothesis of 3). □

THEOREM 3.1. $\dim H^2(\mathcal{E}_q^{(n)},\mathbf{C}(q))=2n$. Moreover a basis for $H^2(\mathcal{E}_q^{(n)},\mathbf{C}(q))$ is given by the cocycles $f_i(\alpha,\alpha')=\delta_{\alpha,-\alpha'}a_iq^{-\underline{a}\cdot\underline{b}}$, $f_{i+n}(\alpha,\alpha')=\delta_{\alpha,-\alpha'}b_iq^{-\underline{a}\cdot\underline{b}}$ where $\alpha,\alpha'\in \mathbf{Z}^n\times$ $\times\mathbf{Z}^n$, $\alpha=(\underline{a},\underline{b})$, $\alpha'=(\underline{a}',\underline{b}')$, $i=1,\dots,n$.

PROOF. By Propositions 3.2, 3.3 we know that $\psi_u\equiv 0$ if $u\neq(\underline{0},\underline{0})$. So we consider $\psi\equiv\psi_{(\underline{0},\underline{0})}$; we set: $\alpha=(\underline{h},\underline{k})$, $\beta=(-\underline{h},\underline{0})$, $\gamma=(\underline{0},-\underline{k})$, and define $a(\underline{h}):=\psi((-\underline{h},\underline{0}))$, $b(\underline{k}):=\psi((\underline{0},-\underline{k}))$.

A routine calculation, which uses only the cocycle condition (C), gives, if $\underline{h}\cdot\underline{k}\neq 0$ $\psi((\underline{h},\underline{k}))=-q^{-\underline{h}\cdot\underline{k}}(a(\underline{h})+b(\underline{k}))$.

Now we choose $\underline{r}$ such that $\underline{r}\cdot\underline{k}\neq\underline{0}$, $\underline{r}\cdot\underline{h}\neq 0$, $\underline{h}\cdot\underline{r}\neq-\underline{r}\cdot\underline{k}$, and consider $\alpha=(\underline{h},\underline{r})$, $\beta=(\underline{k},\underline{0})$, $\gamma=(-\underline{h}-\underline{k},-\underline{r})$ (we assume $\underline{h}\neq-\underline{k}$).

We have $\lambda(\alpha,\beta)=q^{\underline{r}\cdot\underline{k}}-1$, $\lambda(\beta,\gamma)=1-q^{-\underline{r}\cdot\underline{k}}$, $\lambda(\gamma,\alpha)=q^{-\underline{h}\cdot\underline{r}}(1-q^{-\underline{r}\cdot\underline{k}})$, and the cocycle condition gives

$$-(q^{\underline{r}\cdot\underline{k}}-1)\,q^{-(\underline{h}+\underline{k})\cdot\underline{r}}(a(\underline{h}+\underline{k})+b(\underline{r}))+$$
$$+(1-q^{-\underline{r}\cdot\underline{k}})q^{-\underline{h}\cdot\underline{r}}(a(\underline{h})+b(\underline{r}))+q^{-\underline{h}\cdot\underline{r}}(1-q^{-\underline{r}\cdot\underline{k}})\,a(\underline{k})=0$$

that is

$$(***)\qquad a(\underline{h}+\underline{k})=a(\underline{h})+a(\underline{k}).$$

Note that (***) holds even if $u=(\underline{h},-\underline{h})$ – direct verification –.

With a symmetric argument, we obtain an analogous relation for b. But for a function c which verifies (***) it holds $c(\underline{h})=c((h_1,0,\dots,0)+(0,h_2,0,\dots,0)+\dots+(0,0,\dots,h_n))=h_1\,c(1,0,\dots,0)+h_2\,c(0,1,0,\dots,0)+\dots+h_n\,c(0,0,\dots,1)$.

If now $\alpha=(\underline{h},\underline{k})$ with $\underline{h}\cdot\underline{k}=0$, we consider $\alpha=(\underline{h},\underline{k})$, $\beta=(\underline{r},\underline{s})$, $\gamma=(-\underline{h}-\underline{r},-\underline{s}-\underline{k})$ where $\underline{r}$, $\underline{s}$, are such that $\underline{r}\cdot\underline{s}\neq 0$, $\underline{h}\cdot\underline{s}\neq\underline{k}\cdot\underline{r}$, $\underline{h}\cdot\underline{s}+\underline{k}\cdot\underline{r}+\underline{r}\cdot\underline{s}\neq 0$.

Applying the cocycle condition, we can express $\psi(\alpha)$ by means of $\psi(\beta)$, $\psi(\gamma)$; but β and γ are included in the previous case. What we have seen until now shows that $\dim Z^2_N(\mathcal{E}_q^{(n)},\mathbf{C}(q))=\dim H^2(\mathcal{E}_q^{(n)},\mathbf{C}(q))\leq 2n$.

Finally, we note that the cocycles f_i, $i=1,\dots,2n$, are linearly independent and normalized, so that $\dim H^2(\mathcal{E}_q^{(n)},\mathbf{C}(q))=2n$ as asserted. □

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