Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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On motions with bursting characters for Lagrangian mechanical systems with a scalar control. II. A geodesic property of motions with bursting characters for Lagrangian systems

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **3** (1992), n.1, p. 35–42.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1992_9_3_1_35_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1992.

Rend. Mat. Acc. Lincei
s. 9, v. 3:35-42 (1992)

Meccanica. — *On motions with bursting characters for Lagrangian mechanical systems with a scalar control. II. A geodesic property of motions with bursting characters for Lagrangian systems.* Nota di ALDO BRESSAN e MARCO FAVRETTI, presentata (*) dal Corrisp. A. Bressan.

ABSTRACT. — This *Note* is the continuation of a previous paper with the same title. Here (Part II) we show that for every choice of the sequence $u_a(\cdot)$, Σ_a's trajectory l_a after the instant $d+\eta_a$ tends in a certain natural sense, as $a \to \infty$, to a certain geodesic l of V_d, with origin at $(\bar{q}, \bar{u})$. Incidentally l is independent of the choice of applied forces in a neighbourhood of $(\bar{q}, \bar{u})$ arbitrarily prefixed.

KEY WORDS: Lagrangian systems; Feedback theory; Bursts.

RIASSUNTO. — *Sui moti per sistemi Lagrangiani con controllo scalare, aventi caratteri di scoppio. II. Una proprietà geodetica di certi moti per sistemi Lagrangiani, con caratteri di scoppio.* In questa *Nota,* che è la Parte II di una precedente *Nota* dallo stesso titolo si mostra che, per ogni scelta della suddetta successione $u_a(\cdot)$, la traiettoria l_a di Σ_a dopo $d+\eta_a$ tende in un certo senso naturale, per $a \to \infty$, a una certa geodetica l della varietà V_d, uscente dal punto $(\bar{q}, \bar{u})$. Tra l'altro la l è indipendente dalla scelta delle forze attive in un intorno di $(\bar{q}, \bar{u})$ comunque prefissato.

4. INTRODUCTORY CONSIDERATIONS. SOME KINEMATIC PRELIMINARIES

This Part II is the continuation of Part I of the *Note* of the same title. Please refer to Part I for definitions, annotations and references (see Rend. Mat. Acc. Lincei, s. 9, vol. 2, 1991, 339-343).

This second part of the work is restricted to systems whose applied forces have Lagrangian components at most *linear* in $\dot{u}$ (but $\dot{u}^2$ occurs in $SHE_{\Sigma,u}$). For these, a certain family of controls $u_{j,\eta}(\cdot)$ is considered as well as the trajectory $l_{j,\eta}$ described by $\Sigma_{u(\cdot)}$'s representative point P in Hertz's space $\mathbf{R}^{3\nu}$, in connection with Σ_u's dynamic motion that solves the Cauchy problem (1.1). Briefly speaking, certain sequences of controls $u_{j,\eta}(\cdot)$ are used along which $|j| \to 0$, $\eta \to 0^+$ and $j^2\eta^{-1} \to +\infty$; Theor. 6.1 asserts that along them $l_{j,\eta}$ tends in a certain sense to a geodesic of the manifold that represents in $\mathbf{R}^{3\nu}$ the possible positions for P at $t=d$.

It is not restrictive to regard Σ as a system of ν mass points P_1 to P_ν having the respective masses m_1 to m_ν and subject to holonomic and frictionless constraints. Let $O\mathbf{c}_1\,\mathbf{c}_2\,\mathbf{c}_3$ be a (physical) orthonormal frame and let x_i, y_i, z_i be P_i's coordinates in it $(i=1,\ldots,\nu)$. We now consider Hertz's space $\mathbf{R}^{3\nu}$, which is referred to the coordinates ξ_1 to $\xi_{3\nu}$:

$$(4.1)\qquad \xi_i=(m_i)^{1/2}x_i\,,\quad \xi_{\nu+i}=(m_i)^{1/2}y_i\,,\quad \xi_{2\nu+i}=(m_i)^{1/2}z_i\,,\qquad (i=1,\ldots,\nu)\,.$$

Thus any configuration $(x_1,y_1,z_1,\ldots,x_\nu,y_\nu,z_\nu)$ of Σ is represented by $P=(\xi_1,\ldots,\xi_{3\nu})$. Furthermore, we fix the intervals I and H, with H compact, and in con-

(*) Nella seduta del 14 giugno 1991.

nection with the typical function $u \in C^2(I, H)$, we consider the system $\Sigma_{u(\cdot)}$ obtained from Σ by adding the (frictionless) constraint $u = u(t)$. For the sake of simplicity, we assume that, for some open set Ω and some function $\mathbb{P}(\cdot,\cdot,\cdot) \in C^2(I \times \Omega \times H, \boldsymbol{R}^{3\nu})$ the manifold $V[V^{u(\cdot)}]$ «allowed» to $\Sigma[\Sigma_{u(\cdot)}]$ by its constraints – or a suitable part of it – is represented by the 1st [2nd] of the equations

$$(4.2)\qquad \begin{cases} P = \mathbb{P}(t, q, u) & \text{for } (t, q, u) \in I \times \Omega \times H, \\ P = P(t, q), \quad \text{where } P(t,q) := \mathbb{P}(t, q, u(t)) & \text{for } (t, q) \in I \times \Omega . \end{cases}$$

We set $V_t := \{P(t, q) | (q, u) \in \Omega \times H\}$ and $V_t^u := \{\mathbb{P}(t, q, u) | q \in \Omega\}$. Now in connection with $\Sigma_{u(\cdot)}$, we consider an ideal fluid $F^{u(\cdot)}$ whose points are represented by Ω's elements and, for every $q \in \Omega$, «$F^{u(\cdot)}$'s point q» undergoes the motion (4.2). Hence, along any given motion $x_1 = x_1(t), \ldots, z_\nu = z_\nu(t)$ for $\Sigma_{u(\cdot)}$, P's motion $q = q(t)$ w.r.t. (with respect to) $F^{u(\cdot)}$ is determined, as well as P's motion

$$(4.3)\qquad P = P(t, q(t)) = \mathbb{P}(t, q(t), u(t))$$

w.r.t. Hertz's space $\mathbf{R}^{3\nu}$. As is well known, P's velocity and acceleration w.r.t. $\mathbf{R}^{3\nu}$ (along P's actual motion) have the expressions ([1])

$$(4.4)\qquad \begin{cases} \boldsymbol{v} = \boldsymbol{v}^{(d)} + \boldsymbol{v}^{(r)} := P_{/0} + P_{/h} \dot{q}^h , \\ \boldsymbol{A} = \boldsymbol{a}^{(d)} + \boldsymbol{a}^{(r)} + \boldsymbol{a}^{(c)} := P_{/00} + (P_{/h} \ddot{q}^h + P_{/hk} \dot{q}^h \dot{q}^k) + 2 P_{/0h} \dot{q}^h . \end{cases}$$

When $\mathbf{R}^{3\nu}$ is regarded as the fixed space, one can call $\boldsymbol{v}^{(d)}$ [$\boldsymbol{a}^{(d)}$] *dragging velocity* [*acceleration*], $\boldsymbol{v}^{(r)}$ [$\boldsymbol{a}^{(r)}$] *relative velocity* [*acceleration*], and $\boldsymbol{a}^{(c)}$ *complementary* (*or generalized Coriolis'*) *acceleration* of P at the instant t.

Having fixed the instant t^*, we say that M^* is (a local) *virtual* motion of P relative to t^* in case M^* is the motion on the manifold $V_{t^*}^{u(t^*)}$ represented in some neighbourhood I of t^* by $t \vdash \mathbb{P}(t^*, q(t), u(t^*))$, see (4.3). Calling $\boldsymbol{v}^* = \boldsymbol{v}^*(t) [\boldsymbol{a}^* = \boldsymbol{a}^*(t)]$ P's velocity [acceleration] w.r.t. $\mathbf{R}^{3\nu}$ along the motion M^* at any $t \in I$, by (4.4) we have

$$(4.5)\qquad \boldsymbol{v}^*(t^*) = \boldsymbol{v}^{(r)}(t^*), \qquad \boldsymbol{a}^*(t^*) = \boldsymbol{a}^{(r)}(t^*) \qquad \text{– see (4.4) and ftn.1.}$$

For $(t, q, u) \in I \times \Omega \times H$, let $T(t, q, u)$ be the tangent space of V_t^u at $P = \mathbb{P}(t, q, u)$ *i.e.* the affine space $P + \text{span}\{\mathbb{P}_{/1}(t, q, u), \ldots, \mathbb{P}_{/N}(t, q, u)\}$ endowed with the norm determined by the metric tensor $a_{hk} := \mathbb{P}_{/h} \times \mathbb{P}_{/k} (h, k = 1, \ldots, N)$. Thus, *e.g.* $v^* = |\boldsymbol{v}^*| = (a^{hk} v_h^* v_k^*)^{1/2}$, being $a^{hk} = (a_{hk})^{-1}$. By projecting $\boldsymbol{a}^*$ and $\boldsymbol{A}^*$ on $V_{t^*}^{u(t^*)}$'s tangent space $T(P^*)$ at $P^* = \mathbb{P}(t^*, q(t^*), u(t^*))$ one obtains

$$(4.6)\qquad \boldsymbol{a}_\sigma^* := (\boldsymbol{a}^* \times P^{/h}) P_{/h} = \left[\begin{Bmatrix} h \\ k \quad l \end{Bmatrix} \dot{q}^k \dot{q}^l + \ddot{q}^h \right] P_{/h}$$

([1]) We set $q^0 = t$, $q^N = u$, $P_{/\alpha} := \partial P / \partial q^\alpha$, $P_{/\alpha\beta} := \partial^2 P / \partial q^\alpha \partial q^\beta$, and briefly we mean definitions $(4.4)_{2-4}$ «termwise»; furthermore, Greek indices run from 0 to N, Latin indices run from 1 to N.

and

$$(4.7)\qquad A_\sigma := (A \times P^{/h}) P_{/h} = [(a^{(d)} + a^{(c)} + a^{(r)}) \times P^{/h}] P_{/h} =$$

$$= \left[\left\{ {h \atop 0\ \ 0} \right\} + 2 \left\{ {h \atop 0\ \ k} \right\} \dot{q}^k + \left\{ {h \atop k\ \ l} \right\} \dot{q}^k \dot{q}^l + \ddot{q}^h \right] P_{/h} .$$

5. Sequences of controls that afford a burst of Σ

In this section, conditions (α) to (β) below are assumed:

(α) $u_a = u_{j_a, \eta_a}$ for some $j_a > 0$, $\eta_a > 0$ $\forall a \in N_* := \{1, 2, 3, \dots\}$,

(β) $z^{(a)}(\cdot) = (q_{(a)}(\cdot), p^{(a)}(\cdot))$ is the (maximal) solution of (2.5) for $u = u_a$ $\forall a \in N_*$.

In the sequel, we set

$$|b| = \left(\sum_{k=1}^{N} b_k^2 \right)^{1/2} \qquad \text{for } b \in \mathbf{R}^{3\nu},$$

$$|p^{(a)}(t)| = \left(\sum_{k=1}^{N} p_k^{(a)}(t)^2 \right)^{1/2}, \qquad \text{and} \qquad |q_{(a)}(t)| = \left(\sum_{k=1}^{N} q_{(a)}^k(t)^2 \right)^{1/2}.$$

Theorem 5.1. (*a*) *For some sequences u_a of controls of the type* (2.8) – *see* (α)

$$(5.1)\qquad |q_{(a)}(d + \eta_a) - \bar{q}| < 1/a, \qquad |\dot{q}^h_{(a)}(d + \eta_a) P_{/h}| > a \qquad (a \in N_*).$$

(*b*) *If* (5.1) *holds and $\bar{\zeta} := (d, \bar{q}, \bar{u}) \in I \times \Omega \times H$, then, by using* «u.v.» *for «unit vector of»*

$$(5.2)\qquad \begin{cases} \lim w_a = \text{u.v. } [2^{-1}(A_{NN,h}(\bar{\zeta}) + 2Q_{hNN}(\bar{\zeta})) P^{/h}], \\ \text{where} \\ w_a = \text{u.v. } [q^h_{(a)}(T_a) \mathrm{P}_{/h}(T_a, q_{(a)}(T_a), u_a(T_a))] \ \text{ with } T_a := d + \eta_a, u_a(T_a) = u + j_a . \end{cases}$$

Proof. Fix the last integer $r > 0$ with $\bar{D} \subseteq I \times \Omega \times H$, where $D := B(d, 1/r) \times B(\bar{q}, 1/r) \times B(\bar{u}, 1/r)$, call $\rho (> 0)$ and $\sigma (> 0)$ the maximum and minimum eigenvalues of the matrix a_{hk} for $(t, q, u) \in \bar{D}$, and call b the maximum value of $|b(t, q, u)|$ for $(t, q, u) \in \bar{D}$. By Theor. 3.1 for any $a \in N_*$ there is a constant C_* and a $j_a \in (0, 1)$ such that for a suitably small $\eta_a \in (0, 1)$ we have (5.1) and (*i*) $|p^{(a)}(d + \eta_a)| > C_* j_a^2 \eta_a^{-1}$. Hence, by rendering η_a smaller, we also have (*ii*) $C_* j_a^2 \eta_a^{-1} > (a \rho \sigma^{-1} + b)$. Furthermore, by $(2.4)_2$, (*iii*) $|\dot{q}_{(a)}(d + \eta_a)| > \rho^{-1}(|p^{(a)}(d + \eta_a)| - b)$; then by (*i*) and (*ii*) $|\dot{q}^h_{(a)}(d + \eta_a) P_{/h}| > \sigma |\dot{q}^h_{(a)}(d + \eta_a)| > \sigma \rho^{-1}(C_* j_a^2 \eta_a^{-1} - b) > \sigma \rho^{-1} a \sigma^{-1} \rho = a$. Hence $(5.1)_2$ also holds. Thus (*a*) is proved. Note that, for any sequence of controls satisfying condition (α) and (5.1), one has (*iv*) $j_a \to 0$, $\eta_a \to 0^+$ and $j_a^2 \eta_a^{-1} \to \infty$ as $a \to \infty$.

To prove (*b*), consider the following transformation $(q_{(a)}(\cdot), p^{(a)}(\cdot)) \vdash (K_{(a)}(\cdot), P^{(a)}(\cdot))$ for any solution $z^{(a)}(\cdot)$ of the ODE (2.5) with $u = u_a$ where $a \in N_*$, (5.1) holds, and for $\tau \in [0,1]$:

$$(5.3)\qquad \begin{cases} K^b_{(a)}(\tau) := q^b_{(a)}(t(\tau)), \qquad P^{(a)}_b(\tau) := p^{(a)}_b(t(\tau))\,\lambda_a, \\ \text{being} \\ t(\tau) := d + \eta_a \tau \qquad \text{and} \qquad \lambda_a = \eta_a j_a^{-2}. \end{cases}$$

It is easy to see that thus, since $u_a = j_a \eta_a^{-1}$ and *e.g.* $\overset{\prime}{P}{}^{(a)}_b := dP^{(a)}_b/d\tau$, problem (2.4) takes the form:

$$(5.4)\qquad \begin{cases} \overset{\prime}{P}{}^{(a)}_b = -\dfrac{j_a^2}{2} P^{(a)}[(a^{-1})_{,b} - 2Q^{(2)}_b]P^{(a)} + \eta_a P^{(a)}[(a^{-1}b)_{,b} + Q^{(1)}_b] + \\ \qquad + \dfrac{1}{2}[A_{NN,b} + 2Q_{bNN}] + [B_b + Q_{bN}]\lambda_a j_a + \\ \qquad + \dfrac{1}{2}\{[b^{-1}ab + 2C]_{,b} + 2Q_{0b}\}\lambda_a \eta_a, \qquad P^{(a)}_b(0) = \bar{p}_b \lambda_a, \\ \overset{\prime}{K}{}^b_{(a)} = j_a a^{bk}(P^{(a)}_k - \lambda_a b_k), \qquad K^b_{(a)}(0) = \bar{q}^b; \end{cases}$$

and $(5.2)_2$ yields the first two among the equalities

$$(5.5)\qquad \begin{cases} W^b_{(a)} = \dfrac{\overset{\prime}{K}{}^b_{(a)}(1)}{|\overset{\prime}{K}{}^b_{(a)}(1)\,P_{/b}|} = \dfrac{a^{bk}(P^{(a)}_k(1) - \lambda_a b_k)}{[a_{kl}a^{ks}(P^{(a)}_s(1) - \lambda_a b_s)\,a^{lm}(P^{(a)}_m - \lambda_a b_m)]^{1/2}}, \\ W^b := \dfrac{a^{bk}P_k(1)}{[a^{kl}P_l(1)P_k(1)]^{1/2}}, \end{cases}$$

where *e.g.* $a_{hk} = a_{hk}[t, q, u_a(t)]$. In addition, first, as $a \to \infty$, $(\lambda_a, \eta_a, j_a) \to \mathbf{0}$, see (*iv*) above (5.3). Furthermore, the solution of ODE (5.4) depends on the parameters λ_a, η_a, and j_a continuously, so that $\sup\{|P^{(a)}_b(\tau) - P_b(\tau)| : \tau \in (0,1)\} \to 0$ as $a \to \infty$ where $(P_1(\cdot), \ldots, P_N(\cdot))$ is the solution of the limit problem

$$(5.6)\qquad \begin{cases} \dot{P}_b = \alpha_b(\tau, \bar{u}, K) := [2^{-1}(A_{NN,b}(\tau, \bar{u}, K) + 2Q_{bNN}(\tau, \bar{u}, K))], \qquad P_b(0) = 0, \\ \dot{K}^b = 0, \qquad K^b(0) = \bar{q}^b. \end{cases}$$

Then by $(5.5)_3$ $W^b_{(a)} \to W^b$ as $a \to \infty$. Furthermore by $(5.6)_{4\text{-}5}$, $K^b(\tau) \equiv \bar{q}^b$, so that $(5.6)_1$ and the inverse of $(5.3)_{3,4}$ yield

$$(5.7)\qquad P_b(1) = \int_0^1 \tilde{\alpha}_b(\tau, \bar{u}, \bar{q})\,d\tau = \lim_{a\to\infty} \frac{1}{\eta_a}\int_a^{T_a} \alpha_b(t, \bar{u}, \bar{q})\,dt = \alpha_b(d, \bar{u}, \bar{q}) \qquad (b = 1, \ldots, N).$$

Then, by $(5.6)_2$ and $(5.5)_3$ one has $(5.2)_1$. Q.E.D.

REMARK. Note that the hypotesis (2.4) on the coefficients of Σ's kinetic energy renders the «q-part» $(2.4)_2$ of the SHE (2.4) independent of $\dot{u}$ in a neighbourhood U of $(d, \bar{q}, \bar{u})$ unlike what happens for the typical choice of $\Sigma_{u(\cdot)}$ (see (11.6) in [3]). By Theor 3.1, one can assume $(t, u_a(t), q_a(t)) \in [d, d + \eta_a] \times [\bar{u}, \bar{u} + j_a] \times B(\bar{q}, 1/a)$ for sufficiently large a. Furthermore, since the motion $t \mapsto (q_{(a)}(t), p^{(a)}(t))$ for $\Sigma_{u(\cdot)}$ is related to a continuous control $u_a(t)$ – see (2.8) –, $p^{(a)}(\cdot)$ is continuous (even where $u(\cdot)$ has a discontinuity) and therefore the R.H.S. of (2.5) is continuous in U. Hence $\dot{q}_{(a)}(\cdot)$ – unlike $\dot{u}_a$ – is continuous everywhere and in particular at d and T_a.

6. On the trajectory of Σ immediately after a burst

In this section we assume

(6.1) $$Q_{hkl}(t,\chi) \equiv 0, \qquad (h,k,l = 1,\dots,N).$$

For every $a \in N_*$, in connection with the motion $z^{(a)}(\cdot)$ for Σ_{u_a} we consider the motion $t \vdash \mathbb{P}(t,q,u_a(t))$ – see (4.2) – of the ideal fluid $F^{u_a(\cdot)}$, and the dynamic motion $P = P_a(t) = \mathbb{P}(t,q_a(t),u_a(t))$ of the representative point P of Σ_{u_a}; see (4.3). Furthermore, for every $a \in N_*$, we denote by l_a P's trajectory in Hertz's space $\mathbf{R}^{3\nu}$, along the motion $P_a(\cdot)$; and we call $v_{(a)}^{(r)}$ P's velocity w.r.t. $F^{u_a(\cdot)}$. In the sequel we replace the time $t \geqslant T_a$ with the arclength w.r.t. $F^{u_a(\cdot)}$ covered by P along the motion $P_a(\cdot)$:

(6.2) $$\sigma = \sigma_a(t) = \int_{T_a}^{t} v_{(a)}^{(r)}(\tau)\, d\tau .$$

Note that $\dot{\sigma} \geqslant 0$ even if P goes onward and backward on a line l of arclength s, in which cases $\dot{\sigma} = \pm\dot{s}$ respectively. However, if $\dot{\sigma}$ never vanishes, it is not restrictive to assume $\sigma = s$. We denote by $\mathrm{q}(\cdot)$ the maximal solution of the problem

(6.3) $$\overset{\prime\prime}{\mathrm{q}}{}^h + \left\{ {h \atop k\;\; l} \right\}(d,\mathrm{q},\bar{u})\, \overset{\prime}{\mathrm{q}}{}^k \overset{\prime}{\mathrm{q}}{}^l = 0;\quad \mathrm{q}^h(0) = \bar{q}^h,\quad \overset{\prime}{\mathrm{q}}{}^h(0) = W^h, \qquad (\overset{\prime}{\mathrm{q}}{}^h := dq^h/ds),$$

where W^h is defined by $(5.5)_3$. The equation $P = \mathbb{P}(d,\mathrm{q}(s),\bar{u})$ for $s \in [0,\lambda_M)$ with $\lambda_M \in (0,+\infty)$ represents a geodesic of the fixed manyfold $V_d^{\bar{u}}$; see below (4.2).

Theorem 6.1. *Let (6.1)-(6.3) hold. Then the sequence l_a of trajectories for P along the motions $P = P_a(t)$ $(a \in N_*)$ tends, as $a \to \infty$, to $V_d^{\bar{u}}$'s geodesic l defined below* (6.3), *in the sense that for any fixed $\lambda \in [0,\lambda_M)$ – see below* (6.3) *– for a large enough,* (*i*) *σ_a's restriction to $[0,\lambda]$ has an inverse $t \vdash t_a(\sigma)$ with $s = \sigma$ and*

(6.4) $$\lim_{a\to\infty} \sup\{|\mathbb{P}(t_a(s),q(t_a(s)),u(t_a(s))) - \mathbb{P}(d,\mathrm{q}(s),\bar{u})| : s \in [0,\lambda]\} = 0,$$

where $\mathbb{P}(\cdot,\cdot,\cdot) \in C^2(I \times \Omega \times H, \mathbf{R}^{3\nu})$ is defined in (4.2).

Proof. Calling $f^i[\boldsymbol{\phi}^i]$ the applied [reaction] force acting on the mass point P_i, in Hertz's space $\mathbf{R}^{3\nu}$, $\Sigma_{u(\cdot)}$'s dynamic equations have the version

(6.5) $$\boldsymbol{A} = \boldsymbol{F} + \boldsymbol{\phi}, \quad \text{where} \quad F_{3i-3+r} = (m_i)^{-1/2} f_r^i, \quad \phi_{3i-3+r} = (m_i)^{-1/2} \phi_r^i,$$
$$(i = 1,\dots,\nu;\ r = 1,2,3)$$

and since constraints are frictionless, $\mathbf{0} = \boldsymbol{\phi}_\sigma \,(= (\boldsymbol{\phi} \times P^{/h})\, P_{/h})$. Then the projection of $(6.5)_1$ on $V_t^{u(t)}$'s tangent space at $P = \mathbb{P}[t,q(t),u(t)]$ reads $\boldsymbol{A}_\sigma = \boldsymbol{F}_\sigma$. Hence by (4.7) and $(4.6)_2$

(6.6) $$\ddot{q}^h = -\left\{ {h \atop r\;\; s} \right\} \dot{q}^r \dot{q}^s + A_r^h \dot{q}^r + B^h, \qquad \text{with } e.g.\ A_r^h = A_r^h[t,q(t),u(t)]$$

where, remembering (2.3) and that Q_{0h}, Q_{hk}, $\left\{\begin{matrix} h \\ r \quad s\end{matrix}\right\}$, $\left\{\begin{matrix} h \\ 0 \quad s\end{matrix}\right\}$, $\left\{\begin{matrix} h \\ 0 \quad 0\end{matrix}\right\}$, a_{rs}, and $(a^{hk}) = (a_{rs})^{-1}$ are C^1-functions of (t, q, u),

(6.7) $$A^h_r(t,q,u) := a^{hl}Q_{lr} - 2\left\{\begin{matrix} h \\ 0 \quad r\end{matrix}\right\}, \qquad B^h := a^{hl}Q_{0l} - 2\left\{\begin{matrix} h \\ 0 \quad 0\end{matrix}\right\}.$$

Note that (6.6) *is the Lagrangian version of the semi-Hamiltonian* ODE (2.4).

Now fix $\lambda \in [0, \lambda_M)$ and $\mu \in (\lambda, \lambda_M)$; furthermore call P_μ l's point whose distance in $V^{\bar{u}}_{\bar{d}}$ from l's origin $P_0 := (d, \bar{q}, \bar{u})$ is μ. Then l's arc $l_\mu := \widehat{P_0 P_\mu}$ lies in some open set

(6.8) $$A := B(d, \varepsilon_1) \times Q \times B(\bar{u}, \varepsilon_2)(\neq \emptyset),$$

whose closure $\bar{A}$ is compact and belongs to the $(n+2)$-dimensional manifold $V \subset \mathbf{R}^{1+3\nu}$. The dynamic motion $P = P_a(t)$ of Σ_{u_a} (immediately) after the burst, *i.e.* for $t > d + \eta_a := T_a$, solves the ODE (6.6) with $u = u_a(t) = v_j(t - \eta_a)$, and satisfies the initial conditions at $T = T_a$

(6.9) $$q(T_a) = q_{(a)}(T_a), \qquad \dot{q}(T_a) = \dot{q}_{(a)}(T_a), \qquad (u_{(a)}(T_a) = v_{j_a}(d) = \bar{u} + j_a)$$

where the R.H.S.s of $(6.9)_{1\text{-}2}$ are constructed with the solution $t \vdash z(t) = (q_{(a)}(t), p^{(a)}(t))$ in $[d, T_a]$ of problem (2.5) for $u = u_a(t)$; see also the Remark below (5.7).

Hence, remembering (5.1-2) and $(4.2)_3$, for a unique $W_a > 0$ – see $(5.2)_3$ – we have that

(6.10) $$\begin{cases} P_a(T_a) = \mathbf{P}(T_a, q_{(a)}(T_a), \bar{u} + j_a), \\ \dot{P}_a(T_a) = W_a \boldsymbol{w}_a = \mathbf{P}_{/h}(T_a, q_{(a)}(T_a), \bar{u} + j_a)\, \dot{q}^h(T_a) \end{cases}$$

and that, as $a \to \infty$, $(j_a \to 0, \eta_a \to 0^+, T_a \to d$ and$)$

(6.11) $$P_a(T_a) \to P_0 = \mathbf{P}(d, \bar{q}, \bar{u}), \qquad W_a \to +\infty \qquad (\boldsymbol{w}_a \to \boldsymbol{w}; \text{ see } (5.2)_2).$$

Now set, for *e.g.* $M^{-1} = W_a$ and $T = T_a$

(6.12) $$\xi = (t - T)M^{-1}, \quad \acute{q} = dq/d\xi = M\dot{q}, \quad \mathfrak{q}(\xi) := q(T + M\xi),$$

so that the point $P(T_a + M_a \xi)$ covers $l_{a,\xi}$ when ξ covers $[0, \mu]$. Then the problem $(6.6) \cup (6.9)$, for $t \geq T_a$ becomes the problem for $\xi \geq 0$ formed by the ODE

(6.13) $$\mathfrak{q}'' = -\left\{\begin{matrix} h \\ r \quad s\end{matrix}\right\} \mathfrak{q}'^r \mathfrak{q}'^s + M A^h_r \mathfrak{q}'^r + M^2 B^h,$$

where $A^h_r = A^h_r[T + M\xi, \mathfrak{q}(\xi), j + v(T + M\xi)]$, $B^h = B^h[T + M\xi, \mathfrak{q}(\xi), j + v(T + M\xi)]$, $M^{-1} = W_a$, and $T = T_a$, coupled with the initial conditions

(6.14) $$\mathfrak{q}^h(0) = q^h_{(a)}(T_a), \qquad \mathfrak{q}'^h(0) = M\dot{q}^h_{(a)}(T_a)(= w^h_a, \text{ where } \boldsymbol{w}_a = w^h_a P_{/h});$$

we regard the R.H.S. of $(6.13)_{1\text{-}2}$ as constructed by means of the solution $q_{(a)}(\cdot)$ of (2.5) – see below (6.9). For some ε_1 small enough, the ODE (6.13) has the form $\mathfrak{q}'' = f(\xi, \mathfrak{q}, \mathfrak{q}', u, M, j)$ with $f \in C^1$ in the compact set $K := [-\varepsilon_1, \mu] \times Q \times S \times B(\bar{u}, \varepsilon_1) \times [0, \varepsilon_1] \times [0, \varepsilon_1]$. Infact for $M = 0$ problem $(6.12) \cup (6.14)_{1\text{-}3}$ coincides with problem (6.3); and the solution of this in $[0, \mu]$ exists in that it represents the geodesic $l_{P_{0,\mu}}$. Incidentally, for $M = 0$, ξ is the arclength on l.

Call $\mathfrak{q}(\cdot, \tilde{\mathfrak{q}}, \tilde{w}, M, j)$ the general solution in $[0,\mu]$ of the second order ODE (6.13), coupled with the initial conditions $\mathfrak{q}^b(0) = \tilde{\mathfrak{q}}^b$ and $\acute{\mathfrak{q}}^b(0) = \tilde{w}^b$. By a well known theorem (of existence and uniqueness in the large), there is some $\eta > 0$ such that for

$$(6.15) \qquad |\tilde{\mathfrak{q}}^b - \bar{q}^b| \leqslant \eta, \qquad |\tilde{w}^b - w^b| \leqslant \eta, \qquad |M| \leqslant \eta, \qquad |j| \leqslant \eta,$$

the above solution in $[0,\mu]$ exists and is (uniformly) continuous and even C^1 in K, together with $\acute{\mathfrak{q}}(\cdot, \tilde{\mathfrak{q}}, \tilde{w}, M, j)$. Hence, given $\varepsilon \in (0,1)$ arbitrarily, there is some $\bar{\eta} > 0$ such that, for $\eta < \bar{\eta}$, $\{(T + M\xi, \mathfrak{q}(\xi, \tilde{\mathfrak{q}}, \tilde{w}, M, j), j + v(T + M\xi)) | \xi \in [0,\mu]\} \subset A$ and

$$(6.16) \qquad |\mathfrak{q}(\xi, \tilde{\mathfrak{q}}, \tilde{w}, M, j) - \mathfrak{q}(\xi, \bar{q}, w, 0, 0)| < \varepsilon, \qquad |\acute{\mathfrak{q}}(\xi, \tilde{\mathfrak{q}}, \tilde{w}, M, j) - \acute{\mathfrak{q}}(\xi, \bar{q}, w, 0, 0)| < \varepsilon.$$

Now, by (6.8)-(6.10), there is an $\alpha \in N_*$ such that for $a > \alpha$ the solution $\mathfrak{q}_{(a)}(\cdot) := = \mathfrak{q}(\cdot, q_{(a)}(T_a), w_a, M_a, j_a)$ of (6.13)-(6.14) fulfils requirements (6.15). Then (6.16) holds for $\mathfrak{q}_{(a)}(\cdot)$; hence, by the continuity of the function $(\xi, \mathfrak{q}, w, M, j) \vdash [a_{hk}(\xi, \mathfrak{q}, u)\, \acute{\mathfrak{q}}^h \acute{\mathfrak{q}}^k]^{1/2}$ in K, for $\varepsilon (> 0)$ arbitrarily fixed, there is an $\bar{\alpha} > \alpha$ such that $\forall \xi \in [0,\mu]$ and $\forall a > \bar{\alpha}$

$$(6.17) \qquad [a_{hk}(\xi, \mathfrak{q}_{(a)}(\xi), u_a(\xi))\, \acute{\mathfrak{q}}^h_{(a)}(\xi)\, \acute{\mathfrak{q}}^k_{(a)}(\xi)]^{1/2} - [a_{hk}(d, \mathfrak{q}(\xi), \bar{u})\, \acute{\mathfrak{q}}^h(\xi)\, \acute{\mathfrak{q}}^k(\xi)]^{1/2} < \varepsilon.$$

Furthermore, by the definition involving (6.3), $\mathfrak{q}(\xi) = \mathfrak{q}(\xi, \bar{q}, w, 0, 0)$ $\forall \xi \in [0,\mu]$, while by (6.2) and $(6.12)_1$, for $t \geqslant T_a$ $(\xi = (t - T_a)/M_a)$

$$(6.18) \qquad |\sigma_a(t) - \xi| = \left| \int_{T_a}^{t} [a_{hk}\, \dot{q}^h_{(a)}(\tau)\, \dot{q}^k_{(a)}(\tau)]^{1/2}\, d\tau - \xi \right| =$$

$$= \left| \int_0^{\xi} \{[a_{hk}\, \acute{\mathfrak{q}}^h_{(a)}(\zeta)\, \acute{\mathfrak{q}}^k_{(a)}(\zeta)]^{1/2} - [a_{hk}\, \acute{\mathfrak{q}}^h(\zeta)\, \acute{\mathfrak{q}}^k(\zeta)]^{1/2}\}\, d\zeta \right| \leqslant$$

$$\leqslant \int_0^{\xi} |[\ldots]^{1/2} - [\ldots]^{1/2}|\, d\zeta \leqslant \varepsilon\mu, \qquad \forall a > \bar{\alpha}.$$

By (6.16), for $\xi \in [0,\mu]$ we have $d\sigma_a/d\xi = [a_{hk}(\xi, \mathfrak{q}_{(a)}(\xi), u_a(\xi))\, \acute{\mathfrak{q}}^h_{(a)}(\xi)\, \acute{\mathfrak{q}}^k_{(a)}(\xi)]^{1/2} = = |\acute{\mathfrak{q}}_{(a)}(\xi, q_{(a)}(T_a), w_a, M_a, j_a)| \geqslant 1 - \varepsilon > 0$. Therefore σ_a is a strictly increasing function of ξ and hence of t. Then the inverse $t = t_a(\sigma)$ of $\sigma = \sigma_a(t)$ exists in $[T_a, T_a + \mu M_a]$ and $s = = \sigma = \sigma_a(t)$. By (6.18) $\sigma_a(t) \in [\xi - \mu\varepsilon, \xi + \mu\varepsilon]$. Hence, for $\varepsilon\mu < \mu - \lambda$, $\{P(T_a + + \xi M_a, \mathfrak{q}_{(a)}(\xi), u_a(\xi)) | \xi \in [0,\mu]\}$ is an arc (of l_a) containing the arc $l_{a,\lambda}$ of l_a that has $\mathbb{P}(T_a, q_{(a)}(T_a), \bar{u} + j_a)$ as an endpoint. Hence the function $s = s_a(\xi) := \sigma_a[t_a(\xi)]$ is defined in $[0,\mu]$, it is strictly increasing, and with $[0,\lambda] \subseteq s_a([0,\mu])$. Furthermore, by $(6.18)_{1\text{-}3}$,

$$(6.19) \qquad |s_a(\xi) - \xi| \leqslant \varepsilon\xi \leqslant \varepsilon\mu \;\; \forall \xi \in [0,\mu], \qquad \text{hence } |s - \xi_s| \leqslant \varepsilon\mu \;\; \forall s \in [0,\lambda], \;\; \forall a > \alpha$$

where ξ_s is the inverse of $\xi \vdash s = s_a(\xi)$. In order to prove (6.4) we set

$$\begin{cases} \tilde{\mathbb{P}}(\xi, \tilde{q}, \tilde{w}, M, j) := \mathbb{P}[T + M\xi, q(\xi, \tilde{q}, \tilde{w}, M, j), M, j, j + v(T + M\xi)] \\ \text{and} \\ \mathbb{P}_a(\xi) := \tilde{\mathbb{P}}(\xi, q_{(a)}(T_a), w_a, M_a, j_a). \end{cases} \tag{6.20}$$

Note that by the definition of $\sigma_a(\xi)$ below (6.18) and by $(6.12)_1$ one has

$$\mathbb{P}[t_a(s), q_{(a)}(t_a(s)), u_a(t_a(s))] = \mathbb{P}_a(\xi_s) \qquad \forall s \in [0, \lambda], \ \ \forall a > \bar{\alpha}. \tag{6.21}$$

By the uniform continuity of $q(\xi, \tilde{q}, \tilde{w}, M, j)$ in the set defined by (6.15) and $\xi \in [0, \mu]$, given $\varepsilon' > 0$ arbitrarily, for $\varepsilon(>0)$ small enough, $(6,19)_3$ and $(6.16)_1 \cup (6.11)$ yield the first and the second of the inequalities below respectively

$$|\mathbb{P}_a(\xi_s) - \mathbb{P}_a(s)| < \varepsilon', \qquad |\mathbb{P}_a(s) - \mathbb{P}(d, q(s), \bar{u})| < \varepsilon' \qquad \forall s \in [0, \lambda], \ \ \forall a > \bar{\alpha}. \tag{6.22}$$

Then for $s \in [0, \lambda] (\subseteq [0, \mu])$ and $a > \bar{\alpha}$ one has

$$|\mathbb{P}_a(\xi_s) - (\mathbb{P}_a(d, q(s), \bar{u}))| \leq |\mathbb{P}_a(\xi_s) - \mathbb{P}_a(s)| + |\mathbb{P}_a(s) - \mathbb{P}(d, q(s), \bar{u})| < \varepsilon' + \varepsilon'. \tag{6.23}$$

Therefore, by (6.21), $\sup\{|\mathbb{P}[t_a(s), q_{(a)}(t_a(s)), u_a(t_a(s))] - \mathbb{P}(d, q(s), \bar{u})| : s \in [0, \lambda]\} < 2\varepsilon'$. By the arbitrariness of $\varepsilon'(>0)$, (6.4) holds. Q.E.D.

This work has been prepared in the activity sphere of the group N. 3 of the Consiglio Nazionale delle Ricerche in the academic years 1988-89 and 1989-90.

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