Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei Matematica e Applicazioni

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Some new results on a Stefan problem in a concentrated capacity

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **3** (1992), n.1, p. 23–34.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1992_9_3_1_23_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1992.

Rend. Mat. Acc. Lincei
s. 9, v. 3:23-34 (1992)

Equazioni a derivate parziali. — *Some new results on a Stefan problem in a concentrated capacity.* Nota (*) del Socio ENRICO MAGENES.

ABSTRACT. — An existence and uniqueness theorem for a nonlinear parabolic system of partial differential equations, connected with the theory of heat conduction with a transition phase in a *concentrated capacity*, is given in sufficiently general hypotheses on the data.

KEY WORDS: Nonlinear parabolic partial differential equations; Heat conduction; Phase changes.

RIASSUNTO. — *Nuovi risultati su un problema di Stefan in una capacità concentrata.* Viene dato un teorema di esistenza e di unicità per un sistema non lineare parabolico di equazioni a derivate parziali, connesso con la teoria della diffusione del calore con cambiamento di fase in una *capacità concentrata*, in condizioni abbastanza generali sui dati del problema.

The theory of heat conduction with phase changes in a *concentrated capacity* (see [1, 4, 10] and references therein) suggests the following problem.

Let Ω be a bounded regular set in $\mathbf{R}^n$, $n \geq 2$, $\Gamma = \partial\Omega$, ν the inward normal to Γ, $Q = \Omega \times]0, T[$, $T > 0$, $\Sigma = \Gamma \times [0, T[$, Δ the Laplace operator in $\mathbf{R}^n$, Δ_g the Laplace-Beltrami operator on Γ (with respect to a Riemannian structure g on Γ). Given a nondecreasing Lipschitz continuous function β: $\mathbf{R} \to \mathbf{R}$, such that $\beta(0) = 0$ and β grows at least linearly at ∞ ($\theta = \beta(u)$ is the constitutive equation relating the temperature θ and the enthalpy density u), and functions u_0, θ_0, f and φ defined on Γ, Ω, Σ and Q respectively, the problem reads as follows: *find the functions u and θ such that*

$$(1) \qquad \begin{cases} \partial\theta/\partial t - \Delta\theta = \varphi \ \text{ in } Q, \quad \theta(0) = \theta_0 \ \text{ in } \Omega, \quad \theta = \beta(u) \ \text{ on } \Sigma, \\ \partial u/\partial t - \Delta_g \theta - \partial\theta/\partial\nu = f \ \text{ on } \Sigma, \quad u(0) = u_0 \ \text{ on } \Gamma, \end{cases}$$

Recently in [9] I have studied the case: $u_0 = 0$, $\theta_0 = 0$, $\varphi = 0$ in some suitable Hilbert spaces. Now, I want to consider the general case, which is not a straightforward consequence of the previous one.

1. Let Ω be an open bounded set of $\mathbf{R}^n$, $n \geq 2$, whose boundary Γ is an oriented connected C^∞ $(n-1)$-manifold. We shall use the usual L^2-Sobolev spaces $H^s(\Omega)$ and $H^s(\Gamma)$, s real (see *e.g.* [6]). In particular $H^0(\Gamma) = L^2(\Gamma)$. As usual we identify $L^2(\Gamma)$ with its dual so that we have $H^1(\Gamma) \subset L^2(\Gamma) \subset H^{-1}(\Gamma)$, with continuous and dense injections. Let $\langle u, v \rangle$ denote either the scalar product in $L^2(\Gamma)$ or the pairing between $H^{-1}(\Gamma)$ and $H^1(\Gamma)$. Let us denote by ν the inward normal to Γ, by ∇ and Δ the gradient vector and the Laplace operator in $\mathbf{R}^n$. Moreover we shall suppose that a (proper) C^∞-Riemannian structure g is defined on Γ and we denote by Δ_g the Laplace-Beltrami operator on Γ with respect to g (see *e.g.* [3]) and by (u, v) the scalar product with respect to g, either for u and $v \in L^2(\Gamma)$ or for $u \in H^{-1}(\Gamma)$ and $v \in H^1(\Gamma)$. We recall that

(*) Presentata nella seduta del 16 novembre 1991.

$u \to \Delta_g u$ is a linear continuous operator from $H^1(\Gamma)$ into $H^{-1}(\Gamma)$ and that

$$-(\Delta_g v, u) = (dv, du) \qquad \forall u,\ v \in H^1(\Gamma), \tag{1.1}$$

where d is the exterior differential on Γ. Moreover we shall use also the spaces $H^{r,s}$, r and s non negative real numbers, as defined in [6, chapt. 4, n. 2.1], namely:

$$H^{r,s}(Q) = L^2(0,T;H^r(\Omega)) \cap H^s(0,T;L^2(\Omega)), \tag{1.2}$$

$$H^{r,s}(\Sigma) = L^2(0,T;H^r(\Gamma)) \cap H^s(0,T;L^2(\Gamma)), \tag{1.3}$$

endowed with their natural norms. In addition let $0 < \sigma < 1/2$ be a fixed number and let B^σ be given by

$$B^\sigma = L^2(0,T;H^1(\Gamma)) \cap {}_0H^{1/2+\sigma}(0,T;L^2(\Gamma)), \tag{1.4}$$

where

$${}_0H^{1/2+\sigma}(0,T;L^2(\Gamma)) = \{v \in H^{1/2+\sigma}(0,T;L^2(\Gamma)), v(\cdot,0) = 0\}. \tag{1.5}$$

B^σ is a Hilbert space with the norm $\|v\|_{H^{1,1/2+\sigma}(\Sigma)}$.

Now let h_0 be fixed in $H^1(\Gamma)$ and let us consider the set

$$B^\sigma_{h_0} = \{h \in H^{1,1/2+\sigma}(\Sigma), h(\cdot,0) = h_0\} \tag{1.6}$$

which is a closed affine manifold of $H^{1,1/2+\sigma}(\Sigma)$. We can obviously represent any $h \in B^\sigma_{h_0}$ by the formula

$$h(x,t) = h_0(x) + v(x,t) \qquad \text{with } v \in B^\sigma. \tag{1.7}$$

Let us consider the following problem:

$$\begin{cases} \textit{given } h_0 \in H^1(\Gamma),\ h \in B^\sigma_{h_0} \textit{ with } (1.7),\ \varphi \in L^2(Q) \textit{ and } \theta_0 \in H^1(\Omega) \\ \textit{with } \theta_0|_\Gamma = h_0,\ \textit{find } \theta \in H^{3/2,3/4}(Q) \textit{ solution of} \\ \partial\theta/\partial t - \Delta\theta = \varphi \textit{ in } Q, \quad \theta = h \textit{ on } \Sigma, \quad \theta(\cdot,0) = \theta_0 \textit{ in } \Omega. \end{cases} \tag{1.8}$$

We have the following

PROPOSITION 1.1. *Problem* (1.8) *has a unique solution* $\theta \in H^{3/2,3/4}(Q)$ *and we can define* $\partial\theta/\partial\nu$ *on* Σ *with* $\partial\theta/\partial\nu \in L^2(\Sigma)$ *and*

$$\|\partial\theta/\partial\nu\|_{L^2(\Sigma)} \le C\{\|h_0\|_{H^1(\Gamma)} + \|v\|_{B^\sigma} + \|\theta_0\|_{H^1(\Omega)} + \|\varphi\|_{L^2(Q)}\}, \tag{1.9}$$

where C *is a positive number independent of* h_0, v, θ_0, φ.

PROOF. We can split (1.8) writing $\theta = \theta_1 + \theta_2 + \theta_3$ where θ_i, $i = 1,2,3$, are respectively the solutions of the following problems:

$$\partial\theta_1/\partial t - \Delta\theta_1 = \varphi \text{ in } Q, \quad \theta_1 = v \text{ on } \Sigma, \quad \theta_1(\cdot,0) = 0 \text{ on } \Omega; \tag{1.10}$$

$$\partial\theta_2/\partial t - \Delta\theta_2 = 0 \text{ in } Q, \quad \theta_2 = 0 \text{ on } \Sigma, \quad \theta_2(\cdot,0) = \theta_{0,1} \text{ on } \Omega; \tag{1.11}$$

$$\partial\theta_3/\partial t - \Delta\theta_3 = 0 \text{ in } Q, \quad \theta_3 = h_0 \text{ on } \Sigma, \quad \theta_3(\cdot,0) = \theta_{0,2} \text{ on } \Omega; \tag{1.12}$$

where $\theta_{0,1}$ and $\theta_{0,2}$ are determined by $\theta_0 = \theta_{0,1} + \theta_{0,2}$ with the following conditions

(1.13) $$\theta_{0,1} \in H_0^1(\Omega),\ \Delta\theta_{0,1} = \Delta\theta_0 \quad \text{in } H^{-1}(\Omega),$$

(1.14) $$\theta_{0,2} \in H^{3/2}(\Omega),\ \Delta\theta_{0,2} = 0 \quad \text{in } \Omega, \qquad \theta_{0,2} = h_0 \quad \text{on } \Gamma.$$

Now (1.10) can be studied by the methods used in [6, chapt. 4], and more precisely in [2, Teor. 3]; let us remark that B^σ is contained, with continuous embedding, in the space B defined by (2.4) of [9]. Then there exists a unique $\theta_1 \in H^{3/2,3/4}(Q)$ solution of (1.10) and we have $\partial\theta_1/\partial\nu \in L^2(\Sigma)$ with

(1.15) $$\left\|\frac{\partial\theta_1}{\partial\nu}\right\|_{L^2(\Sigma)} \le C(\|v\|_{B^\sigma} + \|\varphi\|_{L^2(Q)}).$$

Problem (1.11) is a classical one and it is well known (see *e.g.* [6, chapt. 4]) that there exists a unique solution $\theta_2 \in H^{2,1}(Q)$ such that $\partial\theta_2/\partial\nu \in H^{1/2,1/4}(\Sigma) \subset L^2(\Sigma)$ and

(1.16) $$\left\|\frac{\partial\theta_2}{\partial\nu}\right\|_{L^2(\Sigma)} \le C\|\theta_{0,1}\|_{H^1(\Omega)}.$$

Finally (1.12), recalling (1.14), has the obvious solution $\theta_3(x,t) = \theta_{0,2}(x)$, independent of t and from [6, chapt. 2], we have

(1.17) $$\left\|\frac{\partial\theta_3}{\partial\nu}\right\|_{L^2(\Sigma)} \le C\|h_0\|_{H^1(\Gamma)}.$$

Collecting the previous results we obtain the proof of Proposition 1.1.

Now if we consider φ and θ_0 (and consequently h_0) fixed we can see $\partial\theta/\partial\nu$ on Σ as an operator depending only on h (really only on v); so we shall call the operator by $\mathcal{T}_{\theta_0,\varphi}$, *i.e.* we shall define

(1.18) $$\mathcal{T}_{\theta_0,\varphi}\colon h \to \mathcal{T}_{\theta_0,\varphi} h = \partial\theta/\partial\nu.$$

REMARK 1.1. Note that for a.e. $t \in [0,T]$ the value $(\mathcal{T}_{\theta_0,\varphi} h)(t)$ depends on the values of h (and φ) in the interval $[0,t]$.

Finally let us introduce the function β:

(1.19) $$\beta(\xi) = a(\xi - 1)^+ - b\xi^- \qquad \forall \xi \in \mathbf{R},$$

where a and b are fixed positive numbers (for more general class of functions β see [9, Remark 2.2]).

Now we state the main result:

THEOREM 1.1. *For any* $f \in L^2(\Sigma)$, $\varphi \in L^2(Q)$, $u_0 \in L^2(\Gamma)$ *with* $\beta(u_0) \in H^1(\Gamma)$, $\theta_0 \in H^1(\Omega)$ *with* $\theta_0|_\Gamma = \beta(u_0)$, *there exists a unique function* u, *with*

$$u \in H^1(0,T; H^{-1}(\Gamma)) \cap L^\infty(0,T; L^2(\Gamma)),$$

$$\beta(u) \in L^\infty(0,T; H^1(\Gamma)) \cap H^1(0,T; L^2(\Gamma))$$

such that in the sense of $H^1(0,T; H^{-1}(\Gamma))$ *we have:*

(1.20) $$\partial u/\partial t - \Delta_g \beta(u) - \mathcal{T}_{\theta_0,\varphi}\beta(u) = f, \qquad u(\cdot,0) = u_0.$$

With the above notation the first equation of (1.20) means that for a.e. t in $[0,T]$ we have

$$\langle \partial u(t)/\partial t, \psi\rangle - (d\beta(u(t)), d\psi) - \langle (\mathcal{T}_{\theta_0,\varphi}\,\beta(u))(t), \psi\rangle = -\langle f(t), \psi\rangle \qquad \forall \psi \in H^1(\Gamma).$$

REMARK 1.2. Equation (1.20) gives an exact formulation of Problem (I) as a nonlinear evolution equation on Σ in the unknown u; this formulation can be viewed as a usual Stefan problem on Σ perturbed by the nonlocal (in x and in t) term $\mathcal{T}_{\theta_0,\varphi}\,\beta(u)$. The proof of Theorem 1.1 shall be given following the same techniques used in [9].

2. In order to prove Theorem 1.1 let us introduce a *regularization of problem* (1.20). For any $\varepsilon > 0$, let β_ε be defined, *e.g.*, as follows:

$$\text{(2.1)} \qquad \begin{cases} \beta_\varepsilon(\xi) = \beta(\xi) \ \text{ if } \xi \leq 0, \ \ \beta_\varepsilon(\xi) = \beta(\xi) + \varepsilon \ \text{ if } \xi \geq 1, \\ \beta_\varepsilon \in C^\infty(\mathbf{R}), \ \ \tilde{l}\varepsilon \leq \beta'_\varepsilon(\xi) \leq \tilde{L} \quad \forall \xi \in \mathbf{R}(\tilde{l}, \tilde{L} \text{ positive numbers}). \end{cases}$$

Then it is easy to prove the existence of $u_0^\varepsilon \in H^1(\Gamma)$ and $\theta_0^\varepsilon \in H^{3/2}(\Omega)$ such that

$$\text{(2.2)} \qquad \lim_{\varepsilon\to 0} u_0^\varepsilon = u_0 \ \text{ in } L^2(\Gamma), \qquad \lim_{\varepsilon \to 0} \beta_\varepsilon(u_0^\varepsilon) = \beta(u_0) \quad \text{in } H^1(\Gamma),$$

$$\text{(2.3)} \qquad \theta_0^\varepsilon|_\Gamma = \beta_\varepsilon(u_0^\varepsilon), \qquad \lim_{\varepsilon\to 0} \theta_0^\varepsilon = \theta_0 \quad \text{in } H^1(\Omega).$$

Then, for any $\varepsilon > 0$, the *regularized problem* reads as follows: *find* $u_\varepsilon \in H^{2,1}(\Sigma)$ *such that*

$$\text{(2.4)} \qquad \partial u_\varepsilon/\partial t - \Delta_g\, \beta_\varepsilon(u_\varepsilon) - \mathcal{T}_{\theta_0^\varepsilon,\varphi}\,\beta_\varepsilon(u_\varepsilon) = f, \quad u_\varepsilon(\cdot,0) = u_0^\varepsilon$$

and we shall prove the following

THEOREM 2.1. *For any* $\varepsilon > 0$, *problem* (2.4) *has a unique solution.*

For the sake of simplicity, since no confusion is possible, *we shall omit the subscript* ε *in all functions* occurring in the formulation of the *regularized problem*, everywhere in the present Sect. 2. Moreover, since we consider $u_0^\varepsilon, \theta_0^\varepsilon, \varphi$ fixed, we *shall omit the subscript* $\theta_0^\varepsilon, \varphi$ *in* $\mathcal{T}_{\theta_0^\varepsilon,\varphi}$, so that problem (2.4) reads

$$\text{(2.5)} \qquad \partial u/\partial t - \Delta_g\, \beta(u) - \mathcal{T}\,\beta(u) = f, \qquad u_0(\cdot, 0) = u_0.$$

In order to prove Theorem 2.1, let us introduce the following family of problems: *for any* $w \in B^\sigma_{u_0}$ *and* $\alpha \in [0,1]$ *find* $v \in B^\sigma_{u_0} \cap H^{2,1}(\Sigma)$ *such that*

$$\text{(2.6)} \qquad \partial v/\partial t - \Delta_g\, \beta(v) = \alpha\, \mathcal{T}\,\beta(w) + f, \qquad v(\cdot,0) = u_0.$$

Since $w \in B^\sigma_{u_0}$, also $\beta(w) \in B^\sigma_{\beta(u_0)}$ and (cfr. (1.6), (1.7)) we have $\beta(w) = \beta(u_0) + k$, with $k \in B^\sigma$. Now we can prove the following

PROPOSITION 2.1. *Problem* (2.6) *has a unique solution so that*

$$\text{(2.7)} \qquad \|v\|_{H^{2,1}(\Sigma)} \leq C(\varepsilon)\{\|u_0\|_{L^2(\Gamma)} + \|\beta(u_0)\|_{H^1(\Gamma)} + \|k\|_{B^\sigma} + \|\theta_0\|_{H^1(\Omega)} + \|\varphi\|_{L^2(Q)} + \|f\|_{L^2(\Sigma)}\},$$

where $\beta(w) = \beta(u_0) + k$ *and* $C(\varepsilon)$ *is independent of* $\alpha \in [0,1]$.

PROOF. Proposition 1.1 shows that $F = \alpha\, \mathcal{T}\,\beta(w) + f$ belongs to $L^2(\Sigma)$. Let $\{F_n\}$

and $\{u_n\}$, $n=1,2,\ldots$ be such that $F_n \in \mathcal{D}(\Sigma)$, $u_n \in \mathcal{D}(\Gamma)$ and $\lim_{n\to\infty} F_n = F$ in $L^2(\Gamma)$ and $\lim_{n\to\infty} u_n = u_0$ in $H^1(\Gamma)$. The problem

$$\partial v_n/\partial t - \Delta_g\, \beta(v_n) = F_n\,, \qquad v_n(0) = u_n \tag{2.8}$$

has a unique solution $v_n \in C^\infty(\overline{\Sigma})$ for classical results (see *e.g.* [5]). The following estimates can be easily proved, recalling that $\beta = \beta_\varepsilon$ satisfies (2.1):

$$\|v_n\|_{L^\infty(0,T;L^2(\Gamma))} + \|\beta(u_n)\|_{L^2(0,T;H^1(\Gamma))} \le C\{\|F_n\|_{L^2(\Sigma)} + \|u_n\|_{L^2(\Gamma)}\}\,, \tag{2.9}$$

$$\|\partial v_n/\partial t\|_{L^2(\Sigma)} \le C(\varepsilon)\{\|F_n\|_{L^2(\Sigma)} + \|\beta(u_n)\|_{H^1(\Gamma)}\}\,, \tag{2.10}$$

where C is independent of ε, α and n and $C(\varepsilon)$ is independent of α and n. Indeed (2.9) (resp. (2.10)) can be obtained if we multiply for any t (2.8) by $v_n(t)$ (resp. $\partial\beta(v_n(t))/\partial t$) and integrate on $\Gamma \times]0,t[$ (resp. on Σ). Hence from (2.8) we also obtain

$$\|v_n\|_{L^2(0,T;H^2(\Gamma))} \le C(\varepsilon)\{\|F_n\|_{L^2(\Sigma)} + \|u_n\|_{L^2(\Gamma)} + \|\beta(u_n)\|_{H^1(\Gamma)}\}\,. \tag{2.11}$$

Then, using standard compactness procedures and the monotonicity of β, we obtain the existence of a solution of (2.6) with $v \in H^{2,1}(\Sigma) \cap B^\sigma_{u_0}$. The uniqueness is also easily proved by the monotonicity of β. Moreover (2.7) follows on using also Proposition 1.1 and the fact that $\alpha \in [0,1]$.

Now, letting u_0, θ_0, φ and f be fixed and writing $w = u_0 + \tilde{w}$ with $\tilde{w} \in B^\sigma$ and $v = u_0 + \tilde{v}$, with $\tilde{v} \in B^\sigma$ (recall (1.6) and (1.7)), we can consider $\tilde{v}$ as a function of $\tilde{w}$ and α, namely,

$$\tilde{v} = \Phi(\tilde{w}, \alpha)\,, \qquad \tilde{w} \in B^\sigma\,, \quad \alpha \in [0,1] \tag{2.12}$$

and Φ is an operator acting from $B^\sigma \times [0,1]$ in $B^\sigma \cap H^{2,1}(\Sigma)$. Proposition 2.1 guaranties that Φ maps bounded sets of B^σ in bounded sets of $B^\sigma \cap H^{2,1}(\Sigma)$, uniformly with respect to $\alpha \in [0,1]$. Moreover, we shall prove the following property:

Proposition 2.2. *The operator Φ is continuous in B^σ, uniformly with respect to $\alpha \in [0,1]$.*

Proof. Let $\tilde{w}_i \in B^\sigma$ and $\tilde{v}_i = \Phi(\tilde{w}_i, \alpha)$, $i = 1,2$ and $w_i = u_0 + \tilde{w}_i$, $v_i = u_0 + \tilde{v}_i$. Then $v_2 - v_1 = \tilde{v}_2 - \tilde{v}_1$ satisfies

$$\partial(v_2 - v_1)/\partial t - \Delta_g(\beta(v_2) - \beta(v_1)) = \alpha(\mathcal{T}\beta(w_2) - \mathcal{T}\beta(w_1))\,, \quad v_2(0) - v_1(0) = 0\,, \tag{2.13}$$

which can be written, for any $t \in [0,T]$, as follows

$$\langle v_2(t) - v_1(t), \psi\rangle + \left(d\int_0^t (\beta(v_2(s)) - \beta(v_1(s)))\, ds, d\psi \right) =$$

$$= \alpha \int_\Gamma \int_0^t \{\mathcal{T}\beta(w_2(s)) - \mathcal{T}\beta(w_1(s))\}\, ds\, \psi\, d\sigma, \ \forall \psi \in H^1(\Gamma)\,.$$

Taking $\psi = \beta(v_2(t)) - \beta(v_1(t))$ and integrating from 0 to T, we obtain

$$(2.14)\qquad \int_0^T\int_\Gamma (v_2(t) - v_1(t))(\beta(v_2(t)) - \beta(v_1(t)))\, d\sigma\, dt +$$

$$+\frac{1}{2}\left(\int_0^T(\beta(v_2(t)) - \beta(v_1(t)))\, dt, \int_0^T(\beta(v_2(t)) - \beta(v_1(t)))\, dt\right) =$$

$$= \alpha\int_0^T\int_\Gamma\int_0^t\{\mathcal{T}\beta(w_2(s)) - \mathcal{T}\beta(w_1(s))\}\, ds\, \{\beta(v_2(t)) - \beta(v_1(t))\}\, d\sigma\, dt.$$

Now, since $\beta(=\beta_\varepsilon)$ satisfies (2.1), we have

$$(2.15)\qquad |\beta(\xi) - \beta(\eta)|^2 \le \tilde{L}(\xi - \eta)(\beta(\xi) - \beta(\eta)), \qquad \forall \xi, \eta \in \boldsymbol{R}.$$

Therefore from (2.14) and Proposition 1.1 (note that $\mathcal{T}\beta(w_2) - \mathcal{T}\beta(w_1) = \mathcal{T}_{0,0}(\beta(w_2) - \beta(w_1)))$, we obtain $\|\beta(v_2) - \beta(v_1)\|_{L^2(\Sigma)} \le C\|\mathcal{T}\beta(w_2) - \mathcal{T}\beta(w_1)\|_{L^2(\Sigma)} \le C\|\beta(w_2) - \beta(w_1)\|_{B^\sigma}$ with C independent of α and ε.

Finally, noting again that $\beta(=\beta_\varepsilon)$ satisfies (2.1), we obtain

$$(2.16)\qquad \|\tilde{v}_2 - \tilde{v}_1\|_{L^2(\Sigma)} = \|v_2 - v_1\|_{L^2(\Sigma)} \le C(\varepsilon)\|w_2 - w_1\|_{B^\sigma} = C(\varepsilon)\|\tilde{w}_2 - \tilde{w}_1\|_{B^\sigma}$$

where $C(\varepsilon)$ is independent of α.

Now let us recall that from the *interpolation inequality* (see, *e.g.* [6, chapt. 1]) we have, for any $z \in H^{2,1}(\Sigma) \cap B^\sigma$

$$(2.17)\qquad \begin{cases} \|z\|_{L^2(0,T;H^1(\Gamma))} \le C\|z\|^{1/2}_{L^2(0,T;L^2(\Gamma))} \cdot \|z\|^{1/2}_{L^2(0,T;H^2(\Gamma))}, \\ \|z\|_{H^{1/2+\sigma}(0,T;L^2(\Gamma))} \le C\|z\|^{1/2-\sigma}_{L^2(0,T;L^2(\Gamma))} \cdot \|z\|^{1/2+\sigma}_{H^1(0,T;L^2(\Gamma))}. \end{cases}$$

Hence, using (2.17) for $\tilde{v}_2 - \tilde{v}_1$ and (2.16), (1.6) and Proposition 2.1, we have that if $\tilde{w}_2 \to \tilde{w}_1$ in B^σ, then $\tilde{v}_2 \to \tilde{v}_1$ in B^σ, uniformly with respect to α.

We shall now prove the following

PROPOSITION 2.3. *There exists a positive number $C(\varepsilon)$, independent of α, such that, if $\tilde{v}$ is any fixed point of Φ and $0 \le \alpha \le 1$, then*

$$\|\tilde{v}\|_{B^\sigma} \le C(\varepsilon)\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0\|_{L^2(\Gamma)} + \|\beta(u_0)\|_{H^1(\Gamma)} + \|\theta_0\|_{H^1(\Omega)}\}.$$

PROOF. If $\tilde{v}$ is a fixed point of Φ, then $v = u_0 + \tilde{v}$ is a solution of problem

$$(2.18)\qquad \partial v/\partial t - \Delta_g\beta(v) - \alpha\mathcal{T}\beta(v) = f, \qquad v(0) = u_0$$

and we have $v \in H^{2,1}(\Sigma) \cap B^\sigma_{u_0}$, by Proposition 1.1 and Proposition 2.1. Setting

$$\lambda(\xi) = \int_0^\xi \beta(\eta)\, d\eta \qquad \forall \xi \in \boldsymbol{R},$$

we can multiply (2.18) by $\beta(v(t))$ for a.e. t in $[0, T]$ and integrate on Γ and we obtain

$$(2.19)\qquad \frac{\partial}{\partial t}\int_\Gamma \lambda(v)\, d\sigma + (d\beta(v), d\beta(v)) - \alpha \int_\Gamma \mathcal{T}\beta(v)\,\beta(v)\, d\sigma = \int_\Gamma f\beta(v)\, d\sigma .$$

By definition $\mathcal{T}\beta(v) = \partial\theta/\partial\nu$ when θ is the solution of the following problem

$$(2.20)\qquad \partial\theta/\partial t - \Delta\theta = \varphi \quad \text{in } Q, \qquad \theta = \beta(v) \quad \text{on } \Sigma, \qquad \theta(0) = \theta_0 \quad \text{in } \Omega .$$

Then, for a.e. t in $[0, T]$, we have

$$-\int_\Gamma \mathcal{T}\beta(v)\cdot\beta(v)\, d\sigma = \int_\Omega |\nabla\theta|^2\, dx + \int_\Omega \Delta\theta\cdot\theta\, dx = \int_\Omega |\nabla\theta|^2\, dx + \int_\Omega \frac{\partial\theta}{\partial t}\,\theta\, dx - \int_\Omega \varphi\theta\, dx ,$$

so that (2.19) becomes

$$\frac{\partial}{\partial t}\int_\Gamma \lambda(v)\, d\sigma + (d\beta(v), d\beta(v)) + \alpha\int_\Omega |\nabla\theta|^2\, dx + \frac{\alpha}{2}\,\frac{\partial}{\partial t}\int_\Omega |\theta|^2\, dx = \alpha\int_\Omega \varphi\theta\, dx + \int_\Omega f\beta(v)\, d\sigma .$$

Therefore, integrating from 0 to $\bar t$, $0<\bar t<T$, we get

$$\int_\Gamma \lambda(v(\bar t))\, d\sigma + \int_0^{\bar t} (d\beta(v), d\beta(v))\, dt + \alpha\int_0^{\bar t}\int_\Omega |\nabla\theta|^2\, dx\, dt + \frac{\alpha}{2}\int_\Omega |\theta(\bar t)|^2\, dx =$$

$$= \int_\Gamma \lambda(u_0)\, d\sigma + \frac{\alpha}{2}\int_\alpha |\theta_0|^2\, dx + \alpha\int_0^{\bar t}\int_\Omega \varphi\theta\, dx\, dt + \int_0^{\bar t}\int_\Gamma f\beta(v)\, d\sigma .$$

Hence on using the elementary inequalities $|\beta(\xi)|^2/2\tilde L \le \lambda(\xi) \le \tilde L\xi^2/2$ we obtain finally

$$(2.21)\qquad \|\beta(v)\|_{L^\infty(0,T;L^2(\Gamma))} + \|\beta(v)\|_{L^2(0,T;H^1(\Gamma))} \le C\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0\|_{L^2(\Gamma)} + \|\theta_0\|_{L^2(\Omega)}\} ,$$

where C is independent of α and ε. Recalling again (2.1) (since $\beta = \beta_\varepsilon$) we deduce from (2.21)

$$(2.22)\qquad \|v\|_{L^\infty(0,T;L^2(\Gamma))} \le C\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0\|_{L^2(\Gamma)} + \|\theta_0\|_{L^2(\Omega)}\} + C ,$$

$$(2.23)\qquad \|v\|_{L^2(0,T;H^1(\Gamma))} \le C(\varepsilon)\,\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0\|_{L^2(\Gamma)} + \|\theta_0\|_{L^2(\Omega)}\} ,$$

where C is independent of α and ε and $C(\varepsilon)$ independent of α.

Now we shall also prove the inequality

$$(2.24)\qquad \|\partial v/\partial t\|_{L^2(\Sigma)} \le C(\varepsilon)\,\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0\|_{L^2(\Gamma)} + \|\beta(u_0)\|_{H^1(\Gamma)} + \|\theta_0\|_{H^1(\Omega)}\} ,$$

where $C(\varepsilon)$ is independent of α. Since $v \in H^{2,1}(\Sigma)$, we can multiply (2.18) by $\partial\beta(v)/\partial t$

and integrate on $\Gamma\times[0,\bar t]$, $0<\bar t<T$; we obtain

$$\int_0^{\bar t}\int_\Gamma \frac{\partial v}{\partial t}\,\frac{\partial\beta(v)}{\partial t}\,d\sigma\,dt-\int_0^{\bar t}\int_\Gamma \Delta_g\,\beta(v)\,\frac{\partial\beta(v)}{\partial t}\,d\sigma\,dt-$$

$$-\alpha\int_0^{\bar t}\int_\Gamma \mathcal{T}\,\beta(v)\,\frac{\partial\beta(v)}{\partial t}\,d\sigma\,dt=\int_0^{\bar t}\int_\Gamma f\,\frac{\partial\beta(v)}{\partial t}\,d\sigma\,dt.$$

Assuming $\beta(v)$ sufficiently regular, we have

$$(2.25)\qquad -\int_0^{\bar t}\int_\Gamma \Delta_g\,\beta(v)\,\frac{\partial\beta(v)}{\partial t}\,d\sigma\,dt=\int_0^{\bar t}\left(d\beta(v),d\,\frac{\partial\beta(v)}{\partial t}\right)d\sigma\,dt=$$

$$=\frac{1}{2}\int_0^{\bar t}\frac{\partial}{\partial t}(d\beta(v),d\beta(v))\,dt=\frac{1}{2}(d\beta(v(\bar t)),d\beta(v(\bar t)))-\frac{1}{2}(d\beta(u_0)\,d\beta(u_0)).$$

Since $\beta(v)\in H^{2,1}(\Sigma)\subset C^0([0,T];H^1(\Gamma))$ the final result in (2.25) is still valid (by density and continuity arguments). Similarly, we have

$$-\int_0^{\bar t}\int_\Gamma \mathcal{T}\,\beta(v)\,\frac{\partial\beta(v)}{\partial t}\,d\sigma\,dt=\int_0^{\bar t}\int_\Omega\left(\nabla\theta\nabla\,\frac{\partial\theta}{\partial t}+\Delta\theta\,\frac{\partial\theta}{\partial t}\right)dx\,dt=$$

$$=\frac{1}{2}\int_0^{\bar t}\frac{\partial}{\partial t}\int_\Omega|\nabla\theta|^2\,dx\,dt+\int_0^{\bar t}\int_\Omega\left|\frac{\partial\theta}{\partial t}\right|^2dx\,dt-\int_0^{\bar t}\int_\Omega\varphi\,\frac{\partial\theta}{\partial t}\,dx\,dt=$$

$$=\frac{1}{2}\int_\Omega|\nabla\theta(\bar t)|^2\,dx-\frac{1}{2}\int_\Omega|\nabla\theta_0|^2\,dx+\int_0^{\bar t}\int_\Omega\left|\frac{\partial\theta}{\partial t}\right|^2dx\,dt-\int_0^{\bar t}\int_\Omega\varphi\,\frac{\partial\theta}{\partial t}\,dx\,dt.$$

Collecting the previous estimates we obtain

$$\int_0^{\bar t}\int_\Gamma\frac{1}{\beta'(v)}\left|\frac{\partial\beta(v)}{\partial t}\right|^2d\sigma\,dt+\frac{1}{2}(d\beta(v(\bar t)),d\beta(v(\bar t)))+$$

$$+\frac{\alpha}{2}\int_\Omega|\nabla\theta(\bar t)|^2\,dx+\alpha\int_0^{\bar t}\int_\Omega\left|\frac{\partial\theta}{\partial t}\right|^2dx\,dt=$$

$$=\int_0^{\bar t}\int_\Gamma f\,\frac{\partial\beta(v)}{\partial t}\,d\sigma\,dt+\frac{1}{2}(d\beta(u_0),d\beta(u_0))+\frac{\alpha}{2}\int_\Omega|\nabla\theta_0|^2\,dx+\alpha\int_0^{\bar t}\int_\Omega\varphi\,\frac{\partial\theta}{\partial t}\,dx\,dt.$$

Then, again from (2.1) (since $\beta = \beta_\varepsilon$) we obtain

$$\left\| \frac{\partial \beta(v)}{\partial t} \right\|^2_{L^2(0,\bar{t};L^2(\Gamma))} + (d\beta(v(\bar{t})), d\beta(v(\bar{t}))) + \alpha \|\nabla\theta(\bar{t})\|^2_{L^2(\Omega)} + \alpha \left\| \frac{\partial \theta}{\partial t} \right\|^2_{L^2(0,\bar{t};L^2(\Omega))} \leq$$

$$\leq C\left\{ \|f\|_{L^2(\Sigma)} \left\| \frac{\partial \beta(v)}{\partial t} \right\|_{L^2(0,\bar{t};L^2(\Gamma))} + (d\beta(u_0), d\beta(u_0)) + \alpha \|\nabla\theta_0\|^2_{L^2(\Omega)} + \alpha \|\varphi\|_{L^2(Q)} \cdot \right.$$

$$\left. \cdot \left\| \frac{\partial \theta}{\partial t} \right\|_{L^2(0,\bar{t};L^2(\Omega))} \right\}.$$

Therefore, using also (2.21) and elementary inequalities, we conclude

$$(2.26) \qquad \|\partial\beta(v)/\partial t\|_{L^2(\Sigma)} + \|\beta(v)\|_{L^\infty(0,T;H^1(\Gamma))} + \alpha\|\nabla\theta\|_{L^\infty(0,T;L^2(\Omega))} + \alpha\|\partial\theta/\partial t\|_{L^2(Q)} \leq$$

$$\leq C\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0\|_{L^2(\Gamma)} + \|\beta(u_0)\|_{H^1(\Gamma)} + \|\theta_0\|_{H^1(\Omega)}\},$$

where C is independent of α and ε. And again from (2.1) we obtain in particular (2.24). Finally Proposition 2.3 follows from (2.23) and (2.24) and recalling that $v = u_0 + \tilde{v}$.

Now the existence part of Theorem 2.1 follows from the classical Leray-Schauder theorem applied to (2.12), on using Propositions 2.1, 2.2, 2.3 and the compactness of the injection of $H^{2,1}(\Sigma) \cap B^\sigma$ in B^σ and noting that for $\alpha = 0$ (2.18) has a unique solution and for $\alpha = 1$ (2.18) becomes (2.5).

In order to prove the uniqueness, let $u_i\,(i = 1,2)$ be a solution of (2.5) and denote by θ_i the solution of (2.20) coresponding to $\beta(u_i)$. Then we have, for a.e. t in $[0, T]$,

$$(2.27) \qquad \left\langle \frac{\partial(u_2 - u_1)}{\partial t}, \psi \right\rangle + (d(\theta_2 - \theta_1), d\psi) - \int_\Gamma \frac{\partial(\theta_2 - \theta_1)}{\partial t} \psi \, d\sigma = 0, \qquad \forall \psi \in H^1(\Gamma).$$

Noting that $\partial(\theta_2 - \theta_1)/\partial t - \Delta(\theta_2 - \theta_1) = 0$ in Q, (2.27) becomes

$$\left\langle \frac{\partial(u_2 - u_1)}{\partial t}, \eta \right\rangle + (d(\theta_2 - \theta_1), d\eta) + \int_\Omega \nabla(\theta_2 - \theta_1)\nabla\eta \, dx +$$

$$+ \int_\Omega \frac{\partial(\theta_2 - \theta_1)}{\partial t} \eta \, dx = 0, \qquad \forall \eta \in H^{3/2}(\Omega).$$

Then, if we integrate from 0 to t, $0 < t < T$, we obtain

$$(2.28) \qquad \langle u_2(t) - u_1(t), \eta \rangle + \left(d\int_0^t (\theta_2(s) - \theta_1(s))\, ds, d\eta \right) +$$

$$+ \int_\Omega \nabla \int_0^t (\theta_2(s) - \theta_1(s))\, ds \, \nabla\eta \, dx + \int_\Omega (\theta_2(t) - \theta_1(t))\, \eta \, dx = 0, \qquad \forall \eta \in H^{3/2}(\Omega).$$

We can take $\eta = \theta_2(t) - \theta_1(t)$ in (2.28) and integrate again from 0 to T, obtaining

$$\int_0^T \int_\Gamma (u_2 - u_1)(\beta(u_2) - \beta(u_1))\, d\sigma\, dt + \frac{1}{2}\left(d\int_0^T (\theta_2 - \theta_1)\, dt, d\int_0^T (\theta_2 - \theta_1)\, dt\right) +$$
$$+\frac{1}{2}\int_\Omega \left(\nabla \int_0^T (\theta_2 - \theta_1)\, dt\right)^2 dx + \int_0^T \int_\Omega |\theta_2 - \theta_1|^2\, dx\, dt = 0.$$

Using (2.15) we have

$$\int_0^T \int_\Gamma |\beta(u_2) - \beta(u_1)|^2\, d\sigma\, dt \leq 0,$$

whence $\beta(u_2) - \beta(u_1) = 0$ and $u_2 - u_1 = 0$ a.e. in Σ, because $u_2(\cdot, 0) = u_1(\cdot, 0)$.

REMARK 2.1. The estimate (2.22) can be improved by the following one

$$(2.29) \qquad \|v\|_{L^\infty(0,T;L^2(\Gamma))} \leq C\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0\|_{L^2(\Gamma)} + \|\beta(u_0)\|_{H^1(\Gamma)} + \|\theta_0\|_{H^1(\Omega)}\},$$

where C is independent of α and ε. In fact we can multiply (2.18) by v and integrate on $\Gamma \times [0,t]$, $0 < t < T$, thus obtaining

$$\frac{1}{2}\int_\Gamma |v(t)|^2\, d\sigma \leq \alpha \int_0^t \int_\Gamma \mathcal{T}\beta(v) \cdot v\, d\sigma\, dt + \int_0^t \int_\Gamma fv\, d\sigma\, dt + \frac{1}{2}\int_\Gamma |u_0|^2\, d\sigma \leq$$
$$\leq \{\|\mathcal{T}\beta(v)\|_{L^2(\Sigma)} + \|f\|_{L^2(\Sigma)}\}\|v\|_{L^2(\Sigma)} + \|u_0\|^2_{L^2(\Gamma)}/2.$$

Hence (2.29) follows from Proposition 1.1 and (2.26).

3. The existence of a solution of problem (1.20) follows using the compactness and monotonicity procedures well known for the usual Stefan problems and the estimates which we have proved for the solution of the *regularized problem* (2.4). Indeed, putting again the subscript ε in all the functions, we can summarize the estimates obtained for u_ε and $\beta_\varepsilon(u_\varepsilon)$ (cfr. (2.22) and (2.26)) by

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Gamma))} \leq C\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0^\varepsilon\|_{L^2(\Gamma)} + \|\theta_0^\varepsilon\|_{L^2(\Omega)}\} + C,$$
$$\|\beta_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;H^1(\Gamma))} + \|\partial\beta_\varepsilon(u_\varepsilon)/\partial t\|_{L^2(\Sigma)} \leq$$
$$\leq C\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0^\varepsilon\|_{L^2(\Gamma)} + \|\beta_\varepsilon(u_0^\varepsilon)\|_{H^1(\Gamma)} + \|\theta_0^\varepsilon\|_{H^1(\Omega)}\}$$

where C is independent of ε. Then from (2.4) and Proposition 1.1 we also obtain

$$\|\partial u_\varepsilon/\partial t\|_{L^2(0,T;H^{-1}(\Gamma))} \leq C\{\|f\|_{L^2(\Sigma)} + \|\varphi\|_{L^2(Q)} + \|u_0^\varepsilon\|_{L^2(\Gamma)} + \|\beta_\varepsilon(u_0^\varepsilon)\|_{H^1(\Gamma)} + \|\theta_0^\varepsilon\|_{H^1(\Omega)}\}.$$

Coupling these estimates with (2.2), (2.3), we can pass to the limit in (2.4) thus proving the existence of a solution of (1.20). The uniqueness follows by the same proof used for the uniqueness of the solution of the regularized problem (2.4). We only have to replace (2.15) for β_ε by the analogous estimate for β: $|\beta(\xi)-\beta(\eta)|^2 \le \max(a,b)\cdot$ $\cdot(\xi-\eta)(\beta(\xi)-\beta(\eta))$, $\forall \xi, \eta \in \boldsymbol{R}$ and to recall that, from the results of [6, chapt. 4] and Proposition 1.1, $\theta_2-\theta_1$ belongs at least to the space $H^{3/2,1}(Q)$.

REMARK 3.1. Theorem 1.1 has been proved here under proper assumptions on the initial enthalpy u_0 and temperature $\beta(u_0)$, namely $u_0 \in L^2(\Gamma)$, $\beta(u_0) \in H^1(\Gamma)$. In fact, typically, the enthalpy and the temperature gradient jump in the Stefan problem. The demonstration of existence and uniqueness of the solution of problem (1.20) under the weakest assumption $u_0 \in L^2(\Gamma)$ needs further investigations and seems to rely on a deeper use of the techniques of [6, vol. 2], [2], [11], [12]. The hypotheses on the domain Ω can be obviously weakened by requiring, *e.g.*, that Γ is an oriented connected C^2 $(n-1)$-manifold. The *final comments* of [9] and the theoretical and numerical questions presented in [7] and [8] apply also for the present Problem (I).

REMARK. Let me point out some mistakes in [9]:

page	line	Errata	Corrige
221	21	[24]	[8]
221	21	$\beta_\varepsilon(v_n)(t)$	$v_n(t)$
224	1	We obtain	Setting $\lambda_\varepsilon(\xi)=\int_0^\xi \beta_\varepsilon(\eta)\,d\eta$ we obtain
224	2 and 14	$\frac{1}{\beta'_\varepsilon(v)}\lvert\beta_\varepsilon(v)\rvert^2$	$\lambda_\varepsilon(v)$
224	17 and 20	$\frac{1}{\beta'_\varepsilon(v(\bar{t}))}\lvert\beta_\varepsilon(v(\bar{t}))\rvert^2$	$\lambda_\varepsilon(v(\bar{t}))$
225	1	(3.11)	(3.1)
225	5	$C\lVert f\rVert_{L^2(\Sigma)}$	$C(\lVert f\rVert_{L^2(\Sigma)}+1)$
226	18, 20, 21, 25	α	1
226	21	$\forall\eta$	$=0,\quad \forall\eta$
227	3	α	1
227	17	$C\lVert f\rVert_{L^2(\Sigma)}$	$C(\lVert f\rVert_{L^2(\Sigma)}+1)$

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