Atti Accademia Nazionale Lincei Classe Scienze Fisiche Matematiche Naturali

Rendiconti Lincei
Matematica e Applicazioni

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Regularity of wave and plate equations with interior point control

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Serie 9, Vol. **2** (1991), n.4, p. 307–315.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLIN_1991_9_2_4_307_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni, Accademia Nazionale dei Lincei, 1991.

Rend. Mat. Acc. Lincei
s. 9, v. 2:307-315 (1991)

Teoria dei controlli. — *Regularity of wave and plate equations with interior point control.* Nota di Roberto Triggiani, presentata (*) dal Corrisp. R. Conti.

Abstract. — The regularity of solutions of various dynamical equations (wave, Euler-Bernoulli, Kirchhoff, Schrödinger) in a bounded open domain Ω in R^N, subject to the action of a point control at some point of Ω, is studied. Detailed proofs of the results are contained in the references [8-10].

Key words: Control Theory; Wave and plate equations; Regularity.

Riassunto. — *Regolarità delle equazioni delle onde e delle piastre con controllo puntuale interno.* Si studia la regolarità delle soluzioni di varie equazioni dinamiche (onde, Euler-Bernoulli, Kirchhoff, Schrödinger) in una regione limitata Ω di R^N, sotto l'azione di un controllo esercitato in un punto di Ω. Le dimostrazioni dettagliate si trovano nei riferimenti bibliografici [8-10].

1. Introduction, statement of main problem

Let Ω be an open bounded domain in R^N, $N=1,2,3$, with sufficiently smooth boundary Γ. In this *Note* we announce new sharp results on the regularity of solutions of various dynamical equations, subject to the action of *point control* exercised at an interior point of Ω, which without loss of generality we take to be the origin. We shall consider: wave equations; Euler-Bernoulli (plate) equations and Kirchhoff (plate) equations; and Schrödinger equations under a variety of boundary conditions. The only known result in the literature so far concerns the wave equation with Dirichlet B.C. with $N=\dim\Omega=3$, where three different proofs are in fact available: see [1]: one, due to Y. Meyer, uses harmonic analysis; another one due to L. Nirenberg, uses the classical Kirchhoff formula for the solutions of the Cauchy problem in R^3 as well as finite speed of propagation arguments; a third one, due to J. L. Lions uses a recently established [3-6] property of the normal trace of the homogeneous wave equation. Our approach is different and very general. In particular it does not requires finite speed of propagation arguments or exact solution formulas. For lack of space we omit corresponding duality results, concerning point observation. Details and proofs are given in [8-10].

2. Wave equation with homogeneous Dirichlet B.C.

In this section we consider

(2.1*a*) $$w_{tt}=\Delta w+\delta(x)\,v(t) \quad \text{in } (0,T]\times\Omega\equiv Q$$

(2.1*b*) $$w(0,x)\equiv w_t(0,x)\equiv 0 \quad \text{in } \Omega$$

(2.1*c*) $$w|_\Sigma\equiv 0 \quad \text{in } (0,T]\times\Gamma\equiv\Sigma$$

(*) Nella seduta del 14 giugno 1991.

where $\delta(x)$ is the Dirac mass $+1$ at the interior point 0 (origin). We define the positive, self-adjoint operator A

(2.2) $$Ah = -\Delta h; \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega); \quad \mathcal{D}(A^{1/2}) = H_0^1(\Omega); \quad \mathcal{D}(A^{1/4}) = H_{00}^{1/2}(\Omega).$$

THEOREM 2.1. With reference to problem (2.1), let

(2.3) $$v \in L_2(0, T).$$

Then, continuously:

(*a*) for $N = \dim \Omega = 3$,

(2.4*a*) $$w \in C([0, T]; L_2(\Omega)),$$

(2.4*b*) $$w_t \in C([0, T]; H^{-1}(\Omega) = [\mathcal{D}(A^{1/2})]'),$$

(2.4*c*) $$w_{tt} \in L_2(0, T; H^{-2}(\Omega)),$$

moreover

(2.4*d*) $$\left.\frac{\partial w}{\partial \nu}\right|_\Sigma \in H^{-1}(\Sigma);$$

(*b*) for $N = \dim \Omega = 2$,

(2.5*a*) $$w \in C([0, T]; H_{00}^{1/2}(\Omega) = \mathcal{D}(A^{1/4})),$$

(2.5*b*) $$w_t \in C([0, T]; [H_{00}^{1/2}(\Omega)]' = [\mathcal{D}(A^{1/4})]'),$$

(2.5*c*) $$w_{tt} \in L_2(0, T; [\mathcal{D}(A^{3/4})]') \subset L_2(0, T; [H_{00}^{3/2}(\Omega)]'),$$

moreover

(2.5*d*) $$\left.\frac{\partial w}{\partial \nu}\right|_\Sigma \in H^{-1/2}(\Sigma);$$

(*c*) for $N = \dim \Omega = 1$,

(2.6*a*) $$w \in C([0, T]; H_0^1(\Omega) = \mathcal{D}(A^{1/2})),$$

(2.6*b*) $$w_t \in C([0, T]; L_2(\Omega)),$$

(2.6*c*) $$w_{tt} \in L_2(0, T; H^{-1}(\Omega) = [\mathcal{D}(A^{1/2})]'),$$

moreover

(2.6*d*) $$\left.\frac{\partial w}{\partial \nu}\right|_\Sigma \in L_2(\Sigma). \qquad \square$$

REMARK 2.1. If one studies the regularity of problem (2.1) by using only that, by Sobolev embedding, $\delta \in [H^\alpha(\Omega)]'$, where $\alpha = 3/2 + \varepsilon$ for $N = 3$; $\alpha = 1 + \varepsilon$ for $N = 2$; $\alpha = 1/2 + \varepsilon$ for $N = 1$, then one would obtain a regularity result for, say, w which is *lower* by «$1/2 + \varepsilon$» in space regularity, measured in Sobolev space order, than those of Theorem 2.1: *e.g.*, for $N = 3$ one would get only $w \in H^{-1/2-\varepsilon}(\Omega)$ rather than $L_2(\Omega)$ as in (2.4*a*); for $N = 2$, one would get only $w \in H^{-\varepsilon}(\Omega)$ rather than $H_{00}^{1/2}(\Omega)$ as in (2.5*a*); for $N = 1$ one would get only $w \in H^{1/2-\varepsilon}(\Omega)$ rather than $H_0^1(\Omega)$ as in (2.6*a*). To see this, we use $[H^\alpha(\Omega)]' \subset [\mathcal{D}(A^{\alpha/2})]'$, so that $A^{-\alpha/2}\delta \in L_2(\Omega)$ for the second-order operator A in (2.2). Then the solution w of (2.1) satisfies abstractly

$$A^{(1-\alpha)/2} w(t) = \int_0^t A^{1/2} S(t-\tau) A^{-\alpha/2} \delta v(\tau)\, d\tau \in C([0,T]; L_2(\Omega))$$

by convolution properties between $A^{-\alpha/2}\delta v \in L_2(0,T; L_2(\Omega))$ and $t \to A^{1/2} S(t)$ strongly continuous on $L_2(\Omega)$, where

$$S(t) = \int_0^t C(\tau)\, d\tau$$

and $C(t)$ is the cosine operator generated by $-A$ in (2.2). □

3. Wave equation with homogeneous Neumann B.C.

The same interior regularity results in terms of Sobolev spaces as above hold true for the Neumann problem with $\Gamma = \Gamma_0 \cup \Gamma_1$:

(3.1a) $\quad w_{tt} = \Delta w + \delta(x) v(t) \quad$ in $(0,T] \times \Omega \equiv Q$

(3.1b) $\quad w(0,x) \equiv w_t(0,x) \equiv 0 \quad$ in Ω

(3.1c) $\quad w|_{\Sigma_0} \equiv 0 \quad$ in $(0,T] \times \Gamma_0 = \Sigma_0$

(3.1d) $\quad \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma_1} \equiv 0 \quad$ in $(0,T] \times \Gamma_1 = \Sigma_1$

except that now: for $N=2$, $H_{00}^{1/2}(\Omega)$ in (2.5a) for w is replaced by $H^{1/2}(\Omega)$; for $N=1$, $H_0^1(\Omega)$ in (2.6a) for w is replaced by $H^1(\Omega)$. As to boundary regularity we now have

(3.2)
$$w|_\Sigma \in \begin{cases} H^{\alpha-1}(\Sigma) & N=3 \\ H^{(\alpha+\beta-1)/2}(\Sigma) & N=2 \\ H^{\beta}(\Sigma) & N=1 \end{cases}$$

where

$\alpha = 3/5 - \varepsilon$; $\beta = 3/5$: for a general Ω;

$\alpha = \beta = 2/3$: for Ω being a sphere;

$\alpha = \beta = 3/4$: for Ω being a parallelepiped.

4. Kirchhoff equation with homogeneous boundary conditions $w|_\Sigma \equiv \Delta w|_\Sigma \equiv 0$

In this section we consider

(4.1a) $\quad w_{tt} - k\Delta w_{tt} + \Delta^2 w = \delta(x) v(t) \quad$ in $(0,T] \times \Omega = Q$

(4.1b) $\quad w(0,x) \equiv w_t(0,x) \equiv 0 \quad$ in Ω

(4.1c) $\quad w|_\Sigma \equiv 0 \quad$ in $(0,T] \times \Gamma = \Sigma$

(4.1d) $\quad \Delta w|_\Sigma \equiv 0 \quad$ in Σ

where k is a positive constant with the origin 0 as an interior point of Ω. The constant $k>0$ makes problem (4.1) hyperbolic over the case $k=0$ of the next sect. 5. Throughout this section, let A be the positive self-adjoint operator defined by

$$Ah = \Delta^2 h; \qquad \mathcal{D}(A) = \{h \in H^4(\Omega) : h|_\Gamma = \Delta h|_\Gamma = 0\}. \tag{4.2}$$

We recall that (with equivalent norms)

$$\mathcal{D}(A^{3/4}) = \{h \in H^3(\Omega) : h|_\Gamma = \Delta h|_\Gamma = 0\}; \qquad \mathcal{D}(A^{1/4}) = H_0^1(\Omega); \tag{4.3}$$

$$A^{1/2} h = -\Delta h; \qquad \mathcal{D}(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega); \tag{4.4}$$

$$\mathcal{D}(A^{1/8}) = [\mathcal{D}(A^{1/4}), L_2(\Omega)]_{1/2} = [H_0^1(\Omega), L_2(\Omega)]_{1/2} = H_{00}^{1/2}(\Omega). \tag{4.5}$$

THEOREM 4.1. With reference to problem (4.1), let

$$v \in L_2(0,T). \tag{4.6}$$

Then, continuously,

(a) for $N = \dim \Omega = 3$,

$$w \in C([0,T]; \mathcal{D}(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)), \tag{4.7a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{1/4}) = H_0^1(\Omega)), \tag{4.7b}$$

$$w_{tt} \in L_2(0,T; L_2(\Omega)); \tag{4.7c}$$

(b) for $N = \dim \Omega = 2$,

$$w \in C([0,T]; \mathcal{D}(A^{5/8})) \subset C([0,T]; H^{5/2}(\Omega)), \tag{4.8a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{3/8})) \subset C([0,T]; H^{3/2}(\Omega)), \tag{4.8b}$$

$$w_{tt} \in L_2(0,T; \mathcal{D}(A^{1/8}) = H_{00}^{1/2}(\Omega)); \tag{4.8c}$$

(c) for $N = \dim \Omega = 1$,

$$w \in C([0,T]; \mathcal{D}(A^{3/4})), \tag{4.9a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega)), \tag{4.9b}$$

$$w_{tt} \in C([0,T]; \mathcal{D}(A^{1/4}) = H_0^1(\Omega)). \tag{4.9c}$$

REMARK 4.1. If one studies the regularity of problem (4.1) by using that, by Sobolev embedding, $\delta \in [H^\alpha(\Omega)]'$ where $\alpha = 3/2 + \varepsilon$ for $N=3$; $\alpha = 1 + \varepsilon$ for $N=2$; $\alpha = 1/2 + \varepsilon$ for $N=1$, then one would obtain a regularity result for, say, w which is lower by «$1/8 + \varepsilon$» in space regularity, measured in fractional powers of A (essentially «$1/2 + 4\varepsilon$» measured in Sobolev space order), than those of Theorem 4.1; *e.g.*, for $N=3$ one would get only $w \in \mathcal{D}(A^{3/8-\varepsilon})$ rather than $\mathcal{D}(A^{1/2})$ as in (4.7a); for $N=2$, one would get only $w \in \mathcal{D}(A^{1/2-\varepsilon})$ rather than $\mathcal{D}(A^{5/8})$ as in (4.8a); for $N=1$ one would get only $w \in \mathcal{D}(A^{5/8-\varepsilon})$ rather than $w \in \mathcal{D}(A^{3/4})$ as in (4.9a). □

5. Kirchhoff equation with homogeneous Dirichlet/Neumann boundary conditions ($w|_\Sigma = \partial w/\partial\nu|_\Sigma \equiv 0$)

In this section we consider

$$w_{tt} - k\Delta\, w_{tt} + \Delta^2 w = \delta(x)\, v(t) \quad \text{in } (0,T]\times\Omega = Q, \tag{5.1a}$$

$$w(0,x) \equiv w_t(0,x) \equiv 0 \quad \text{in } \Omega, \tag{5.1b}$$

$$w|_\Sigma \equiv 0 \quad \text{in } (0,T]\times\Gamma = \Sigma, \tag{5.1c}$$

$$\left.\frac{\partial w}{\partial \nu}\right|_\Sigma \equiv 0 \quad \text{in } \Sigma, \tag{5.1d}$$

with k a positive constant and with the origin 0 as an interior of Ω. Throughout this section let A be the positive definite self-adjoint operator on $L_2(\Omega)$ defined by

$$Ah = \Delta^2 h;\ \mathcal{D}(A) = \left\{h \in H^4(\Omega) : h|_\Gamma = \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0\right\}. \tag{5.2}$$

We recall that (with equivalent norms)

$$\begin{cases} \mathcal{D}(A^{3/4}) = \left\{h \in H^3(\Omega) : h|_\Gamma = \left.\frac{\partial h}{\partial \nu}\right|_\Gamma = 0\right\}; \\ \mathcal{D}(A^{1/2}) = H_0^2(\Omega); \qquad \mathcal{D}(A^{1/4}) = H_0^1(\Omega); \end{cases} \tag{5.3}$$

$$\mathcal{D}(A^{3/8}) = [\mathcal{D}(A^{1/2}),\, \mathcal{D}(A^{1/4})]_{1/2} = [H_0^2(\Omega),\, H_0^1(\Omega)]_{1/2} = H_{00}^{3/2}(\Omega); \tag{5.4}$$

$$\mathcal{D}(A^{1/8}) = [\mathcal{D}(A^{1/4}),\, L_2(\Omega)]_{1/2} = [H_0^1(\Omega), L_2(\Omega)]_{1/2} = H_{00}^{1/2}(\Omega). \tag{5.5}$$

Moreover, we introduce the positive definite, self-adjoint operators B and B_k on $L_2(\Omega)$ defined by

$$Bh = -\Delta h; \qquad \mathcal{D}(B) = H^2(\Omega) \cap H_0^1(\Omega); \qquad B_k = (I + kB); \qquad \mathcal{D}(B_k) = \mathcal{D}(B). \tag{5.6}$$

We note that by (5.3) and (5.6) we have (properly):

$$\mathcal{D}(A^{1/2}) \subset \mathcal{D}(B); \qquad \text{hence} \qquad BA^{-1/2} \in \mathcal{L}(L_2(\Omega)), \tag{5.7}$$

while $A^{1/2}B^{-1}$ is an unbounded operator on $L_2(\Omega)$.

$$\mathcal{D}(A^{3/8-\varepsilon}) = \mathcal{D}(B^{3/4-2\varepsilon}) = H^{3/2-4\varepsilon}(\Omega) \cap H_0^1(\Omega), \qquad \varepsilon > 0; \tag{5.8}$$

in particular we explicitly note

$$\mathcal{D}(B_k^{1/2}) = \mathcal{D}(B^{1/2}) = \mathcal{D}(A^{1/4}) = H_0^1(\Omega); \tag{5.9}$$

$$\mathcal{D}(B_k^{1/4}) = \mathcal{D}(B^{1/4}) = \mathcal{D}(A^{1/8}) = H^{1/2}(\Omega) = H_0^{1/2}(\Omega); \tag{5.10}$$

$$\mathcal{D}(A^{3/8}) = [\mathcal{D}(A^{1/2}),\, \mathcal{D}(A^{1/4})]_{1/2} \subset [\mathcal{D}(B),\, \mathcal{D}(B^{1/2})]_{1/2} = \mathcal{D}(B^{3/4}) = \mathcal{D}(B_k^{3/4}). \tag{5.11}$$

The fact that under the boundary conditions in (5.2), the operator $BA^{-1/2}$ is *not* an isomorphism on $L_2(\Omega)$ as noted in (5.7) is a major technical difference over the case of the preceding section, and is responsible for additional technical difficulties, which are reflected in the following regularity result (compare with Theorem 4.1, particularly the case $N=3$).

THEOREM 5.1. With reference to problem (5.1) let

$$v \in L_2(0,T). \tag{5.12}$$

Then, continuously

(*a*) for $N = \dim \Omega = 3$,

$$w \in C([0,T]; \mathcal{D}(A^{1/2}) = H_0^2(\Omega)), \tag{5.13a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{1/4}) = H_0^1(\Omega)), \tag{5.13b}$$

$$Bw_{tt} \in L_2(0,T; [\mathcal{D}(A^{1/2})]' = H^{-1}(\Omega)); \tag{5.13c}$$

(*b*) for $N = \dim \Omega = 2$,

$$w \in C([0,T]; \mathcal{D}(A^{5/8})), \tag{5.14a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{3/8}) = H_{00}^{3/2}(\Omega)), \tag{5.14b}$$

$$Bw_{tt} \in L_2(0,T; [\mathcal{D}(A^{3/8})]'); \tag{5.14c}$$

(*c*) for $N = \dim \Omega = 1$,

$$w \in C([0,T]; \mathcal{D}(A^{3/4}) = H^3(\Omega) \cap H_0^2(\Omega)), \tag{5.15a}$$

$$w_t \in C([0,T]; \mathcal{D}(A^{1/2}) = H_0^2(\Omega)), \tag{5.15b}$$

$$w_{tt} \in L_2(0,T; \mathcal{D}(A^{1/4}) = H_0^1(\Omega)). \quad \square \tag{5.15c}$$

6. EULER-BERNOULLI EQUATION WITH HOMOGENEOUS DIRICHLET/NEUMANN B.C.

In this section we consider

$$w_{tt} + \Delta^2 w = \delta(x)\, v(t) \quad \text{in } (0,T] \times \Omega \equiv Q, \tag{6.1a}$$

$$w(0,x) \equiv w_t(0,x) \equiv 0 \quad \text{in } \Omega, \tag{6.1b}$$

$$w|_\Sigma \equiv 0 \quad \text{in } (0,T] \times \Gamma \equiv \Sigma, \tag{6.1c}$$

$$\left.\frac{\partial w}{\partial \nu}\right|_\Sigma \equiv 0 \quad \text{in } \Sigma, \tag{6.1d}$$

with the origin 0 as an interior point of Ω. Let A the positive, self-adjoint operator defined in (5.2).

THEOREM 6.1. With reference to problem (6.1), let

$$v \in L_2(0,T). \tag{6.2}$$

Then, continuously:

(*a*) for $N = \dim \Omega = 3$,

$$w \in C([0,T]; H_{00}^{1/2}(\Omega) = \mathcal{D}(A^{1/8})), \tag{6.3a}$$

$$w_t \in C([0,T]; [H_{00}^{3/2}(\Omega)]' = [\mathcal{D}(A^{3/8})]'), \tag{6.3b}$$

$$w_{tt} \in L_2(0,T; [\mathcal{D}(A^{7/8})]'); \tag{6.3c}$$

(*b*) for $N = \dim \Omega = 2$,

(6.4*a*) $$w \in C([0,T];\ H_0^1(\Omega) = \mathcal{D}(A^{1/4})),$$

(6.4*b*) $$w_t \in C([0,T];\ H^{-1}(\Omega) = [\mathcal{D}(A^{1/4})]'),$$

(6.4*c*) $$w_{tt} \in L_2(0,T;\ [\mathcal{D}(A^{3/4})]');$$

(*c*) for $N = \dim \Omega = 1$,

(6.5*a*) $$w \in C([0,T];\ H_{00}^{3/2}(\Omega) = \mathcal{D}(A^{3/8})),$$

(6.5*b*) $$w_t \in C([0,T];\ [H_{00}^{1/2}(\Omega)]' = [\mathcal{D}(A^{1/8})]'),$$

(6.5*c*) $$w_{tt} \in L_2(0,T;\ [\mathcal{D}(A^{5/8})]').$$

Remark 6.1. It was noted in Remark 2.1 that in the case of the wave problems (2.1) and (3.1) the present sharp approach produces regularity results which are «$1/2 + \varepsilon$» higher in space regularity (measured in Sobolev space order) than the one directly obtained by simply using that $\delta \in [H^\alpha(\Omega)]'$, $\alpha = 3/2 + \varepsilon$, $1 + \varepsilon$, $1/2 + \varepsilon$, for $N = 3, 2, 1$ respectively. The same gain in regularity is obtained in the case of Kirchhoff problems (also hyperbolic) as noted in Remark 4.1. Instead, in the case of Euler-Bernoulli problems (both problem (6.1) as well the subsequent problem (7.1)) the present sharp approach produces only an «ε-improvement» over Theorem 6.1. To see this, we use $\delta \in [H^\alpha(\Omega)]' \subset [\mathcal{D}(A^{\alpha/4})]'$, equivalently that $A^{-\alpha/4}\delta \in L_2(\Omega)$ for the fourth-order operator A in (5.2). Then the solution w to, say, problem (6.1) satisfies

$$A^{1/2-\alpha/4} w(t) = \int_0^t A^{1/2} S(t-\tau) A^{-\alpha/4} \delta v(\tau)\, d\tau \in C([0,T];\ L_2(\Omega)),$$

by the usual convolution properties. This yields results which are «ε-worse», *i.e.*, $w \in \mathcal{D}(A^{1/8-\varepsilon})$, $\mathcal{D}(A^{1/4-\varepsilon})$, $\mathcal{D}(A^{3/8-\varepsilon})$, in space regularity over those in (6.3*a*), (6.4*a*), (6.5*a*) respectively. □

7. Euler-Bernoulli equation with $w|_\Sigma = \Delta w|_\Sigma = 0$

If we now consider the Euler-Bernoulli equation (6.1*a*), (6.1*b*), (6.1*c*) with B.C. (6.1*d*) replaced by

(7.1) $$\Delta w|_\Sigma \equiv 0,$$

then the same regularity results as in Theorem 6.1 hold true if we replace the operator A defined by (5.2) with the operator A in (4.2).

8. Schrödinger equations

In this section we consider

(8.1*a*) $$y_t = i\Delta y + \delta(x)\, v(t) \quad \text{in } Q = (0,T] \times \Omega;$$

(8.1*b*) $$y(0,x) \equiv 0 \quad \text{in } \Omega;$$

(8.1*c*) $$y|_\Sigma \equiv 0 \quad \text{in } \Sigma = (0,T] \times \Gamma;$$

with the origin 0 an interior point of Ω. We define the positive self-adjoint operator A on $L_2(\Omega)$ by

(8.2) $$A = -\Delta;\quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega);\quad \mathcal{D}(A^{1/2}) = H_0^1(\Omega);\quad \mathcal{D}(A^{1/4}) = H_{00}^{1/2}(\Omega).$$

THEOREM 8.1. With reference to problem (8.1), let $v \in L_2(0, T)$. Then continuously

(*a*) for $N = \dim \Omega = 3$,

(8.3*a*) $$y \in C([0, T];\ [\mathcal{D}(A^{3/4})]') \subset C([0, T];\ H^{-3/2-\varepsilon}(\Omega));$$

(8.3*b*) $$y_t \in L_2(0, T;\ [\mathcal{D}(A^{7/4})]' \cap H^{-7/2-\varepsilon}(\Omega));$$

(*b*) for $\dim \Omega = 2$,

(8.4*a*) $$y \in C([0, T];\ [\mathcal{D}(A^{1/2})]' = H^{-1}(\Omega));$$

(8.4*b*) $$y_t \in L_2(0, T;\ [\mathcal{D}(A^{3/2})]' \cap H^{-3}(\Omega));$$

(*c*) for $N = \dim \Omega = 1$,

(8.5*a*) $$y \in C([0, T];\ [\mathcal{D}(A^{1/4})]' = [H_{00}^{1/2}(\Omega)]') \subset C([0, T];\ H^{-1/2-\varepsilon}(\Omega));$$

(8.5*b*) $$y_t \in L_2(0, T;\ [\mathcal{D}(A^{5/4})]' \cap H^{-5/2-\varepsilon}(\Omega)).$$

REMARK 8.1. The above results are «ε-smoother» in space regularity over the ones that can be obtained directly by simply using the property that, by Sobolev embedding, $\delta \in [H^\alpha(\Omega)]'$, $\alpha = 3/2 + \varepsilon;\ 1 + \varepsilon;\ 1/2 + \varepsilon$, for $N = 3, 2, 1$ respectively. □

FINAL REMARK. It can be shown that exact controllability, as well as uniform stabilization, in the explicitly identified, sharp regularity spaces noted above are not possible for all of the preceding problems with (finitely many) interior point controls in $L_2(0, T)$, where $\dim \Omega = N \geq 2$. □

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