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Convex approximations of functionals with curvature

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Analisi numerica. — *Convex approximations of functionals with curvature.* Nota di Giovanni Bellettini, Maurizio Paolini e Claudio Verdi, presentata (*) dal Socio E. Magenes.

Abstract. — We address the numerical minimization of the functional $\mathcal{F}(v) = \int_\Omega |Dv| + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx$, for $v \in BV(\Omega; \{-1, 1\})$. We note that $\mathcal{F}$ can be equivalently minimized on the larger, convex, set $BV(\Omega, [-1, 1])$ and that, on that space, $\mathcal{F}$ may be regularized with a sequence $\left\{ \mathcal{F}_\varepsilon(v) = \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx \right\}_\varepsilon$ of regular functionals. Then both $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ can be discretized by continuous linear finite elements. The convexity of the functionals in $BV(\Omega; [-1, 1])$ is useful for the numerical minimization of $\mathcal{F}$. We prove the Γ-$L^1(\Omega)$-convergence of the discrete functionals to $\mathcal{F}$ and present a few numerical examples.

Key words: Calculus of variations; Surfaces with prescribed mean curvature; Finite elements; Convergence of discrete approximations.

Riassunto. — *Approssimazioni convesse di funzionali con curvatura.* Si studia la minimizzazione numerica del funzionale $\mathcal{F}(v) = \int_\Omega |Dv| + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx$, per $v \in BV(\Omega; \{-1, 1\})$, i cui minimi relativi sono funzioni caratteristiche di insiemi $A \subseteq \Omega \subset \boldsymbol{R}^n$ con frontiera di curvatura media $\varkappa$ ed angolo di contatto arccos (μ) all'intersezione con $\partial\Omega$. Si osserva che $\mathcal{F}$ può essere equivalentemente minimizzato sullo spazio convesso $BV(\Omega, [-1, 1])$, dove viene regolarizzato con una successione di funzionali regolari $\left\{ \mathcal{F}_\varepsilon(v) = \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx \right\}_\varepsilon$. Sia $\mathcal{F}$ che $\mathcal{F}_\varepsilon$ vengono quindi discretizzati con elementi finiti continui lineari. La convessità dei funzionali in $BV(\Omega, [-1, 1])$ gioca un ruolo importante nella minimizzazione numerica di $\mathcal{F}$. Si dimostra la Γ-convergenza dei funzionali discreti a $\mathcal{F}$ in $L^1(\Omega)$ e si presentano, infine, alcuni esempi numerici.

0. Introduction

Several geometrical type problems in the calculus of variations arising, for instance, in phase transition theories [5] and computer vision theory [16], fall within the general setting proposed by E. De Giorgi [7, 1]. These problems usually involve unknown interfaces, obtained as minima of functionals defined on the space $BV(\Omega; \{-1, 1\})$ of the characteristic functions of sets of finite perimeter in Ω. The numerical minimization of such functionals seems quite difficult, because of the lack of convexity and regularity (see, *e.g.*, [2, 3]).

In this paper we address the numerical minimization of a model functional [10, 12, 13] via convex approximations. More precisely, given an open bounded

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set $\Omega \subset \boldsymbol{R}^n$ $(n \geq 2)$, a function $\varkappa \in L^\infty(\Omega)$, and $\mu \in L^\infty(\partial\Omega; [-1, 1])$, we consider the minimum problem:

$$\min_{v \in BV(\Omega; \{-1,1\})} \mathcal{F}(v), \qquad \text{where} \qquad \mathcal{F}(v) := \int_\Omega |Dv| + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx .$$

It is well known [10] that any minimum of $\mathcal{F}$ is the characteristic function of a set $A \subseteq \Omega$ whose boundary has prescribed mean curvature $\varkappa$ and contact angle $\arccos(\mu)$ at $\partial\Omega$.

Noting that $\mathcal{F}$ can be equivalently minimized on the larger, convex, set $BV(\Omega; [-1, 1])$, the (nonstrict) convexity of $\mathcal{F}$ can be exploited for the numerical minimization via linear finite element discretizations. Since the numerical algorithms perform better for strictly convex regular functionals, $\mathcal{F}$ is preliminarly regularized by

$$\mathcal{F}_\varepsilon(v) = \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx, \qquad \forall v \in BV(\Omega; [-1, 1]),$$

which, in turn, is discretized by continuous linear finite elements.

The main result of this paper is the Γ-$L^1(\Omega)$-convergence of the discrete functionals $\{\mathcal{F}_h\}_h$ and $\{\mathcal{F}_{\varepsilon,h}\}_{\varepsilon,h}$ to $\mathcal{F}$. More specifically, we prove the following diagram of convergence:

$$\text{(0.1)} \qquad \begin{array}{ccc} \mathcal{F}_\varepsilon & \xrightarrow[\varepsilon \to 0]{\text{uniformly in } BV(\Omega; [-1,1])} & \mathcal{F} \\ \Gamma\text{-}L^1(\Omega) \Big\uparrow h \to 0 & & \Gamma\text{-}L^1(\Omega) \Big\uparrow h \to 0, \\ & \text{uniformly in } V_h \subset BV(\Omega; [-1,1]) & \\ & \text{and with respect to } h & \\ \mathcal{F}_{\varepsilon,h} & \xrightarrow[\varepsilon \to 0]{} & \mathcal{F}_h \end{array}$$

where V_h is the finite element space. Hence, letting ε and h go to 0 independently, it follows that Γ-$\lim \mathcal{F}_{\varepsilon,h} = \mathcal{F}$ in $L^1(\Omega)$. In view of basic properties of the Γ-convergence [8], any family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ of discrete absolute minima admits a subsequence converging to a minimum point u of $\mathcal{F}$ in $L^1(\Omega)$ and $\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h})$ converges to $\mathcal{F}(u)$. We stress that no relation between ε and h is required for the limit procedure, whereas the nonconvex approximation via double well potential, first proposed in [15], Γ-converges if $h = o(\varepsilon)$ [2].

The outline of the paper is as follows. In § 1 we state precisely the functionals and recall some basic properties. For the sake of completeness, in § 2 we show the semicontinuity of both $\mathcal{F}$ and $\mathcal{F}_\varepsilon$. The demonstration of the convergence results is given in § 3. The paper concludes in § 4 with some numerical examples.

1. The setting

Let $\Omega \subset \boldsymbol{R}^n$ $(n \geqslant 2)$ be an open bounded set with Lipschitz-continuous boundary and denote by $|\cdot|$ the n-dimensional Lebesgue measure and by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\boldsymbol{R}^n$ [9]. Let $BV(\Omega)$ be the space of the bounded variation functions in Ω and set $\widetilde{\mathcal{K}} := BV(\Omega; \{-1, 1\})$, $\mathcal{K} := BV(\Omega; [-1, 1])$. Let us denote by $v \in L^1(\partial\Omega)$ the trace of $v \in BV(\Omega)$ on $\partial\Omega$. Let $\int_\Omega |Dv|$ denote the *total variation* in Ω and $\int_\Omega \sqrt{1 + |Dv|^2}$ the *area* of any function $v \in BV(\Omega)$ [12, Definitions 1.1 and 14.1]. For any set $E \subseteq \Omega$, let $\chi_E(x) := 1$ if $x \in E$, $\chi_E(x) := -1$ if $x \in \Omega \setminus E$, be its characteristic function. It is well known that $v \in \widetilde{\mathcal{K}}$ if and only if v is the characteristic function χ_E of a set $E \subseteq \Omega$ of finite perimeter in Ω, and $P(E, \Omega) := \frac{1}{2}\int_\Omega |D\chi_E|$ is the perimeter of E in Ω [12]. Finally, for any $v \in BV(\Omega)$, set $\{v > t\} := \{x \in \Omega : v(x) > t\}$ and note that $\chi_{\{v>t\}} \in \widetilde{\mathcal{K}}$, for a.e. $t \in \boldsymbol{R}$ [12].

1.1. *The original functional.* Given $\mu \in L^\infty(\partial\Omega; [-1, 1])$ and $\varkappa \in L^\infty(\Omega)$, let us define

$$\mathcal{F}(v) := \int_\Omega |Dv| + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx, \qquad \forall v \in BV(\Omega). \tag{1.1}$$

It is well known that $\mathcal{F}$ admits at least a minimum point $u \in \mathcal{K}$, because $\mathcal{F}$ is bounded from below and $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$ [14, Proposition 1.2]. We stress that,

> *if $u \in \mathcal{K}$ is a minimum point of $\mathcal{F}$ in $\mathcal{K}$, then, for a.e. $t \in [-1, 1]$, the characteristic function $\chi_{\{u>t\}} \in \widetilde{\mathcal{K}}$ is a minimum point of $\mathcal{F}$ in $\widetilde{\mathcal{K}}$.*

In fact, using the coarea formula [12, Theorem 1.23] and the Cavalieri formula, we get

$$\mathcal{F}(v) = \frac{1}{2}\int_{-1}^{1} \mathcal{F}(\chi_{\{v>t\}}) \, dt, \qquad \text{that is} \qquad \int_{-1}^{1} (\mathcal{F}(\chi_{\{v>t\}}) - \mathcal{F}(v)) \, dt = 0,$$

for all $v \in \mathcal{K}$. Then, the minimality of u in $\mathcal{K}$ entails $\mathcal{F}(\chi_{\{u>t\}}) - \mathcal{F}(u) \geqslant 0$, whence $\mathcal{F}(u) = \mathcal{F}(\chi_{\{u>t\}})$, for a.e. $t \in [-1, 1]$. Hence, $\min_{v \in \widetilde{\mathcal{K}}} \mathcal{F}(v) = \min_{v \in \mathcal{K}} \mathcal{F}(v)$ and the minimization of $\mathcal{F}$ on $\widetilde{\mathcal{K}}$ is equivalent to minimize $\mathcal{F}$ on the convex closed space $\mathcal{K}$, which reads as a (nonstrictly) convex problem. Note that $\mathcal{F}$ may exhibit relative minima in $\widetilde{\mathcal{K}}$; in view of the convexity of $\mathcal{K}$, they are no longer relative minima of $\mathcal{F}$ in $\mathcal{K}$. Moreover, $\mathcal{F}$ has a unique minimum point in $\widetilde{\mathcal{K}}$ if and only if $\mathcal{F}$ has a unique minimum point in $\mathcal{K}$, and they coincide.

1.2. *The regularized functionals.* For any $\varepsilon > 0$, the regularized functional reads:

$$\mathcal{F}_\varepsilon(v) := \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \mu v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx, \qquad \forall v \in \mathcal{K}. \tag{1.2}$$

Since $\mathcal{F}_\varepsilon$ is bounded from below and $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$ [13, §3.8, Theorem 11], $\mathcal{F}_\varepsilon$ has a minimum point $u_\varepsilon \in \mathcal{K}$. Moreover, since $\mathcal{F}_\varepsilon$ is strictly convex in $\mathcal{K} \cap W^{1,1}_{\mathrm{loc}}(\Omega)/\boldsymbol{R}$, its minimum is unique up to a possible additive constant. More precisely, u_ε is unique if and only if either $\int_\Omega \varkappa \neq \int_{\partial\Omega} \mu$ or $\sup_\Omega u_\varepsilon = 1$ and $\inf_\Omega u_\varepsilon = -1$. If u_ε is regular, then it satisfies the following variational inequality with Neumann boundary conditions:

$$\int_\Omega \frac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \cdot \nabla(v - u_\varepsilon) \, dx + \int_{\partial\Omega} \mu(v - u_\varepsilon) \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa(v - u_\varepsilon) \, dx \geqslant 0,$$

for all $v \in H^1(\Omega, [-1, 1])$ (see, *e.g.*, [12, 13]). Note that, with no further assumptions (*e.g.*, Ω convex), the minimum u_ε of $\mathcal{F}_\varepsilon$ is, in general, just a bounded variation function (see [12, Example 12.15]; if $\Omega \subset \boldsymbol{R}^2$ is an annulus with boundaries Γ_1 and Γ_2, $\mu = 1$ on Γ_1, $\mu = -1$ on Γ_2, and $\varkappa = 0$, then

$$\int_{\partial\Omega} \mu v \, d\mathcal{H}^1 = \int_{\partial\Omega} |v + \mu| \, d\mathcal{H}^1 - \mathcal{H}^1(\partial\Omega),$$

and our minimum problem corresponds to the Dirichlet problem for the area functional, with boundary datum $-\mu/\varepsilon$ suggested by Giusti as an example of nonexistence of classical solutions).

1.3. *The discrete functionals.* Let $\{\mathcal{S}_h\}_{h>0}$ denote a *regular* family of partitions of Ω into simplices [6, p. 132]. Let $h_S \leqslant h$ denote the diameter of any $S \in \mathcal{S}_h$. Let $V_h \subset H^1(\Omega, [-1, 1]) \subset \mathcal{K}$ be the piecewise linear finite element space over $\mathcal{S}_h$ with values in $[-1, 1]$ and Π_h be the usual Lagrange interpolation operator. For the sake of simplicity, we assume that $\bar{\Omega} = \bigcup_{S \in \mathcal{S}_h} S$. We approximate μ and $\varkappa$ by continuous piecewise linear functions μ_h and $\varkappa_h$, respectively, so that [6]

$$\|\mu_h\|_{L^\infty(\partial\Omega)} \leqslant 1, \qquad \|\nabla \mu_h\|_{L^1(\partial\Omega)} = o(h^{-1}), \qquad \mu_h \xrightarrow[h \to 0]{} \mu \quad \text{in } L^1(\partial\Omega), \tag{1.3}$$

$$\|\varkappa_h\|_{L^\infty(\Omega)} \leqslant \|\varkappa\|_{L^\infty(\Omega)}, \qquad \|\nabla \varkappa_h\|_{L^1(\Omega)} = o(h^{-1}), \qquad \varkappa_h \xrightarrow[h \to 0]{} \varkappa \quad \text{in } L^1(\Omega). \tag{1.4}$$

We define the discrete functionals as follows:

$$\mathcal{F}_h(v) := \int_\Omega |\nabla v| dx + \int_{\partial\Omega} \Pi_h(\mu_h v) \, d\mathcal{H}^{n-1} - \int_\Omega \Pi_h(\varkappa_h v) \, dx, \qquad \forall v \in V_h, \tag{1.5}$$

$$\mathcal{F}_{\varepsilon,h}(v) := \int_\Omega \sqrt{\varepsilon^2 + |\nabla v|^2} \, dx + \int_{\partial\Omega} \Pi_h(\mu_h v) \, d\mathcal{H}^{n-1} - \int_\Omega \Pi_h(\varkappa_h v) \, dx, \qquad \forall v \in V_h. \tag{1.6}$$

Since $\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$ are continuous over a compact subset of a finite dimensional space, they admit a minimum point. Since $\mathcal{F}_{\varepsilon,h}$ is strictly convex in $V_h/\mathbf{R}$, its minimum is unique up to a possible additive constant.

The quadrature formulae in (1.5) and (1.6) allow the direct implementation on a computer of the minimization of $\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$. Implementation details will appear in [4].

2. Semicontinuity

Just for the sake of completeness, we show here the lower semicontinuity of both functionals $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ in $\mathcal{K}$ with respect to the $L^1(\Omega)$-topology (see also [13, 14]).

We give a unified proof for both functionals $\mathcal{F}$ and $\mathcal{F}_\varepsilon$, considering $\mathcal{F} = \mathcal{F}_\varepsilon$ with $\varepsilon = 0$. Hence, let $\varepsilon \geqslant 0$ be fixed. First, we approximate $\mu \in L^\infty(\partial\Omega; [-1,1])$ by a sequence of piecewise constant functions $\{\mu^k\}_{k \in N}$, so that $\mu^k \to \mu$ in $L^1(\partial\Omega)$, as $k \to \infty$. Denoting by $\mathcal{F}_\varepsilon^k$ the functional involving μ^k, we have $|\mathcal{F}_\varepsilon^k(v) - \mathcal{F}_\varepsilon(v)| \leqslant \|\mu^k - \mu\|_{L^1(\partial\Omega)}$, for all $v \in \mathcal{K}$, namely, $\mathcal{F}_\varepsilon^k \to \mathcal{F}_\varepsilon$ uniformly in $\mathcal{K}$, as $k \to \infty$. The assertion is thus reduced to prove that, for any k, $\mathcal{F}_\varepsilon^k$ is semicontinuous in $\mathcal{K}$. Since no confusion is possible, we omit the superscript k. Then, let μ be a piecewise constant function with values $-1 = \mu_0 < \mu_1 < \ldots < \mu_N = 1$ and set $\nu_i := (\mu_i - \mu_{i-1})/2$ and $G_i := \{\mu \geqslant \mu_i\} \subseteq \partial\Omega$, for all $1 \leqslant i \leqslant N$ ($G_1 = \partial\Omega$ and $G_N = \emptyset$ are allowed). Since

$$\sum_{i=1}^N \nu_i = 1 \qquad \text{and} \qquad \mu(x) = \sum_{i=1}^N \nu_i \chi_{G_i}(x), \quad \text{for all } x \in \partial\Omega,$$

$\mathcal{F}_\varepsilon$ can be decomposed as a convex combination of functionals $\mathcal{F}_\varepsilon^i$ defined by:

$$\mathcal{F}_\varepsilon(v) = \sum_{i=1}^N \nu_i \left[\int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} \chi_{G_i} v \, d\mathcal{H}^{n-1} - \int_\Omega \varkappa v \, dx \right] =: \sum_{i=1}^N \nu_i \mathcal{F}_\varepsilon^i(v).$$

Hence, we finally have to show that, for any $1 \leqslant i \leqslant N$, $\mathcal{F}_\varepsilon^i$ is semicontinuous. Given a ball B containing $\overline{\Omega}$, let $\tilde{\chi}_i \in W^{1,1}(B \setminus \overline{\Omega}; [-1,1])$ be a function with trace $-\chi_{G_i}$ on $\partial\Omega$ [11, Theorem 1.II; 12, Theorem 2.16]. If, for any $v \in \mathcal{K}$ we define $v_i \in BV(B; [-1,1])$ by $v_i(x) := v(x)$ if $x \in \Omega$, $v_i(x) := \tilde{\chi}_i(x)$ if $x \in B \setminus \Omega$, and set

$$C_i := \int_{B \setminus \overline{\Omega}} \sqrt{\varepsilon^2 + |\nabla \tilde{\chi}_i|^2} \, dx,$$

we have [12, §14.4]

$$\int_B \sqrt{\varepsilon^2 + |Dv_i|^2} = \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial\Omega} |v + \chi_{G_i}| \, d\mathcal{H}^{n-1} + C_i.$$

Hence, noting that

$$\int_{\partial\Omega} \chi_{G_i} v \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} |v + \chi_{G_i}| \, d\mathcal{H}^{n-1} - \mathcal{H}^{n-1}(\partial\Omega),$$

we get

$$\mathcal{F}_\varepsilon^i(v) = \int_B \sqrt{\varepsilon^2 + |Dv_i|^2} - \int_\Omega \varkappa v \, dx - C_i - \mathcal{H}^{n-1}(\partial\Omega),$$

and the semicontinuity of $\mathcal{F}_\varepsilon^i$ follows from the $L^1(\Omega)$-lower semicontinuity of the total variation and the area in $BV(\Omega)$ [12, Theorems 1.9 and 14.2].

REMARK 2.1. For any $\varepsilon \geq 0$, the functional $\mathcal{F}_\varepsilon$ is not $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$ if $\mu \notin L^\infty(\partial\Omega; [-1, 1])$. In fact, let $\mu(x) > 1$ for a.e. $x \in \partial\Omega \cap B$, for some ball B. Set $B_\Omega := \Omega \cap B \neq \emptyset$ and $B_{\partial\Omega} := \partial\Omega \cap B$. Let $\{B_k \subset B_\Omega\}_{k \in N}$ be a sequence of sets of finite perimeter in Ω, so that $\partial B_k \cap \partial\Omega = B_{\partial\Omega}$, $\lim_{k\to\infty} |B_k| = 0$, and $\lim_{k\to\infty} P(B_k, \Omega) = \mathcal{H}^{n-1}(B_{\partial\Omega})$. Let $\{v_k := -\chi_{B_k}(\chi_{B_\Omega} + 1)/2\}_k$ be a sequence converging to $v := (\chi_{B_\Omega} + 1)/2$ in $L^1(\Omega)$, as $k \to \infty$. Then, noting that

$$\mathcal{F}_\varepsilon(v) = P(B_\Omega, \Omega) + \varepsilon|\Omega| + \int_{B_{\partial\Omega}} \mu - \int_\Omega \varkappa v$$

and

$$\mathcal{F}_\varepsilon(v_k) = P(B_\Omega, \Omega) + \varepsilon|\Omega| + 2P(B_k, \Omega) - \int_{B_{\partial\Omega}} \mu - \int_\Omega \varkappa v_k,$$

we have $\mathcal{F}_\varepsilon(v) > \liminf_{k\to\infty} \mathcal{F}_\varepsilon(v_k)$.

3. CONVERGENCE

We shall prove the diagram of convergence (0.1). First, we note that $\lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon = \mathcal{F}$ *uniformly in* $\mathcal{K}$ and $\lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon,h} = \mathcal{F}_h$ *uniformly in* $\mathbf{V}_h$ *and with respect to* h. In fact, since

$$0 \leq \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} - \int_\Omega |Dv| = \varepsilon\left(\int_\Omega \sqrt{1 + \left|D\left(\frac{v}{\varepsilon}\right)\right|^2} - \int_\Omega \left|D\left(\frac{v}{\varepsilon}\right)\right| \right) \leq \varepsilon|\Omega|,$$

for all $v \in BV(\Omega)$, we have

$$|\mathcal{F}_\varepsilon(v) - \mathcal{F}(v)| \leq \varepsilon|\Omega|, \quad \forall v \in \mathcal{K}, \qquad \text{and} \qquad |\mathcal{F}_{\varepsilon,h}(v) - \mathcal{F}_h(v)| \leq \varepsilon|\Omega|, \quad \forall v \in \mathbf{V}_h. \tag{3.1}$$

THEOREM 3.1. *Γ-$\lim_{h\to 0} \mathcal{F}_h = \mathcal{F}$ and Γ-$\lim_{h\to 0} \mathcal{F}_{\varepsilon,h} = \mathcal{F}_\varepsilon$, in $L^1(\Omega)$.*

PROOF. The functionals $\mathcal{F}$ and $\mathcal{F}_\varepsilon$ ($\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$, respectively) are set to $+\infty$ in $L^1(\Omega) \setminus \mathcal{K}$ ($L^1(\Omega) \setminus V_h$, respectively). We give a unified proof for both cases $\varepsilon > 0$ and $\varepsilon = 0$, considering $\mathcal{F}_h = \mathcal{F}_{\varepsilon,h}$ and $\mathcal{F} = \mathcal{F}_\varepsilon$ with $\varepsilon = 0$. Hence, let $\varepsilon \geq 0$ be fixed. We prove [8] that:

(*i*). For any $v \in L^1(\Omega)$ and any sequence $\{v_h \in L^1(\Omega)\}_h$ converging to v in $L^1(\Omega)$, as $h \to 0$, we have $\mathcal{F}_\varepsilon(v) \leq \liminf_{h\to 0} \mathcal{F}_{\varepsilon,h}(v_h)$.

(*ii*). For any $v \in L^1(\Omega)$ there exists a sequence $\{v_h \in L^1(\Omega)\}_h$ converging to v in $L^1(\Omega)$, as $h \to 0$, such that $\mathcal{F}_\varepsilon(v) = \lim_{h\to 0} \mathcal{F}_{\varepsilon,h}(v_h)$.

Preliminarly, we decompose $\mathcal{F}_{\varepsilon,h}(v_h)$, for all $v_h \in V_h$, as follows:

(3.2) $$\mathcal{F}_{\varepsilon,h}(v_h) = \mathcal{F}_\varepsilon(v_h) + \int_{\partial\Omega} [\Pi_h(\mu_h v_h) - \mu\, v_h]\, d\mathcal{H}^{n-1} - \int_\Omega [\Pi_h(\varkappa_h v_h) - \varkappa\, v_h]\, dx,$$

which reads as $\mathcal{F}_{\varepsilon,h}(v_h) =: \mathcal{F}_\varepsilon(v_h) + I + II$, and split I and II as follows:

$$I = \int_{\partial\Omega} [\Pi_h(\mu_h v_h) - \mu_h\, v_h]\, d\mathcal{H}^{n-1} + \int_{\partial\Omega} (\mu_h - \mu)\, v_h\, d\mathcal{H}^{n-1} =: I_1 + I_2,$$

$$II = \int_\Omega [\Pi_h(\varkappa_h v_h) - \varkappa_h v_h]\, dx + \int_\Omega (\varkappa_h - \varkappa)\, v_h\, dx =: II_1 + II_2.$$

Since $|v_h| \leq 1$ in $\overline{\Omega}$, we have $|I_2| \leq \|\mu_h - \mu\|_{L^1(\partial\Omega)}$ and $|II_2| \leq \|\varkappa_h - \varkappa\|_{L^1(\Omega)}$. In view of basic properties of the interpolation operator Π_h, and using the local inverse inequality $\|\nabla v_h\|_{L^\infty(T)} \leq C h_S^{-1} \|v_h\|_{L^\infty(T)}$, where either $T = \partial S$ or $T = S \in \mathcal{S}_h$ and $v_h \in V_h$ [6, p. 140], we get

$$|I_1| \leq C \sum_{S \in \mathcal{S}_h} h_S^2 \|D^2(\mu_h v_h)\|_{L^1(\partial S \cap \partial\Omega)} \leq C \sum_{S \in \mathcal{S}_h} h_S^2 \|\nabla\mu_h \cdot \nabla v_h\|_{L^1(\partial S \cap \partial\Omega)} \leq Ch \|\nabla\mu_h\|_{L^1(\partial\Omega)}$$

and, similarly, $|II_1| \leq Ch \|\nabla\varkappa_h\|_{L^1(\Omega)}$. Hence, using (1.3) and (1.4), for any sequence $\{v_h \in V_h\}_h$, we obtain

(3.3) $$\lim_{h\to 0} [|I| + |II|] = 0.$$

Proof of step (*i*). Let $v \in L^1(\Omega)$ and $\{v_h \in L^1(\Omega)\}_h$ be any sequence so that $\lim_{h\to 0} v_h = v$ in $L^1(\Omega)$. We can assume that $v_h \in V_h$, for any h. Then, from the lower semicontinuity of $\mathcal{F}_\varepsilon$, (3.2), and(3.3), we conclude

$$\mathcal{F}_\varepsilon(v) \leq \liminf_{h\to 0} \mathcal{F}_\varepsilon(v_h) = \liminf_{h\to 0} \mathcal{F}_{\varepsilon,h}(v_h).$$

Proof of step (*ii*). We can assume that $v \in \mathcal{K}$. Given a ball B containing $\overline{\Omega}$, let $\tilde{v} \in W^{1,1}(B \setminus \overline{\Omega}; [-1,1])$ be a function with trace v on $\partial\Omega$ [11] and denote again by $v \in BV(B; [-1,1])$ the function $v(x) := v(x)$ if $x \in \Omega$, $v(x) := \tilde{v}(x)$ if $x \in B \setminus \Omega$. Let $\eta_h = o(h^{-1})$ and $\{\delta_h\}_h$ be a family of mollifiers defined by $\delta_h(x) := \eta_h^n\, \delta(\eta_h x)$. Set $\hat{v}_h(x) := (v * \delta_h)(x)$, for all $x \in B$, where v is extended to 0 outside B. It is well known [12, Proposition 1.15] that

(3.4) $$\lim_{h\to 0} \|\hat{v}_h - v\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{h\to 0} \int_\Omega |\nabla\hat{v}_h|\, dx = \int_\Omega |Dv|.$$

We claim that the sequence $\{v_h\}_h$ for Step (*ii*) can be defined by $v_h := \Pi_h \hat{v}_h \in V_h$. In fact, noting that $\|D^2 \hat{v}_h\|_{L^1(\Omega)} \leq \eta_h$, using well known properties of Π_h, we have $\|v_h - \hat{v}_h\|_{W^{1,1}(\Omega)} \leq C \|D^2 \hat{v}_h\|_{L^1(\Omega)} [h^2 + h] = o(1)$. Hence, since

$$\left| \int_\Omega |\nabla v_h|\, dx - \int_\Omega |\nabla\hat{v}_h|\, dx \right| \leq \int_\Omega |\nabla(v_h - \hat{v}_h)|\, dx,$$

in view of (3.4) we obtain

$$\lim_{h\to 0} \|v_h - v\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \lim_{h\to 0} \int_\Omega |\nabla v_h|\, dx = \int_\Omega |Dv|.$$

This entails [12, Theorem 2.11]

$$\lim_{h\to 0} \int_{\partial\Omega} |v_h - v| \, d\mathcal{H}^{n-1} = 0$$

and, using the inequality

$$\left| \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} - \int_\Omega \sqrt{\varepsilon^2 + |\nabla v_h|^2}\, dx \right| \leq \left| \int_\Omega |Dv| - \int_\Omega |\nabla v_h| \, dx \right|,$$

(3.2), and (3.3), gives

$$\mathcal{F}_\varepsilon(v) = \lim_{h\to 0} \mathcal{F}_\varepsilon(v_h) = \lim_{h\to 0} \mathcal{F}_{\varepsilon,h}(v_h).$$

A straightforward consequence of (3.1) and Theorem 3.1 is the following Γ-convergence result for $\mathcal{F}_{\varepsilon,h}$, as ε and h go to 0 independently.

COROLLARY 3.1. *Γ-$\lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h} = \mathcal{F}$, in $L^1(\Omega)$.*

PROOF. (*i*) Let $v \in L^1(\Omega)$ and $\{v_{\varepsilon,h} \in L^1(\Omega)\}_{\varepsilon,h}$ be any sequence converging to v in $L^1(\Omega)$, as $(\varepsilon, h) \to (0,0)$. Using Theorem 3.1, Step (*i*), for $\mathcal{F}_h$, and (3.1), we get $\mathcal{F}(v) \leq \liminf \mathcal{F}_h(v_{\varepsilon,h}) = \liminf \mathcal{F}_{\varepsilon,h}(v_{\varepsilon,h})$, as $(\varepsilon,h) \to (0,0)$.

(*ii*). Let $v \in \mathcal{X}$ and $\{v_h \in V_h\}_h$ be the sequence constructed in Step (*ii*) of Theorem 3.1, for $\mathcal{F}_h$. Then, using (3.1), we get $\mathcal{F}(v) = \lim_{h\to 0} \mathcal{F}_h(v_h) = \lim_{(\varepsilon,h)\to(0,0)} \mathcal{F}_{\varepsilon,h}(v_h)$.

REMARK 3.1. Let $u_{\varepsilon,h}$ be a minimum of $\mathcal{F}_{\varepsilon,h}$. We have $\mathcal{F}_{\varepsilon,h}(u_{\varepsilon,h}) \leq \mathcal{F}_{\varepsilon,h}(0) = \varepsilon|\Omega|$, whence, using (1.3) and (1.4),

$$\int_\Omega |Du_{\varepsilon,h}| \leq \int_\Omega \sqrt{\varepsilon^2 + |Du_{\varepsilon,h}|^2} \leq \mathcal{H}^{n-1}(\partial\Omega) + |\Omega|(1 + \|\varkappa\|_{L^\infty(\Omega)}),$$

for all $0 \leq \varepsilon \leq 1$ and $h > 0$. Then, by the compactness theorem in $BV(\Omega)$ [12, Theorem 1.19], the family $\{u_{\varepsilon,h}\}_{\varepsilon,h}$ admits a subsequence converging to some $u \in \mathcal{X}$ in $L^1(\Omega)$. Corollary 3.1 entails that u is a minimum point of $\mathcal{F}$.

4. NUMERICAL EXPERIMENTS

Implementation details on the minimization algorithm can be found in [4]. Here we simply present a couple of numerical examples. The unique discrete absolute minimum $u_{\varepsilon,h}$ of $\mathcal{F}_{\varepsilon,h}$ is approximated by Newton-like iterations. A quasiuniform mesh is used.

Example 1. Let $\Omega := (-2,2)^2$, $\mu := 1$ (tangential contact at $\partial\Omega$), and $\varkappa := 1$. The functional $\mathcal{F}$ has one absolute minimum, $A := \{([|x_1| - 1]_+)^2 + ([|x_2| - 1]_+)^2 \leq 1\}$. Figure 4.1 shows both the exact minimum (dashed lines) and the computed one $A_{\varepsilon,h} := \{u_{\varepsilon,h} > 0\}$ (solid lines). Here $\varepsilon = 0.2$ and $h = 0.14$; the initial guess is the empty set, which is a relative minimum of $\mathcal{F}$ in $\tilde{\mathcal{X}}$! Note that, using the approximation via double well potential [3], the discrete minimizing set presents no contact with $\partial\Omega$, because the relaxed solution forms a transition

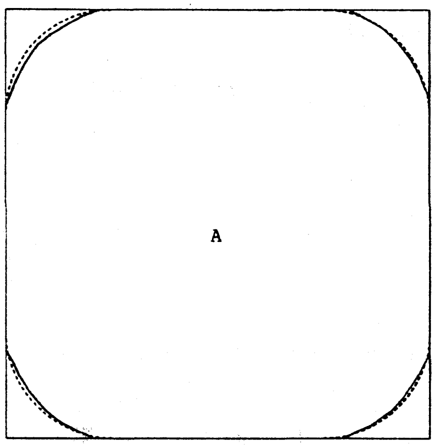

Fig. 4.1. – Ex. 1: Exact (dashed lines) and computed (solid lines) minimum.

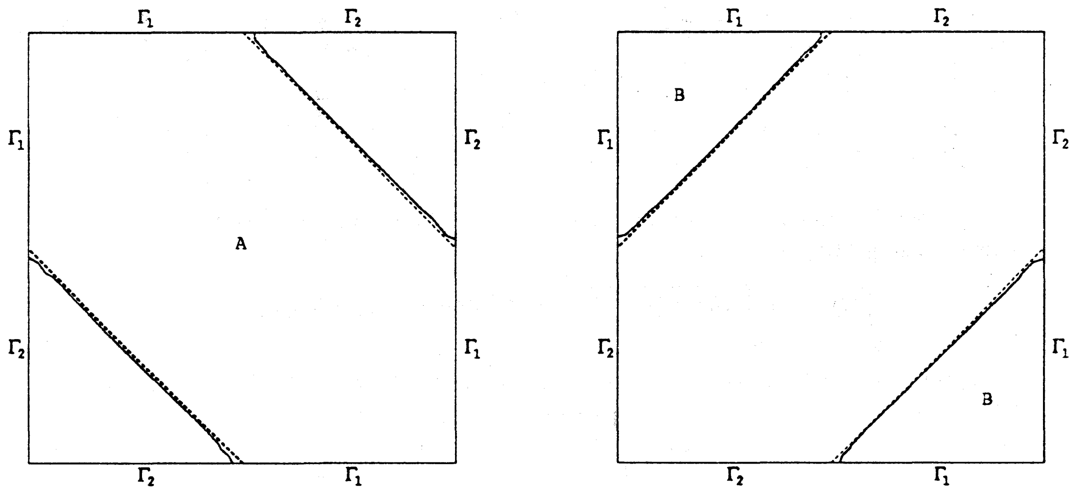

Fig. 4.2. – Ex. 2: Exact (dashed lines) and computed (solid lines) minima.

layer across the interface. This effect is absent in our convex approximations which, in turn, exhibits higher accuracy.

Example 2. Let $\Omega := (-2,2)^2$ and $\{\Gamma_1, \Gamma_2\}$ be the partition of $\partial\Omega$ defined by $\Gamma_1 := \partial\Omega \cap \{x_1 x_2 \leq 0\}$ and $\Gamma_2 := \partial\Omega \setminus \Gamma_1$. Let $\mu := -1$ on Γ_1, $\mu := 1$ on Γ_2, and $\varkappa := 0$. Set $\mu_h := \Pi_h(\mu)$. The functional $\mathcal{F}$ has two absolute minima in $\widetilde{\mathcal{X}}$, A and B, shown in fig. 4.2 (dashed lines). The computed minima are obtained from the unique discrete minimum $u_{\varepsilon,h}$ (the initial guess is the empty set) as $A_{\varepsilon,h} := \{u_{\varepsilon,h} > -0.5\}$ and $B_{\varepsilon,h} := \{u_{\varepsilon,h} > 0.5\}$ whereas, in [3], they where obtained iterating from different initial guesses. Here $\varepsilon = 0.2$ and $h = 0.14$.

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