Convex approximations of functionals with curvature

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ABSTRACT. — We address the numerical minimization of the functional $\mathcal{F}(v) = \int |Dv| + \int_{\Omega} \mu v d\mathcal{H}^{n-1} - \int_{\partial \Omega} xv dx$, for $v \in BV(\Omega; \{-1, 1\})$. We note that $\mathcal{F}$ can be equivalently minimized on the larger, convex, set $BV(\Omega, [-1, 1])$ and that, on that space, $\mathcal{F}$ may be regularized with a sequence $\mathcal{F}_\epsilon(v) = \int \sqrt{\epsilon^2 + |Dv|^2} + \int_{\Omega} \mu v d\mathcal{H}^{n-1} - \int_{\partial \Omega} xv dx$ of regular functionals. Then both $\mathcal{F}$ and $\mathcal{F}_\epsilon$ can be discretized by continuous linear finite elements. The convexity of the functionals in $BV(\Omega, [-1, 1])$ is useful for the numerical minimization of $\mathcal{F}$. We prove the $\Gamma$-convergence of the discrete functionals to $\mathcal{F}$ and present a few numerical examples.

KEY WORDS: Calculus of variations; Surfaces with prescribed mean curvature; Finite elements; Convergence of discrete approximations.

0. Introduction

Several geometrical type problems in the calculus of variations arising, for instance, in phase transition theories [5] and computer vision theory [16], fall within the general setting proposed by E. De Giorgi [7,1]. These problems usually involve unknown interfaces, obtained as minima of functionals defined on the space $BV(\Omega; \{-1, 1\})$ of the characteristic functions of sets of finite perimeter in $\Omega$. The numerical minimization of such functionals seems quite difficult, because of the lack of convexity and regularity (see, e.g., [2,3]).

In this paper we address the numerical minimization of a model functional [10,12,13] via convex approximations. More precisely, given an open bounded

set $\Omega \subset \mathbb{R}^n (n \geq 2)$, a function $x \in L^\infty (\Omega)$, and $\mu \in L^\infty (\partial \Omega; [-1,1])$, we consider the minimum problem:

$$\min_{v \in BV[\Omega;[-1,1])] \mathcal{F}(v), \quad \text{where} \quad \mathcal{F}(v) := \int_{\Omega} |Dv| + \int_{\partial \Omega} \mu v \, d\mathcal{H}^{n-1} - \int_{\Omega} xv \, dx.$$  

It is well known [10] that any minimum of $\mathcal{F}$ is the characteristic function of a set $A \subset \Omega$ whose boundary has prescribed mean curvature $x$ and contact angle $\arccos (\mu)$ at $\partial \Omega$.

Noting that $\mathcal{F}$ can be equivalently minimized on the larger, convex, set $BV[\Omega;[-1,1])$, the (nonstrict) convexity of $\mathcal{F}$ can be exploited for the numerical minimization via linear finite element discretizations. Since the numerical algorithms perform better for strictly convex regular functionals, $\mathcal{F}$ is preliminarily regularized by

$$\mathcal{F}_\varepsilon(v) = \int_{\Omega} \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial \Omega} \mu v \, d\mathcal{H}^{n-1} - \int_{\Omega} xv \, dx, \quad \forall v \in BV[\Omega;[-1,1]),$$

which, in turn, is discretized by continuous linear finite elements.

The main result of this paper is the $\Gamma$-convergence of the discrete functionals $\{\mathcal{F}_b\}_b$ and $\{\mathcal{F}_{\varepsilon,b}\}_{\varepsilon,b}$ to $\mathcal{F}$. More specifically, we prove the following diagram of convergence:

\[ \begin{array}{c}
\mathcal{F}_\varepsilon \quad \xrightarrow{\varepsilon \to 0} \quad \mathcal{F} \\
\uparrow \quad \uparrow \quad \uparrow \\
(0.1) \quad \Gamma\text{-}L^1(\Omega) \quad b \to 0 \quad \Gamma\text{-}L^1(\Omega) \quad b \to 0, \\
\mathcal{F}_{\varepsilon,b} \quad \xrightarrow{\varepsilon \to 0} \quad \mathcal{F}_b \\
\text{uniformly in } \mathcal{V}_b \subset BV[\Omega;[-1,1]) \quad \text{and with respect to } b 
\end{array} \]

where $\mathcal{V}_b$ is the finite element space. Hence, letting $\varepsilon$ and $b$ go to 0 independently, it follows that $\Gamma\text{-}\lim \mathcal{F}_{\varepsilon,b} = \mathcal{F}$ in $L^1(\Omega)$. In view of basic properties of the $\Gamma$-convergence [8], any family $\{u_{\varepsilon,b}\}_{\varepsilon,b}$ of discrete absolute minima admits a subsequence converging to a minimum point $u$ of $\mathcal{F}$ in $L^1(\Omega)$ and $\mathcal{F}_{\varepsilon,b}(u_{\varepsilon,b})$ converges to $\mathcal{F}(u)$. We stress that no relation between $\varepsilon$ and $b$ is required for the limit procedure, whereas the non-convex approximation via double well potential, first proposed in [15], $\Gamma$-converges if $b = o(\varepsilon)$ [2].

The outline of the paper is as follows. In § 1 we state precisely the functionals and recall some basic properties. For the sake of completeness, in § 2 we show the semicontinuity of both $\mathcal{F}$ and $\mathcal{F}_\varepsilon$. The demonstration of the convergence results is given in § 3. The paper concludes in § 4 with some numerical examples.
Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be an open bounded set with Lipschitz-continuous boundary and denote by $| \cdot |$ the $n$-dimensional Lebesgue measure and by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ [9]. Let $BV(\Omega)$ be the space of the bounded variation functions in $\Omega$ and set $\mathcal{K} := BV(\Omega; \{-1, 1\})$, $\mathcal{X} := BV(\Omega; [-1, 1])$. Let us denote by $v \in L^1(\partial \Omega)$ the trace of $v \in BV(\Omega)$ on $\partial \Omega$. Let $\int_{\partial \Omega} |Dv|$ denote the total variation in $\Omega$ and $\int_{\Omega} \sqrt{1 + |Dv|^2}$ the area of any function $v \in BV(\Omega)$ [12, Definitions 1.1 and 14.1]. For any set $E \subset \Omega$, let $\chi_E(x) := 1$ if $x \in E$, $\chi_E(x) := -1$ if $x \in \Omega \setminus E$, be its characteristic function. It is well known that $v \in \mathcal{K}$ if and only if $v$ is the characteristic function $\chi_E$ of a set $E \subset \Omega$ of finite perimeter in $\Omega$, and $P(E, \Omega) := \frac{1}{2} \int_{\Omega} |D\chi_E|$ is the perimeter of $E$ in $\Omega$ [12]. Finally, for any $v \in BV(\Omega)$, set $\{v > t\} := \{x \in \Omega : v(x) > t\}$ and note that $\chi_{\{v > t\}} \in \mathcal{K}$, for a.e. $t \in \mathbb{R}$ [12].

1.1. The original functional. Given $\mu \in L^\infty(\partial \Omega; [-1, 1])$ and $\kappa \in L^\infty(\Omega)$, let us define

$$\mathcal{F}(v) := \int_{\partial \Omega} |Dv| + \int_{\partial \Omega} \mu v \, d\mathcal{H}^{n-1} - \int_{\Omega} \kappa v \, dx, \quad \forall v \in BV(\Omega).$$

It is well known that $\mathcal{F}$ admits at least a minimum point $u \in \mathcal{K}$, because $\mathcal{F}$ is bounded from below and $L^1(\Omega)$-lower semicontinuous in $\mathcal{K}$ [14, Proposition 1.2]. We stress that,

*if $u \in \mathcal{K}$ is a minimum point of $\mathcal{F}$ in $\mathcal{K}$, then, for a.e. $t \in [-1, 1]$, the characteristic function $\chi_{\{u > t\}} \in \mathcal{K}$ is a minimum point of $\mathcal{F}$ in $\mathcal{K}$.*

In fact, using the coarea formula [12, Theorem 1.23] and the Cavalieri formula, we get

$$\mathcal{F}(v) = \frac{1}{2} \int_{-1}^{1} \mathcal{F}(\chi_{\{v > t\}}) \, dt, \quad \text{that is} \quad \int_{-1}^{1} (\mathcal{F}(\chi_{\{v > t\}}) - \mathcal{F}(v)) \, dt = 0,$$

for all $v \in \mathcal{K}$. Then, the minimality of $u$ in $\mathcal{K}$ entails $\mathcal{F}(\chi_{\{u > t\}}) - \mathcal{F}(u) \geq 0$, whence $\mathcal{F}(u) = \mathcal{F}(\chi_{\{u > t\}})$, for a.e. $t \in [-1, 1]$. Hence, $\min_{v \in \mathcal{K}} \mathcal{F}(v) = \min_{v \in \mathcal{K}} \mathcal{F}(v)$ and the minimization of $\mathcal{F}$ on $\mathcal{K}$ is equivalent to minimize $\mathcal{F}$ on the convex closed space $\mathcal{K}$, which reads as a (nonstrictly) convex problem. Note that $\mathcal{F}$ may exhibit relative minima in $\mathcal{K}$; in view of the convexity of $\mathcal{K}$, they are no longer relative minima of $\mathcal{F}$ in $\mathcal{K}$. Moreover, $\mathcal{F}$ has a unique minimum point in $\mathcal{K}$ if and only if $\mathcal{F}$ has a unique minimum point in $\mathcal{K}$, and they coincide.
1.2. The regularized functionals. For any $s > 0$, the regularized functional reads:

$$\mathcal{F}_s(v) := \int_{\Omega} \sqrt{\varepsilon^2 + |Dv|^2} + \mu v \, d\mathcal{C}^{n-1} - \int_{\partial\Omega} \varphi(v) \, d\mathcal{H}^1, \quad \forall v \in \mathcal{H}.$$  

Since $\mathcal{F}_s$ is bounded from below and $L^1(\Omega)$-lower semicontinuous in $\mathcal{H}$ [13, §3.8, Theorem 11], $\mathcal{F}_s$ has a minimum point $u_\varepsilon \in \mathcal{H}$. Moreover, since $\mathcal{F}_s$ is strictly convex in $\mathcal{H} \cap W^{1,1}(\Omega)/\mathbb{R}$, its minimum is unique up to a possible additive constant. More precisely, $u_\varepsilon$ is unique if and only if either $\int_{\partial\Omega} \varphi \neq \mu$ or $\sup_{\partial\Omega} u_\varepsilon = 1$ and $\inf_{\partial\Omega} u_\varepsilon = -1$. If $u_\varepsilon$ is regular, then it satisfies the following variational inequality with Neumann boundary conditions:

$$\int_{\Omega} \frac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \cdot (v - u_\varepsilon) \, dx + \mu (v - u_\varepsilon) \, d\mathcal{C}^{n-1} - \int_{\partial\Omega} \varphi(v) \, d\mathcal{H}^1 \geq 0,$n$$

for all $v \in H^1(\Omega, [-1, 1])$ (see, e.g., [12, 13]). Note that, with no further assumptions (e.g., $\Omega$ convex), the minimum $u_\varepsilon$ of $\mathcal{F}_s$ is, in general, just a bounded variation function (see [12, Example 12.15]; if $\Omega \subset \mathbb{R}^2$ is an annulus with boundaries $\Gamma_1$ and $\Gamma_2$, $\mu = 1$ on $\Gamma_1$, $\mu = -1$ on $\Gamma_2$, and $x = 0$, then

$$\int_{\partial\Omega} \mu v \, d\mathcal{H}^1 = \int_{\partial\Omega} |v + \mu| \, d\mathcal{H}^1 - \mathcal{C}^1(\partial\Omega),$$

and our minimum problem corresponds to the Dirichlet problem for the area functional, with boundary datum $-\mu/\varepsilon$ suggested by Giusti as an example of nonexistence of classical solutions).

1.3. The discrete functionals. Let $\{S_h\}_{h > 0}$ denote a regular family of partitions of $\Omega$ into simplices [6, p. 132]. Let $b_h \leq h$ denote the diameter of any $S \in S_h$. Let $V_h \subset H^1(\Omega, [-1, 1]) \subset \mathcal{H}$ be the piecewise linear finite element space over $S_h$ with values in $[-1, 1]$ and $\Pi_h$ be the usual Lagrange interpolation operator. For the sake of simplicity, we assume that $\Omega = \bigcup_{S \in S_h} S$. We approximate $\mu$ and $x$ by continuous piecewise linear functions $\mu_h$ and $x_h$, respectively, so that [6]

$$\|\mu_h\|_{L^\infty(\partial\Omega)} \leq 1, \quad \|\nabla \mu_h\|_{L^1(\partial\Omega)} = o(h^{-1}), \quad \mu_h \xrightarrow{H^{-1}} \mu \quad \text{in} \ L^1(\partial\Omega),$$

$$\|x_h\|_{L^\infty(\Omega)} \leq \|x\|_{L^\infty(\Omega)}, \quad \|\nabla x_h\|_{L^1(\Omega)} = o(h^{-1}), \quad x_h \xrightarrow{H^{-1}} x \quad \text{in} \ L^1(\Omega).$$

We define the discrete functionals as follows:

$$\mathcal{F}_h(v) := \int_{\Omega} |\nabla v| \, dx + \int_{\partial\Omega} \Pi_h(\mu_h v) \, d\mathcal{C}^{n-1} - \int_{\partial\Omega} \Pi_h(x_h v) \, dx, \quad \forall v \in V_h,$$

$$\mathcal{F}_{\varepsilon,h}(v) := \int_{\Omega} \sqrt{\varepsilon^2 + |\nabla v|^2} \, dx + \int_{\partial\Omega} \Pi_h(\mu_h v) \, d\mathcal{C}^{n-1} - \int_{\partial\Omega} \Pi_h(x_h v) \, dx, \quad \forall v \in V_h.$$}

Since $\mathcal{F}_h$ and $\mathcal{F}_{\varepsilon,h}$ are continuous over a compact subset of a finite dimensional...
space, they admit a minimum point. Since $\mathcal{F}_{\varepsilon, b}$ is strictly convex in $V_b/R$, its minimum is unique up to a possible additive constant.

The quadrature formulae in (1.5) and (1.6) allow the direct implementation on a computer of the minimization of $\mathcal{F}_{\varepsilon}$ and $\mathcal{F}_{\varepsilon, b}$. Implementation details will appear in [4].

2. Semicontinuity

Just for the sake of completeness, we show here the lower semicontinuity of both functionals $\mathcal{F}$ and $\mathcal{F}_{\varepsilon}$ in $\mathcal{X}$ with respect to the $L^1(\Omega)$-topology (see also [13, 14]).

We give a unified proof for both functionals $\mathcal{F}$ and $\mathcal{F}_{\varepsilon}$, considering $\mathcal{F} = \mathcal{F}_{\varepsilon}$ with $\varepsilon = 0$. Hence, let $\varepsilon \geq 0$ be fixed. First, we approximate $\mu \in L^\infty(\partial \Omega; [-1, 1])$ by a sequence of piecewise constant functions $\{\mu^k\}_{k \in N}$, so that $\mu^k \to \mu$ in $L^1(\partial \Omega)$, as $k \to \infty$. Denoting by $\mathcal{F}_{\varepsilon}^k$ the functional involving $\mu^k$, we have $|\mathcal{F}_{\varepsilon}^k(v) - \mathcal{F}_{\varepsilon}(v)| \leq \|\mu^k - \mu\|_{L^1(\partial \Omega)}$, for all $v \in \mathcal{X}$, namely, $\mathcal{F}_{\varepsilon}^k \to \mathcal{F}_{\varepsilon}$ uniformly in $\mathcal{X}$, as $k \to \infty$. The assertion is thus reduced to prove that, for any $k$, $\mathcal{F}_{\varepsilon}^k$ is semicontinuous in $\mathcal{X}$. Since no confusion is possible, we omit the superscript $k$. Then, let $\mu$ be a piecewise constant function with values $-1 = -\mu_0 < \mu_1 < \ldots < \mu_N = 1$ and set $v_i := (\mu_i - \mu_{i-1})/2$ and $G_i := \{\mu \geq \mu_i\} \subset \partial \Omega$, for all $1 \leq i \leq N$ ($G_1 = \partial \Omega$ and $G_N = \emptyset$ are allowed). Since

$$\sum_{i=1}^N v_i = 1 \quad \text{and} \quad \mu(x) = \sum_{i=1}^N v_i \chi_{G_i}(x), \text{ for all } x \in \partial \Omega,$$

$\mathcal{F}_{\varepsilon}$ can be decomposed as a convex combination of functionals $\mathcal{F}_{\varepsilon}^i$ defined by:

$$\mathcal{F}_{\varepsilon}(v) = \sum_{i=1}^N v_i \left[ \int_{\Omega} \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial \Omega} \chi_{G_i} v \, d\mathcal{H}^{n-1} - \frac{1}{2} \int_{\Omega} \varepsilon v \, dx \right] =: \sum_{i=1}^N v_i \mathcal{F}_{\varepsilon}^i(v).$$

Hence, we finally have to show that, for any $1 \leq i \leq N$, $\mathcal{F}_{\varepsilon}^i$ is semicontinuous. Given a ball $B$ containing $\overline{\Omega}$, let $\overline{\chi}_i \in W^{1,1}(B \setminus \overline{\Omega}; [-1, 1])$ be a function with trace $-\chi_{G_i}$ on $\partial \Omega$ [11, Theorem 1.11; 12, Theorem 2.16]. If, for any $v \in \mathcal{X}$ we define $\nu_i \in BV(B; [-1, 1])$ by $\nu_i(x) := v(x)$ if $x \in \Omega$, $\nu_i(x) := \overline{\chi}_i(x)$ if $x \in B \setminus \Omega$, and set

$$C_i := \int_{B \setminus \overline{\Omega}} \sqrt{\varepsilon^2 + |\nabla \overline{\chi}_i|^2} \, dx,$$

we have [12, §14.4]

$$\int_{B} \sqrt{\varepsilon^2 + |Dv|^2} = \int_{\Omega} \sqrt{\varepsilon^2 + |Dv|^2} + \int_{\partial \Omega} |v + \chi_{G_i}| \, d\mathcal{H}^{n-1} + C_i.$$

Hence, noting that

$$\int_{\partial \Omega} \chi_{G_i} v \, d\mathcal{H}^{n-1} = \int_{\partial \Omega} |v + \chi_{G_i}| \, d\mathcal{H}^{n-1} - \mathcal{H}^{n-1}(\partial \Omega),$$
we get
\[ \mathcal{F}_\varepsilon(v) = \int_B \sqrt{\varepsilon^2 + |Dv|^2} - \int_\Omega xv \, dx - C_1 - \mathcal{H}^{n-1} (\partial \Omega), \]
and the semicontinuity of \( \mathcal{F}_\varepsilon \) follows from the \( L^1(\Omega) \)-lower semicontinuity of the total variation and the area in \( BV(\Omega) \) [12, Theorems 1.9 and 14.2].

**Remark 2.1.** For any \( \varepsilon \geq 0 \), the functional \( \mathcal{F}_\varepsilon \) is not \( L^1(\Omega) \)-lower semicontinuous in \( \mathcal{X} \) if \( \mu \not\in L^\infty (\partial \Omega; [-1,1]) \). In fact, let \( \mu(x) > 1 \) for a.e. \( x \in \partial \Omega \cap B \), for some ball \( B \). Set \( B_\Omega := \Omega \cap B \neq \emptyset \) and \( B_{30} := \partial \Omega \cap B \). Let \( \{B_k \subset B_\Omega\}_{k \in N} \) be a sequence of sets of finite perimeter in \( \Omega \), so that \( \partial B_k \cap \partial \Omega = B_{30} \), \( \lim_{k \to \infty} |B_k| = 0 \), and \( \lim P(B_k, \Omega) = \mathcal{H}^{n-1}(B_{30}) \). Let \( \{v_k := -\chi_{B_k} (\chi_{B_\Omega} + 1/2)\}_k \) be a sequence converging to \( v := (\chi_{B_\Omega} + 1/2) \) in \( L^1(\Omega) \), as \( k \to \infty \). Then, noting that
\[ \mathcal{F}_\varepsilon(v) = P(B_\Omega, \Omega) + \varepsilon|\Omega| + \int_{B_{30}} \mu - \int_\Omega xv, \]
and
\[ \mathcal{F}_\varepsilon(v_k) = P(B_\Omega, \Omega) + \varepsilon|\Omega| + 2P(B_k, \Omega) - \int_{B_{30}} \mu - \int_\Omega xv_k, \]
we have \( \mathcal{F}_\varepsilon(v) > \liminf_{k \to \infty} \mathcal{F}_\varepsilon(v_k) \).

### 3. Convergence

We shall prove the diagram of convergence (0.1). First, we note that \( \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon = \mathcal{F} \)
uniformly in \( \mathcal{X} \) and \( \lim_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,b} = \mathcal{F}_b \) uniformly in \( \mathcal{V}_b \) and with respect to \( \varepsilon \) and \( b \). In fact, since
\[ 0 \leq \int_\Omega \sqrt{\varepsilon^2 + |Dv|^2} - |Dv| \leq \varepsilon|\Omega| + \int_\Omega \left( \int_\Omega 1 + \left| D \left( \frac{v}{\varepsilon} \right) \right|^2 - \left| D \left( \frac{v}{\varepsilon} \right) \right|^2 \right) \leq \varepsilon|\Omega|, \]
for all \( v \in BV(\Omega) \), we have
\begin{align*}
(3.1) & \quad |\mathcal{F}_\varepsilon(v) - \mathcal{F}(v)| \leq |\Omega|, \quad \forall v \in \mathcal{X}, \quad \text{and} \quad |\mathcal{F}_{\varepsilon,b}(v) - \mathcal{F}_b(v)| \leq |\Omega|, \quad \forall v \in \mathcal{V}_b. \end{align*}

**Theorem 3.1.** \( \Gamma_\varepsilon \lim_{\varepsilon \to 0} \mathcal{F}_b = \mathcal{F} \) and \( \Gamma_\varepsilon \lim_{\varepsilon \to 0} \mathcal{F}_{\varepsilon,b} = \mathcal{F}_b \), in \( L^1(\Omega) \).

**Proof.** The functionals \( \mathcal{F} \) and \( \mathcal{F}_\varepsilon \) (\( \mathcal{F}_b \) and \( \mathcal{F}_{\varepsilon,b} \), respectively) are set to \( +\infty \) in \( L^1(\Omega \setminus \mathcal{X} \setminus L^1(\Omega \setminus \mathcal{V}_b) \). We give a unified proof for both cases \( \varepsilon > 0 \) and \( \varepsilon = 0 \), considering \( \mathcal{F}_b = \mathcal{F}_{\varepsilon,b} \) and \( \mathcal{F} = \mathcal{F}_\varepsilon \) with \( \varepsilon = 0 \). Hence, let \( \varepsilon \geq 0 \) be fixed. We prove [8] that:

(i). For any \( v \in L^1(\Omega) \) and any sequence \( \{v_b \in L^1(\Omega)\}_b \) converging to \( v \) in \( L^1(\Omega) \), as \( b \to 0 \), we have \( \mathcal{F}_\varepsilon(v) \leq \liminf_{b \to 0} \mathcal{F}_{\varepsilon,b}(v_b) \).

(ii). For any \( v \in L^1(\Omega) \) there exists a sequence \( \{v_b \in L^1(\Omega)\}_b \) converging to \( v \) in \( L^1(\Omega) \), as \( b \to 0 \), such that \( \mathcal{F}_\varepsilon(v) = \lim_{b \to 0} \mathcal{F}_{\varepsilon,b}(v_b) \).
Preliminarily, we decompose $\mathcal{F}_{\varepsilon}(v_h)$, for all $v_h \in V_h$, as follows:

(3.2) $\mathcal{F}_{\varepsilon}(v_h) = \mathcal{F}_{\varepsilon}(v) + \int_{\partial \Omega} \left[ \Pi_b (\mu_b v_h - \mu v_h) d\mathcal{H}^{n-1} - \int_{\partial \Omega} \Pi_b (\chi_b v_h - \chi v_h) dx \right],$

which reads as $\mathcal{F}_{\varepsilon}(v_h) = \mathcal{F}_{\varepsilon}(v) + I + II$, and split $I$ and $II$ as follows:

$I = \int_{\partial \Omega} \left[ \Pi_b (\mu_b v_h - \mu v_h) d\mathcal{H}^{n-1} + \int_{\partial \Omega} (\mu_b - \mu) v_h d\mathcal{H}^{n-1} =: I_1 + I_2,$

$II = \int_{\partial \Omega} \left[ \Pi_b (\chi_b v_h - \chi v_h) dx + \int_{\partial \Omega} (\chi_b - \chi) v_h dx =: II_1 + II_2 \right].$

Since $|v_h| \leq 1$ in $\overline{\Omega}$, we have $|I_2| \leq \| \mu_b - \mu \|_{L^1(\partial \Omega)}$ and $|II_2| \leq \| \chi_b - \chi \|_{L^1(\partial \Omega)}$. In view of basic properties of the interpolation operator $\Pi_b$, and using the local inverse inequality $\| \nabla v_h \|_{L^\infty(T)} \leq C b_3^{-1} \| v_h \|_{L^\infty(T)}$, where either $T = \partial S$ or $T = S \in S_h$ and $v_h \in V_h$ [6, p. 140], we get

$|I_1| \leq C \sum_{S \in \partial S} b_2^2 \| \nabla (\mu_b v_h) \|_{L^1(\partial S \cap \partial \Omega)} \leq C \sum_{S \in \partial S} b_3^2 \| \nabla \mu_b \cdot \nabla v_h \|_{L^1(\partial S \cap \partial \Omega)} \leq C b \| \nabla \mu_b \|_{L^1(\partial \Omega)}$

and, similarly, $|II_1| \leq C b \| \nabla \chi_b \|_{L^1(\partial \Omega)}$. Hence, using (1.3) and (1.4), for any sequence $\{v_h \in V_h \}_h$, we obtain

(3.3) $\lim_{h \to 0} \left[ |I| + |II| \right] = 0.$

**Proof of Step (i).** Let $v \in L^1(\Omega)$ and $\{v_h \in L^1(\Omega)\}_h$ be any sequence so that $\lim_{h \to 0} v_h = v$ in $L^1(\Omega)$. We can assume that $v_h \in V_h$, for any $h$. Then, from the lower semi­continuity of $\mathcal{F}_{\varepsilon}$, (3.2), and (3.3), we conclude

$\mathcal{F}_{\varepsilon}(v) \leq \inf_{h \to 0} \mathcal{F}_{\varepsilon}(v_h) = \inf_{h \to 0} \mathcal{F}_{\varepsilon,b}(v_h).$

**Proof of Step (ii).** We can assume that $v \in \mathcal{K}$. Given a ball $B$ containing $\overline{\Omega}$, let $\tilde{v} \in W^{1,1}(B \setminus \overline{\Omega}; [-1,1])$ be a function with trace $v$ on $\partial \Omega$ [11] and denote again by $v \in BV(B; [-1,1])$ the function $v(x) := v(x)$ if $x \in \Omega$, $v(x) := \tilde{v}(x)$ if $x \in B \setminus \Omega$. Let $\eta_b = o(b^{-1})$ and $\{\eta_b\}_b$ be a family of mollifiers defined by $\eta_b(x) := \eta_b'(\eta_b x)$, $x \in B$. It is well known [12, Proposition 1.15] that

(3.4) $\lim_{b \to 0} \| v_b - v \|_{L^1(\Omega)} = 0$ and $\lim_{b \to 0} \int_{\Omega} |\nabla v_b| \ dx = \int_{\Omega} |Dv|.$

We claim that the sequence $\{v_h\}_b$ for Step (ii) can be defined by $v_b := \Pi_b \tilde{v} \in V_b$. In fact, noting that $\| D^2 \tilde{v}_b \|_{L^1(\Omega)} \leq C b_3$, using well known properties of $\Pi_b$, we have $\| v_b - \tilde{v}_b \|_{W^{1,1}(\Omega)} \leq C \| D^2 \tilde{v}_b \|_{L^1(\Omega)} [b^2 + b] = o(1)$. Hence, since

$\int_{\Omega} |\nabla v_b - \nabla \tilde{v}_b| \ dx \leq \int_{\Omega} |\nabla( v_b - \tilde{v}_b)| \ dx,$

in view of (3.4) we obtain

(3.5) $\lim_{b \to 0} \| v_b - v \|_{L^1(\Omega)} = 0$ and $\lim_{b \to 0} \int_{\Omega} |\nabla v_b| \ dx = \int_{\Omega} |Dv|.$
This entails [12, Theorem 2.11]

\[ \lim_{h \to 0} \int_{\Omega} |v_h - v| \, d\Omega^{n-1} = 0 \]

and, using the inequality

\[ \left| \int_{\Omega} \sqrt{\varepsilon^2 + |Dv|^2} - \int_{\Omega} \sqrt{\varepsilon^2 + |\nabla v_h|^2} \, dx \right| \leq \left| \int_{\Omega} |Dv| - \int_{\Omega} |\nabla v_h| \, dx \right|, \]

(3.2), and (3.3), gives

\[ \mathcal{F}_\varepsilon(v) = \lim_{h \to 0} \mathcal{F}_\varepsilon(v_h) = \lim_{h \to 0} \mathcal{F}_{\varepsilon, b}(v_h). \]

A straightforward consequence of (3.1) and Theorem 3.1 is the following \( \Gamma \)-convergence result for \( \mathcal{F}_{\varepsilon, b} \), as \( \varepsilon \) and \( b \) go to 0 independently.

**Corollary 3.1.** \( \Gamma \)-\( \lim \mathcal{F}_{\varepsilon, b} = \mathcal{F} \) in \( L^1(\Omega) \).

**Proof.** (i) Let \( \nu \in L^1(\Omega) \) and \( \{v_{\varepsilon, b} \in L^1(\Omega)\}_{\varepsilon, b} \) be any sequence converging to \( \nu \) in \( L^1(\Omega) \), as \( (\varepsilon, b) \to (0, 0) \). Using Theorem 3.1, Step (i), for \( \mathcal{F}_b \), and (3.1), we get

\[ \mathcal{F}(\nu) \leq \lim \inf \mathcal{F}_b(v_{\varepsilon, b}) = \lim \inf \mathcal{F}_{\varepsilon, b}(v_{\varepsilon, b}), \]

as \( (\varepsilon, b) \to (0, 0) \).

(ii). Let \( \nu \in \mathfrak{X} \) and \( \{v_{b} \in V_b\}_{b} \) be the sequence constructed in Step (ii) of Theorem 3.1, for \( \mathcal{F}_b \). Then, using (3.1), we get

\[ \mathcal{F}(\nu) = \lim_{b \to 0} \mathcal{F}_b(v_b) = \lim_{b \to 0} \mathcal{F}_{\varepsilon, b}(v_b). \]

**Remark 3.1.** Let \( u_{\varepsilon, b} \) be a minimum of \( \mathcal{F}_{\varepsilon, b} \). We have \( \mathcal{F}_{\varepsilon, b}(u_{\varepsilon, b}) \leq \mathcal{F}_{\varepsilon, b}(0) = \varepsilon |\Omega| \), whence, using (1.3) and (1.4),

\[ \int_{\Omega} |Du_{\varepsilon, b}| \leq \int_{\Omega} \sqrt{\varepsilon^2 + |Du_{\varepsilon, b}|^2} \leq 3\varepsilon^{\alpha-1}(\partial\Omega) + |\Omega|(1 + \|\omega\|_{L^\infty(\Omega)}), \]

for all \( 0 \leq \varepsilon \leq 1 \) and \( b > 0 \). Then, by the compactness theorem in \( BV(\Omega) \) [12, Theorem 1.19], the family \( \{u_{\varepsilon, b}\}_{\varepsilon, b} \) admits a subsequence converging to some \( u \in \mathfrak{X} \) in \( L^1(\Omega) \). Corollary 3.1 entails that \( u \) is a minimum point of \( \tilde{\mathcal{F}} \).

### 4. Numerical experiments

Implementation details on the minimization algorithm can be found in [4]. Here we simply present a couple of numerical examples. The unique discrete absolute minimum \( u_{\varepsilon, b} \) of \( \mathcal{F}_{\varepsilon, b} \) is approximated by Newton-like iterations. A quasiuniform mesh is used.

**Example 1.** Let \( \Omega := (-2, 2)^2 \), \( \mu := 1 \) (tangential contact at \( \partial\Omega \)), and \( \kappa := 1 \). The functional \( \mathcal{F} \) has one absolute minimum, \( A := \{(|x_1| - 1)_+|^2 + (|x_2| - 1)_+|^2 \leq 1\} \). Figure 4.1 shows both the exact minimum (dashed lines) and the computed one \( A_{\varepsilon, b} := \{u_{\varepsilon, b} > 0\} \) (solid lines). Here \( \varepsilon = 0.2 \) and \( b = 0.14 \); the initial guess is the empty set, which is a relative minimum of \( \mathcal{F} \) in \( \mathfrak{X} \). Note that, using the approximation via double well potential [3], the discrete minimizing set presents no contact with \( \partial\Omega \), because the relaxed solution forms a transition
layer across the interface. This effect is absent in our convex approximations which, in turn, exhibits higher accuracy.

**Example 2.** Let \( \Omega := (-2,2)^2 \) and \( \{\Gamma_1, \Gamma_2\} \) be the partition of \( \partial \Omega \) defined by \( \Gamma_1 := \partial \Omega \cap \{x_1 x_2 \leq 0\} \) and \( \Gamma_2 := \partial \Omega \setminus \Gamma_1 \). Let \( \mu := -1 \) on \( \Gamma_1 \), \( \mu := 1 \) on \( \Gamma_2 \), and \( x := 0 \). Set \( \mu_b := \Pi_b (\mu) \). The functional \( \mathcal{F} \) has two absolute minima in \( \mathcal{X} \), \( A \) and \( B \), shown in fig. 4.2 (dashed lines). The computed minima are obtained from the unique discrete minimum \( u_{\epsilon,b} \) (the initial guess is the empty set) as \( A_{\epsilon,b} := \{u_{\epsilon,b} > -0.5\} \) and \( B_{\epsilon,b} := \{u_{\epsilon,b} > 0.5\} \) whereas, in [3], they were obtained iterating from different initial guesses. Here \( \epsilon = 0.2 \) and \( b = 0.14 \).
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