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Extension of a regularity result concerning the dam problem


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Equazioni a derivate parziali. — *Extension of a regularity result concerning the dam problem.* Nota di Gianni Gilardi e Stephan Luckhaus, presentata (*) dal Socio E. Magenes.

**Abstract.** — One proves, in the case of piecewise smooth coefficients, that the time derivative of the solution of the so called dam problem is a measure, extending the result proved by the same authors in the case of Lipschitz continuous coefficients.

**Key words:** Regularity; Porous media; Non negative subharmonic functions.

**Riassunto.** — *Estensione di un risultato di regolarità sul problema della diga.* Si dimostra, nel caso di coefficienti regolari a tratti, che la derivata rispetto alla variabile temporale della soluzione del cosiddetto problema della diga è una misura, estendendo il risultato che gli stessi autori hanno già dimostrato nel caso di coefficienti lipschitziani.

**Introduction**

We deal with the solution to the dam problem, that is

\begin{align}
\tag{0.1} & u \in L^2(0, T; H^1(Q)) \quad \text{and} \quad \chi \in L^\infty(Q); \\
\tag{0.2} & u \geq 0, \ 0 \leq \chi \leq 1, \ u(1-\chi) = 0 \quad \text{in} \ Q; \\
\tag{0.3} & u = g \quad \text{on} \ \Sigma_D; \\
\tag{0.4} & \int_Q (-\chi \partial_t v + a(\nabla u + \chi e) \cdot \nabla v) \leq 0 
\end{align}

for every \( v \in H^1(Q) \) such that

\begin{align}
\tag{0.5} & v \geq 0 \quad \text{on} \ \Sigma_D, \quad v = 0 \quad \text{on} \ \Sigma_D \cap \{g > 0\}, \quad v(\cdot, 0) = v(\cdot, T) = 0 \quad \text{in} \ \Omega; \\
\tag{0.6} & \chi(\cdot, 0) = \chi^0 \quad \text{in} \ \Omega. \quad \square
\end{align}

Here \( \Omega \) is a connected bounded open set in \( \mathbb{R}^n \) with Lipschitz boundary and represents the porous medium. The boundary \( \partial \Omega \) consists of the pervious part \( \Gamma_D \) and the impervious part \( \Gamma_N \), whose closures are \( C^{1,1} \) manifolds intersecting in a smooth \((n-2)\)-dimensional submanifold of \( \partial \Omega \). \( a = (a_{ij}) \) is the permeability matrix and \( e \in \mathbb{R}^n \) is a given unit vector, which takes gravity into account in the physical model. The function \( u \) represents the unknown pressure and the velocity is given by \(-a(\nabla u + \chi e)\) by Darcy’s law. Finally the following notations have been used

\begin{align}
\tag{0.7} & Q = \Omega \times ]0, T[ \quad \text{and} \quad \Sigma_D = \Gamma_D \times ]0, T[.
\end{align}

We assume the following regularity for the data

\begin{align}
\tag{0.8} & g \in C^{0,1}(\mathbb{R}^{n+1}), \quad g \geq 0, \quad \chi^0 \in L^\infty(\Omega), \quad 0 \leq \chi^0 \leq 1. \quad \square
\end{align}

This problem has been studied by several authors from different points of view:

results deal mainly with the stationary case, but the evolution problem is also considered (see [1, 3-6, 8-11, 13, 14]; see also the lists of references of [2, 5, 7, 12]).

In [11] it is proved that \( \partial_t u \) is a measure whose negative part is a function, under the assumption that \( a \) is a uniformly elliptic matrix with Lipschitz continuous coefficients. Here we extend that result to the case of piecewise smooth coefficients, in order to cover the case of layered materials. More precisely we assume

\[
\frac{\partial}{\partial t} u \in L^\infty(\Omega), \quad |a_{ij}| \leq 1, \quad \sum_j a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n,
\]

(0.10) \[ \text{div} (ae) \geq 0 \text{ in } \Omega \quad \text{and} \quad ae \cdot \mathbf{v} < 0 \text{ on } \Gamma_N, \]

where \( \lambda > 0 \) is fixed, and \( \mathbf{v} \) is the outer unit normal vector to \( \partial \Omega \). Moreover we assume the following regularity condition: there exists a smooth manifold with boundary, whose interior \( \Gamma_0 \) lies in \( \Omega \), such that \( \partial \Gamma_0 \subset \partial \Omega \) and

\[
a_{ij} \in C^{0,1}(\Omega_k) \quad \forall i,j,k \quad \text{and} \quad ae \cdot \mathbf{v} \neq 0 \text{ on } \Gamma_0,
\]

where \( \Omega_k \) are the connected components of \( \Omega \setminus \Gamma_0 \), which we assume to be \( C^{0,1} \).

Notice that no more than two \( \Omega_k \)'s can have nonempty intersection. For technical reasons we assume that \( a \) is continuous in a neighbourhood of \( \Gamma_N \).

1. KNOWN RESULTS FOR SMOOTH \( a \)

In the paper [11] the idea of the proof was to use an inequality from below on the term \( \text{div} (a(\nabla u + e)) \) in terms of the measure of the set where \( u = 0 \). We give a few of the inequalities from this paper without proof.

We work on sets of the type described in the following definition.

1.1. DEFINITION. Assume \( \Lambda > 0 \) and \( \varphi > 0 \). A set \( \Omega \subset \mathbb{R}^n \) is said to be a \( \Lambda, \varphi \)-set if there exists a connected set \( \mathbb{U} \) such that

\[
diam \mathbb{U} \leq \Lambda \varphi;
\]

(1.1) \[ \Omega = B_\varphi(\mathbb{U}), \quad \text{i.e. } \Omega = \bigcup_{x \in \mathbb{U}} B_\varphi(x). \]

A connected set \( \mathbb{U} \) satisfying (1.1) and (1.2) is called a core of \( \Omega \). □

Assuming that \( a \) is a uniformly elliptic matrix with \( L^\infty \) coefficients, for these sets the Harnack inequality holds. Moreover, if \( a \) is globally \( C^{0,1} \), the Hopf maximum principle holds too.

The first Lemma holds for \( L^\infty \) coefficients (see [11], Lemma 1.5):

1.2. LEMMA. Let \( \Omega \) to be a \( \Lambda, \varphi \)-set with core \( \mathbb{U} \). Suppose that \( u \in H^1(\Omega) \) is nonnegative and satisfies

\[
\text{div} (a(\nabla u + e)) \geq 0 \quad \text{in } \Omega,
\]

(1.3) \[ \text{and that } w \in u + H^1(\Omega) \text{ satisfies}
\]

\[
\text{div} (a(\nabla w + e)) = 0 \quad \text{in } \Omega.
\]

(1.4)
Then for any \( \beta \geq 0 \) we have

\[
\int_{\Omega} \text{div} (a(\nabla u + e)) \geq c_1 \beta \{|w - u \geq \beta\} \cap \Omega |^{1 - 2/n}
\]

where \( c_1 \) depends only on \( \lambda \).

Furthermore, if for some \( x \in \Omega \)

\[
\int_{B_{\rho/2}(x)} u \leq \frac{1}{2} \int_{B_{\rho/2}(x)} w
\]

then, for any \( y \in \Omega \) and every \( \rho' \leq \rho \), we have the estimate

\[
\int_{\Omega} \text{div} (a(\nabla u + e)) \geq c_2 \left( \int_{B_{\rho/2}(y)} u - c_3 \rho \right) \rho'^{-2},
\]

where \( c_2 \) and \( c_3 \) depend only on \( \lambda \) and \( \Lambda \). \( \square \)

Using 1.2. and also the Hopf maximum principle, the main results for «subharmonic» functions are (see [11], Lemmas 1.9 and 1.12):

1.3. **Main Lemma.** Assume \( \Omega \) to be a \( \Lambda, \rho \)-set, with \( \rho \leq 1 \) and core \( \Omega \), and \( a \) to be Lipschitz continuous. If \( u \in H^1(\Omega) \) is nonnegative and satisfies (1.3), then, for every \( x \in \Omega \) and \( \rho' \leq \rho \), we have the estimate

\[
\int_{\Omega} \text{div} (a(\nabla u + e)) \geq c_1 \left( \frac{1}{\rho} \int_{B_{\rho/2}(x)} u - c_2 \right) \{|u = 0\} \cap \Omega |^{1 - 1/n}
\]

where \( c_1 \) and \( c_2 \) depend only on \( \lambda, \Lambda \), and the norm of \( a \) in \( C^{0,1} \). \( \square \)

1.4. **Main Lemma** (for Neumann boundary points). Assume that \( a \) is Lipschitz continuous and \( x_0 \in \Gamma_N \), and let \( \Omega \) be a \( \Lambda, \rho \)-set with core \( \Omega \) such that

\[
B_{\rho}(x_0) \cap \Omega \subset \Omega \subset B_{4\rho}(x_0) \cap \Omega;
\]

\[
\partial \Omega \cap \partial \Omega \subset \Gamma_N.
\]

Moreover assume that \( u \in H^1(\Omega) \) is non negative and satisfies (1.3) and (in a suitable sense, see [11])

\[
|a \nabla u \cdot \nu| \leq K \quad \text{on} \quad \partial \Omega \cap \partial \Omega.
\]

Then for any \( x \in B_{\rho/2}(x_0) \cap \Omega \) and any \( \rho' < \rho/2 \) we have

\[
\int_{\Omega} \text{div} (a(\nabla u + e)) \geq c_1 \left( \frac{1}{\rho} \int_{B_{\rho}(x)} u - c_2 (1 + K) \right) \{|u = 0\} \cap \Omega |^{1 - 1/n}
\]

where \( c_1 \) and \( c_2 \) depend only on \( \lambda, \Lambda, \Omega \), and the norm of \( a \) in \( C^{0,1} \). \( \square \)
From Lemma 1.10 and Remark 1.11 of [11] we derive the following:

1.5. **MAIN LEMMA** (for **DIRICHLET BOUNDARY POINTS**). Assume that \( a \) is Lipschitz continuous and \( x_0 \in \Gamma_D \), and let \( \hat{\partial} \) be a \( \Lambda, \varphi \)-set whose core \( \mathcal{U} \) is contained in \( \Omega \). We set \( \partial = \Omega \cap \hat{\partial} \) and assume

\begin{align}
B_{\epsilon}(x_0) \cap \Omega &\subseteq \hat{\partial} \subseteq B_{2\epsilon}(x_0) \cap \Omega; \\
|v(x) - v(x')| &\leq A|x - x'|/\rho \quad \text{for } x, x' \in \Omega \cap \partial \partial.
\end{align}

Finally let \( u \in H^1(\Omega) \) be nonnegative and satisfy (1.3). Then

\begin{align}
\int_\partial \text{div}(a(\nabla u + e)) &\geq c_1 \left( \frac{1}{\rho} \int_{B_\rho(x_0) \cap \partial \Omega} u \, d\mathcal{H}^{n-1} - c_2 \right) \cdot |\{u = 0\} \cap \partial|^1/\rho; \\
\int_\partial \text{div}(a(\nabla u + e)) &\geq c_3 \left( \frac{1}{\rho} \int_{B_\rho(x_0) \cap \partial \Omega} u \, d\mathcal{H}^{n-1} - c_4 \right) \cdot \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} |A_\varepsilon|
\end{align}

where \( A_\varepsilon = \{d(x, \partial \Omega) < \varepsilon\} \cap \{u = 0\} \) and \( c_1, \ldots, c_4 \) depend only on \( \lambda, \Lambda, \) and the norm of \( a \) in \( C^{0,1} \).

We can apply these results because the following statements hold for the solution \((u, \chi)\) of the dam problem with \( L^\infty \) coefficients (see [11], Proposition 2.2 and Lemma 2.4):

1.6. **PROPOSITION.** The following inequalities hold

\begin{align}
\text{div}(a(\nabla u + e)) &\geq 0 \quad \text{and} \quad \partial_t (1 - \chi) - \text{div}((1 - \chi) a e) \leq 0. \quad \square
\end{align}

1.7. **LEMMA.** Assume that \( \psi \) is the characteristic function of an open subset of \( \Omega \times \mathbb{R} \). Assume moreover \( \psi \in BV(\mathbb{R}^{n+1}) \) and

\begin{align}
\partial_t \psi - a e \cdot \nabla \psi &\leq 0 \quad \text{and} \quad 0 \leq \psi \leq \chi_{\Omega \times \mathbb{R}}.
\end{align}

Then we have

\begin{align}
\partial_t \int_\Omega \psi(t) (1 - \chi(t)) &\leq -\int_\Omega \text{div}(a(\nabla u(t) + e)) \psi(t).
\end{align}

In particular the left hand side of (1.19) is nonpositive. \( \square \)

1.8. **REMARK.** The condition (1.18) can be replaced by the following one: (1.18) holds in an open set \( \omega_1 \) of \( \mathbb{R}^{n+1}, \chi = 1 \) in an open set \( \omega_2 \), and \( \omega_1 \cup \omega_2 \supseteq \Omega \times \mathbb{R}. \) \( \square \)
2. THE PROOF OF THE RESULT

Here, as in [11], we use the following approximation of $\partial_t u$:

\[ \partial_t^h u(x, t) = \frac{2}{3h} \left( u(x, t) - \frac{1}{b} \int_{t-2b}^{t-b} u(x, s) \, ds \right). \]

for $x \in \Omega$, $h > 0$, and $2b < t < T$.

In the whole section we use the assumptions and the notations of the introduction and denote by $(u, \chi)$ the solution of the dam problem (0.1),..., (0.6).

2.1. THEOREM. Suppose that $\Omega_t \subset \Omega$ is a $\Lambda, \rho$-set such that $a_{ij} \in C^{0,1}(\overline{\Omega})$ and suppose that $\psi(\cdot, t) := \chi_0 \psi(t)$ satisfies (1.18). Assume $2b < t < T$. Then there exists a constant $c_1$, depending only on $\lambda, \Lambda, \|a\|_{C^{0,1}(\overline{\Omega})}$, and $\psi/b$, such that either

\[ \int_{B_{\rho}(x)} \partial_t^h u(y, t) \, dy \geq -c_1 \text{ for all } B_{\rho}(x) \subseteq B_{\rho/2}(x) \subseteq \bigcap_{t-2b < \tau < t} \Omega_{\tau} \text{ or} \]

\[ \chi(t, \tau) = 1 \text{ in } \Omega_{\tau} \text{ for } t-h < \tau < t. \]

PROOF. The proof of a quite similar statement is given in [11]. Here we sketch the proof again. By 1.7 and 1.3 we get

\[ \partial_t \int_{\Omega} (1 - \chi(s)) \leq - \int_{\Omega} \text{div} (a(\nabla u(s) + e)) \leq -c_2 \left( \frac{1}{\rho} \int_{B_{\rho}(x)} u(y, s) \, dy - c_3 \right)^+ |\Omega_{\tau} \cap \{u(s) = 0\}|^{1-1/\rho}. \]

Let $c_1$ be any positive number and assume (2.2) to be false for some ball $B_{\rho}(x)$: we prove that if $c_1$ is large enough then (2.3) holds. Our assumption implies

\[ \int_{B_{\rho}(x)} \left( \int_{t-2b}^{t-h} u(y, s) \, ds \right) \, dy \geq 3c_1b^2/2. \]

But setting $f(s) = \int_{\Omega} (1 - \chi(s))$, we have $f(s) \leq |\Omega_{\tau} \cap \{u(s) = 0\}|$; so the previous inequalities give

\[ f'(s) \leq -c_2 \left( \frac{1}{\rho} \int_{B_{\rho}(x)} u(y, s) \, dy - c_3 \right)^+ (f(s))^{1-1/\rho}. \]

From this it follows that either $f(t-h) = 0$ or

\[ f^{1/\rho}(t-2b) - f^{1/\rho}(t-h) \geq \frac{c_2}{n} \left( \frac{1}{\rho} \int_{B_{\rho}(x)} \left( \int_{t-2b}^{t-h} u(y, s) \, ds \right) \, dy - c_3 b \right). \]
Using (2.4) this gives $f^{1/n}(t-2h) \geq (c_2/n) [(3/2) c_1 h \rho^{-1} - c_3] h$. As $f^{1/n}(t-2h) \leq c_4 \rho$, this gives for $c_1$ an upper bound depending on $\rho/h$. So, if $c_1$ is large enough (depending on $\rho/h$), we have $f(t-2h) = 0$ and the monotonicity of $f$ gives the result. \hfill \Box

For points on $\Gamma_0$ we have to split the argument into two parts: «above» and «below» the interface.

2.2. THEOREM. Assume $2h < t < T$, $x_0 \in \Gamma_0$, e.g. $x_0 \in \overbar{\Omega}_1 \cap \overbar{\Omega}_2$. Then there exists a constant $c \geq 1$ depending on the geometry such that, if $B_{c_3}(x_0) \subset \overbar{\Omega}_1 \cap \overbar{\Omega}_2$, then either

\begin{equation}
\int_{B_{c_3}(x_0)} \partial_t u(y, t) \, dy \geq -c_1 \quad \text{for all } B_{c_3}(x) \subset B_{c_2}(x_0) \text{ or }
\end{equation}

\begin{equation}
\chi(\tau) = 1 \quad \text{in } B_{c_2}(x_0) \quad \text{for } t-h < \tau < t. \quad \Box
\end{equation}

PROOF. We can assume $a(x_0) \cdot v < 0$ where $v$ is the outer normal at $x_0$ to $\partial \Omega_1$. For $t-2h < \tau < t$ let $\partial(\tau)$ be a $\Lambda, \rho$-set whose core $\mathcal{U}(\tau)$ contains $x_0$ and satisfying (1.18) with $\psi(\tau) = \chi_{\partial(\tau)}$. Let also $\partial_1(\tau)$ be a $\Lambda, \rho$-set whose core $\mathcal{U}_1(\tau)$ intersects $\mathcal{U}(\tau)$ and assume that also $\psi(\tau) = \chi_{\partial_1(\tau)}$ satisfies (1.18). Assume moreover that $\partial_1(\tau) \subset \Omega_1$ for $t-2h < \tau < s$ and that $\partial_1(\tau)$ contains a $|\beta(\tau-s)|$-neighbourhood of $\Gamma_0 \cap \partial(\tau)$ for $s < \tau < t-h$.

Here $\Lambda$ depends essentially on $n$, on the norm of $a$, and on the infimum of $a \mathbf{e} \cdot v$ in a neighbourhood of $x_0$; the constant $c$ must be larger than a uniform $\Lambda; \beta$ obviously depends on $\inf a \mathbf{e} \cdot v$; $\Lambda$ will be chosen later.

Take $c_1 > 0$ and assume (2.5) to be false for some $x$ and $\rho'$ as in the statement. Thus (for some $c_2$ depending on $\Lambda$)

$$
\int_{t-2h}^{t-h} \left( \int_{\partial(\tau)} u(y, \tau) \, dy \right) \, d\tau \geq c_2 c_1 h^2.
$$

Choose $s$ such that

\begin{equation}
\int_{t-2h}^{t} \left( \int_{\partial(\tau)} u(y, \tau) \, dy \right) \, d\tau = c_2 c_1 h^2 / 2.
\end{equation}

Define $u(\cdot, \tau) \in H^1(\partial(\tau))$ by $\text{div}(a(\nabla u(\cdot, \tau) + e)) = 0$ in $\partial(\tau)$, and call $T$ the set of all $\tau \in \{t-2h, t-h\}$ such that

$$
\left\{ x \in \partial(\tau) : u(x, \tau) - u(x, \tau) > \alpha \int_{\partial(\tau)} u(y, \tau) \, dy \right\} > \varphi^n
$$

where $\alpha$ will be specified later. From (1.5) and (1.19) we get

$$
c_3 \alpha \int_{T} \int_{\partial(\tau)} u(\tau) \cdot \mathbf{a}^{1-2/n} \varphi^{n-2} \, d\tau \leq \int_{t-2h}^{t-h} \int_{\partial(\tau)} \text{div}(a(\nabla u(\tau) + e)) \, d\tau \leq \int_{t-2h}^{t-h} \partial_{\tau} \left( 1 - \chi(\tau) \right) \, d\tau \leq c_4 \varphi^n.
$$
Hence
\[ \int_{\mathcal{S}, \mathcal{O}(\tau)} u(\tau) \, d\tau \leq c_5 \alpha^{(2/n) - 2} \rho^2 \]
and, if \( c_1 > (c_3/4c_2) \alpha^{(2/n) - 2} (\rho/b)^2 \), then
\[ (2.7) \quad \int_{\mathcal{S}, \mathcal{O}(\tau)} u(\tau) \, d\tau \geq \frac{c_2 c_1}{4} h^2, \quad i = 1, 2, \]
where \( \mathcal{S}_1 = ]t - 2h, s[ \setminus \mathcal{I} \) and \( \mathcal{S}_2 = ]s, t - h[ \setminus \mathcal{I} \).

Suppose that \( y' \) is such that \( B_{b/4} (y') \subset \mathcal{O}(\tau) \); then by Harnack's inequality
\[ w(y', \tau) \geq c_6 \int_{\mathcal{O}(\tau)} u(\tau) - c_7 \rho. \]
Therefore, if \( \alpha < c_6/2, \alpha < |B_{1/4}|, \tau \notin \mathcal{I}, \) and \( B_{b/2} (y) \subset \mathcal{O}(\tau) \), then there exists \( y' \in B_{b/4} (y) \) such that
\[ u(y', \tau) \geq w(y', \tau) - \alpha \int_{\mathcal{O}(\tau)} u(\tau) \geq \frac{c_6}{2} \int_{\mathcal{O}(\tau)} u(\tau) - c_7 \rho. \]

Taking \( y \in \mathcal{U}_1 (\tau) \cap \mathcal{U}(\tau) \), by the Main Lemma 1.3 we obtain
\[ \int_{\mathcal{O}(\tau)} \text{div} (a \nabla u(\tau) + e) \geq \left( \frac{c_8}{\rho} \int_{\mathcal{O}(\tau)} u(\tau) - c_9 \right)^+ \left( \int_{\mathcal{O}_1 (\tau)} (1 - \chi(\tau)) \right)^{1-1/n}. \]
As in the previous proof we derive the differential inequality
\[ \partial_t \int_{\mathcal{O}_1 (\tau)} (1 - \chi(t)) \leq -\left( \frac{c_8}{\rho} \int_{\mathcal{O}(\tau)} u(\tau) - c_9 \right)^+ \left( \int_{\mathcal{O}_1 (\tau)} (1 - \chi(\tau)) \right)^{1-1/n}, \]
which gives as a consequence, after integration over \( \mathcal{S}_1 \), either
\[ (2.8) \quad \frac{1}{n} |\mathcal{O}_1 (t - 2b)|^{1/n} \geq \int_{\mathcal{S}_1} \left( \frac{c_8}{\rho} \int_{\mathcal{O}(\tau)} u(\tau) - c_9 \right) \, d\tau \text{ or} \]
\[ (2.9) \quad \int_{\mathcal{O}_1 (\tau)} (1 - \chi(\tau)) = 0 \quad \text{for } s < \tau < t. \]

By (2.7), from (2.8) it follows that
\[ c_{10} (\Lambda) \rho \geq c_{11} c_1 b^2 / \rho - c_9 b \quad i.e. \quad c_1 \leq (\rho/b)(c_{12} \rho/b + c_{13}). \]
So, if \( c_1 \) is larger than the last number, then (2.9) holds, \( i.e. \chi(\tau) = 1 \) in \( \mathcal{O}_1 (\tau) \) for \( s < \tau < t. \)

Now we construct \( \mathcal{O}_2 (\tau) \subset \Omega_2 \) such that \( \chi(\tau) = 1 \) in \( \mathcal{O}_2 (\tau) \) for \( t - h < \tau < t. \) A naive
construction of such an \( \partial_2 (\tau) \) satisfying (1.18) will produce a gap between \( \partial_2 (\tau) \) and \( \partial_1 (\tau) \) for \( \tau > s \), since \( \beta \) is small. This can be avoided by constructing \( \partial_2 (\tau) \subset \Omega_2 \) such that we have (1.18) outside of \( \partial_1 (\tau) \) (with \( \psi (\cdot , \tau) = \chi_{\partial_1 (\tau)} \)).

There is a difficult in constructing the «upper» boundary of \( \partial_2 (\tau) \): changing coordinates, fix \( a (x_0) e \) to be parallel to the \( n \)-th axis and represent \( \partial_2 \) locally by \( x_n = \phi (x') \) (where \( x = (x', x_n) \)). Then give the «upper» boundary by \( x'^* = \phi(x') - \beta (\tau - s) \) \( + \frac{1}{2} \left( \phi (\tau - s) \right)^2 \). We have \( \partial_2 \phi (x', \tau) \leq - c_{15} \) at those \( x' \) such that \( \phi (x') - \phi (x', \tau) \geq \beta (\tau - s) \). So we choose \( c_{15} \) according to the velocity of the solutions of \( x = - a (x) e \), and \( c_{14} \) large enough, namely \( c_{14} \geq c_{15} \beta^{1/2} \cdot (2 \beta^2 / \rho)^{1/2} + 1 \). So, applying Remark 1.8, we conclude the proof.

Now we deal with points on \( \Gamma_0 \cap \Gamma_D \) and distinguish between Dirichlet boundary values which are larger or smaller than a number of order \( \beta \).

2.3. Lemma. Let \( x \in \Gamma_0 \cap \Gamma_D \). Then for every \( \alpha > 0 \) there exists \( \gamma > 0 \) such that if \( u (x, t) > \gamma b \) then \( \chi = 1 \) in \( (\partial \cap B_{\alpha b} (x)) \times [t - h, t + h] \) provided \( 2 \alpha b < d(x, \Gamma_N) \). □

Proof. Let \( \Omega_1 \) and \( \Omega_2 \) be the components of \( \Omega \setminus \Gamma_0 \) whose closures contain \( x \) and choose open subsets \( \partial_1 \subset \Omega_1 \) and \( \partial_2 \subset \Omega_2 \) satisfying the following conditions (where \( \Lambda \) will depend e.g. in the \( C^{1,1} \) structure of \( \Gamma_0 \): \( |\psi (x) - \psi (x')| \leq \Lambda (\alpha b)^{-1} |x - x'| \) \( \forall x, x' \in \Omega_1 \cap \partial \partial_1, i = 1, 2; \ B_{\alpha b} (x) \cap \Omega \subset \partial_1 \cup \partial_2 \subset B_{\lambda b} (x); \ \exists x_i \in \partial_1 \cap \Gamma_D : B_{\alpha b} (x_i) \cap \Omega \cap \partial_i, i = 1, 2 \). If \( \gamma \) is large enough, \( u (x, \tau) > \gamma b / 2 \) for \( |\tau - t| < 2 \beta \). Then we can apply Lemma 1.5 with \( \rho = \alpha b \) and deduce

\[
\int_{\partial_1} \text{div} (a (\nabla u (\tau) + e)) \geq c_1 \left( \frac{1}{h} \inf_{\partial_1 \cap \Gamma_D} u (\cdot , \tau) - c_2 \right) \cdot \left( \chi (\cdot , \tau) < 1 \right) \cap \partial_1 \right|^{1 - 1/n}
\]

\[
\int_{\partial_1} \text{div} (a (\nabla u (\tau) + e)) \geq c_3 \left( \frac{1}{h} \inf_{\partial_1 \cap \Gamma_D} u (\cdot , \tau) - c_4 \right) \cdot \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} |A_{\varepsilon} (\tau)|
\]

where \( A_{\varepsilon} = \{ d (\cdot , \Omega \cap \partial \partial_1) < \varepsilon \} \cap \{ u (\tau) = 0 \} \). Now we argue as in the smooth case and deduce

\[
\partial_1 \int (1 - \chi) = - \int_{\partial_1} \text{div} (a (\nabla u + e)) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{ d (\cdot , \Omega \cap \partial \partial_1) < \varepsilon \} \cap \{ u (\tau) = 0 \}} |a (\cdot , \partial \partial_1)| d x 
\]

\[
\leq c_3 \left( \frac{1}{h} \inf_{\partial_1 \cap \Gamma_D} u (\cdot , \tau) - c_4 \right) \cdot \left( \int_{\partial_1} (1 - \chi) \right)^{1 - 1/n} \leq - c_7 \left( \int_{\partial_1} (1 - \chi) \right)^{1 - 1/n}.
\]

From this differential inequality we conclude the result as in [11]. □

As in [11] we can find a «parallel» boundary to \( \partial \Omega \cap \{ g < \beta b \} \) such that \( \partial_0 u \) is bounded there.
2.4. **Lemma.** Assume \( \alpha, \beta > 0 \) and define \( w \in H^1(\Omega) \) by
\[
\text{div}(a(\nabla w + e)) = 0 \quad \text{in} \; \Omega; \quad w = g \quad \text{on} \; \Gamma_D \quad \text{and} \quad a(\nabla w + e) \cdot v = 0 \quad \text{on} \; \Gamma_N.
\]
Then there exists a smooth \( \Omega_b \subset \Omega \) such that \( d(\Omega_b, \{g \leq \beta h\}) > \alpha h \) and
\[
\int_{\Omega \cap \partial \Omega_b} |w|^\epsilon \leq cb^\epsilon
\]
for some \( \epsilon > 0 \), where \( c \) depends on \( \alpha \) and \( \beta \). \( \Box \)

2.5. **Remark on the Proof.** The estimate follows from an estimate on \( \partial_s w \). Near
the points of \( \Gamma_D \cap \partial \Omega \), \( \partial_s w \) has a singularity which can be estimate by the \( \epsilon \)-Hölder con­
tinuity of \( w \). \( \Box \)

2.6. **Main Theorem.** Under the assumptions of the introduction, let \( (u, \chi) \) be the so­
lution to the dam problem. Then \( \partial_s u \) is a measure on \( \Omega \times [0, +\infty[ \) whose negative part is a
locally bounded function.

More precisely, for every \( T > 0 \) and \( \delta > 0 \) there exists a constant \( c \) which depends only
on \( T, \delta, \lambda \), on the norm of \( a \) in \( C^{0,1} \) of the components \( \Omega_k \), on the norm of \( g \) in \( C^{0,1} \), and on
the geometry, such that
\[
(2.10) \quad \partial_s u \geq -c \quad \text{in} \; Q(T, \delta), \quad \text{where}
\]
\[
(2.11) \quad Q(T, \delta) = \{ (x, t) \in \Omega \times [0, T] : d(x, \Gamma_D \cap \{g < \delta\}) > \delta \}. \quad \Box
\]

**Proof.** Arguing as in [11], but using Theorems 2.1 and 2.2 in the interior,
a similar statement for \( \Gamma_N \) (see [11] Lemma 2.9), and Lemma 2.3, we see that
\( w_b := (\partial_s^b u) \wedge (-c_1) \) fulfills
\[
\begin{cases}
\text{div}(a \nabla w_b) \leq 0 & \text{in} \; \Omega_b, \\
w_b = (\partial_s^b u) \wedge (-c_1) & \text{on} \; \Gamma_D \cap \partial \Omega_b (s), \\
a \nabla w_b \cdot v \geq 0 & \text{on} \; \Gamma_N,
\end{cases}
\]
where \( \Omega_b \) is as in the Lemma 2.4. So one can estimate \( w_b \) from below by the solution of
the corresponding equalities.

Since on \( \Omega \cap \partial \Omega_b \) is estimated by \(-w/b \) from below, with \( w \) given in Lemma 2.4,
one finally gets the estimate
\[
\int_{\Omega_b} (w_b^-)^\epsilon \leq c_2
\]
and, more precisely, \( w_b \) is bounded pointwise from below away from the set where
\( g \leq O(b) \). \( \Box \).

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