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Fisica matematica. — *An existence result in nonlinear theory of electromagnetic fields.* Nota di Dorin Iesan e Antonio Scalia, presentata (*) dal Socio D. Graffi.

Abstract. — This paper is concerned with the nonlinear theory of equilibrium for materials which do not conduct electricity. An existence and uniqueness result is established.

Key words: Electromagnetism; Nonlinear theory; Existence and uniqueness.

Riassunto. — *Un risultato di esistenza nella teoria nonlineare dell'elettromagnetismo.* In questo lavoro si affronta lo studio dell'equilibrio per materiali non lineari e non conduttori di elettricità. Per tale problema vengono formulati teoremi di esistenza e di unicità.

1. Introduction

The equations of electromagnetic theory have been the subject of many investigations (see, for example, [1-5]).

This paper is concerned with the nonlinear theory of equilibrium for materials which do not conduct electricity. Moreover, only isothermal processes are considered.

The aim of this paper is to establish an existence theorem for a boundary-value problem by using results on the nonlinear operators presented in [6, 7].

2. Basic equations

We assume that a bounded region R of three dimensional Euclidean space $\mathcal{E}_3$ is occupied by a rigid body which does not move. We let $\overline{R}$ denote the closure of R and call ∂R the boundary of R. We assume that ∂R is sufficiently smooth for the divergence theorem and Friedrichs' inequality to be applicable. Letters in boldface stand for vector fields. We write v_i for the components of $\boldsymbol{v}$ in the underlying rectangular Cartesian coordinate frame. We shall employ the usual summation and differentiation conventions: the subscripts are understood to range over the integers (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate.

In the case of equilibrium, the field equations of the electromagnetic theory reduce to

$$\text{curl } \boldsymbol{E} = \boldsymbol{0}, \quad \text{curl } \boldsymbol{H} = \boldsymbol{0}, \tag{2.1}$$

$$\text{div } \boldsymbol{D} = \rho, \quad \text{div } \boldsymbol{B} = 0, \tag{2.2}$$

where $\boldsymbol{E}$ is the electric intensity, $\boldsymbol{H}$ is the magnetic intensity, $\boldsymbol{D}$ is the electric induction, $\boldsymbol{B}$ is the magnetic induction, and ρ is the density of charge.

(*) Nella seduta del 15 dicembre 1990.

The material at each place $\boldsymbol{x}$ in R is specified by the following constitutive equations

$$\zeta = \hat{\zeta}(\boldsymbol{E}, \boldsymbol{H}), \qquad \boldsymbol{D} = -\frac{\partial \hat{\zeta}}{\partial \boldsymbol{E}}, \qquad \boldsymbol{B} = -\frac{\partial \hat{\zeta}}{\partial \boldsymbol{H}}, \tag{2.3}$$

where ζ is the enthalpy density. We assume that $\hat{\zeta}$ is a smooth function. We restrict our attention to materially homogeneous bodies.

It follows from (2.1) that

$$E_i = \varphi_{,i}, \qquad H_i = \psi_{,i}, \tag{2.4}$$

where $-\varphi$ is the potential of the electric field and $-\psi$ is the magnetic potential. The constitutive equation can be written in the form

$$\zeta = \hat{\zeta}(\varphi_{,i}, \psi_{,i}), \qquad D_i = -\frac{\partial \hat{\zeta}}{\partial \varphi_{,i}}, \qquad B_i = -\frac{\partial \hat{\zeta}}{\partial \psi_{,i}}. \tag{2.5}$$

We consider the boundary conditions

$$\varphi = \tilde{\varphi}, \qquad \psi = \tilde{\psi} \quad \text{on } \partial R, \tag{2.6}$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are prescribed functions.

The problem consists in finding the functions φ and ψ which satisfy the equations (2.2) and (2.5) in R and the boundary conditions (2.6) on ∂R.

3. Existence theorems

In order to derive existence theorems, we first recall some results established by Langenbach [7]. These results have been used in [8] to establish existence theorems for the first boundary-value problem of elastostatics.

Let Ω be a bounded region of $\mathbf{R}^n$, with the boundary surface $\partial\Omega$, and let $\mathcal{H}(\Omega)$ be a Hilbert space on Ω. The boundary $\partial\Omega$ is assumed to be sufficiently smooth for the divergence theorem to be applicable.

Let P be an operator P: $D(P) \to \mathcal{H}(\Omega)$, $D(P) \subset \mathcal{H}(\Omega)$, $D(P)$ being a linear subset, dense in $\mathcal{H}(\Omega)$. We assume that P has a linear Gâteaux differential on $\omega \subset D(P)$, *i.e.* there exists an operator (DP) such that (DP): $\omega \to L(D(P), \mathcal{H}(\Omega))$, and

$$\lim_{t \to 0} \frac{1}{t}[P(x + th) - P(x)] = (DP)(x)\,h, \qquad x \in \omega, \; h \in D(P),$$

where $L(D(P), \mathcal{H}(P))$ is the set of all linear operators from $D(P)$ in $\mathcal{H}(\Omega)$. The connection between P and (DP) is given by

$$Px - Px_0 = \int_0^1 (DP)(x_0 + t(x - x_0))(x - x_0)\,dt.$$

The operator P is monotone if for all $u, v \in D(P)$, $\langle Pu - Pv, u - v \rangle \geqslant 0$. The operator P is said to be strictly monotone if it is monotone and $\langle Pu - Pv, u - v \rangle = 0$ only for $u = v$.

We consider the equation

$$Pu = f, \tag{3.1}$$

with the linear and homogeneous boundary-value conditions

$$L_i u = 0, \qquad (i = 1, 2, \dots, p). \tag{3.2}$$

Let $D_0(P) = \{u \in D(P); L_i u = 0\}$, and $f \in \mathcal{H}(\Omega)$. The next three theorems are established in [6] (see also [7]).

Theorem 3.1. If

i) $D_0(P)$ and $D(P)$ are linear sets, and $D_0(P)$ is dense in $\mathcal{H}(\Omega)$;

ii) for all $u, h \in D(P)$, P has linear Gâteaux differential, and $(DP)(u)h$ is a continuous mapping of u in every two-dimensional hyperplane which contains the point u;

iii) $P(0) = 0$;

iv) for all $u \in D(P)$, $h, g \in D_0(P)$, we have $\langle (DP)(u)h, g\rangle = \langle (DP)(u)g, h\rangle$;

v) for all $u \in D(P)$, $h \in D_0(P)$, $h \neq 0$, $\langle (DP)(u)h, h\rangle > 0$,

then

α) if there exists a solution $u_0 \in D_0(P)$ of the eq. (3.1), it is unique and attains on $D_0(P)$ the minimum of the functional

$$\Phi(u) = \int_0^1 \langle P(tu), u\rangle dt - \langle f, u\rangle; \tag{3.3}$$

β) conversely, if an element of $D_0(P)$ attains on $D_0(P)$ the minimum of the functional (3.3), then it is a solution of (3.1).

This theorem allow us to associate a variational problem with the boundary-value problem considered in the section 2.

Theorem 3.2. If the condition (*v*) of Theorem 3.1 is changed into $\langle (DP)(u)h, h\rangle \geqslant c|h|^2$, $u \in D(P)$, $h \in D_0(P)$, $c = \text{const.}$, $c > 0$, then

i) the functional (3.3) is bounded below on $D_0(P)$;

ii) the functional (3.3) is strictly convex on $D_0(P)$;

iii) any minimizing sequence of the functional (3.3) is convergent in $\mathcal{H}(\Omega)$.

The limit of a minimizing sequence of the functional (3.3) is called generalized solution of the boundary-value problem (3.1), (3.2). It is known that the generalized solution is unique (cf. [6]).

Theorem 3.3. If there exists $u_0 \in D_0(P)$ such that $\langle (DP)(u)h, h\rangle \geqslant \geqslant c_1 \langle (DP)(u_0)h, h\rangle \geqslant c_2 \|h\|^2$, where c_1, c_2 are positive constants, then the generalized solution of (3.1), (3.2) is an element of the energetic space of the linear operator $(DP)(u_0)$.

We now consider the boundary-value problem (2.2), (2.5) and (2.6). The field equations can be written in the form

$$\left(\frac{\partial \hat{\zeta}}{\partial \varphi_{,i}}\right)_{,i} = -\rho, \qquad \left(\frac{\partial \hat{\zeta}}{\partial \psi_{,i}}\right)_{,i} = 0, \qquad \text{on } R. \tag{3.4}$$

Let V be the space of all two-dimensional vector fields $\boldsymbol{u} = (\varphi, \psi)$ defined on $\bar{R}$. We introduce the notations

$$M_1 \boldsymbol{u} = -\left(\frac{\partial \hat{\zeta}}{\partial \varphi_{,i}}\right)_{,i}, \qquad M_2 \boldsymbol{u} = -\left(\frac{\partial \hat{\zeta}}{\partial \psi_{,i}}\right)_{,i}, \tag{3.5}$$

on V, and the notations

$$M\boldsymbol{u} = (M_1 \boldsymbol{u}, M_2 \boldsymbol{u}), \qquad \boldsymbol{F} = (\rho, 0). \tag{3.6}$$

Clearly, the equations (3.4) take the form

$$M\boldsymbol{u} = \boldsymbol{F} \quad \text{on } R. \tag{3.7}$$

Let $\boldsymbol{v} \in V$, $\boldsymbol{v} = (v_1, v_2)$ such that $v_1 = \tilde{\varphi}$, $v_2 = \tilde{\psi}$ on ∂R.

We now define $\boldsymbol{w}$ and $A\boldsymbol{w}$ by

$$\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}, \qquad A\boldsymbol{w} \equiv (A_1 \boldsymbol{w}, A_2 \boldsymbol{w}) = M(\boldsymbol{w} + \boldsymbol{v}) - M\boldsymbol{v}. \tag{3.8}$$

The boundary value problem (2.2), (2.5) and (2.6) becomes

$$A\boldsymbol{w} = \boldsymbol{f} \quad \text{on } R, \tag{3.9}$$

$$\boldsymbol{w} = \boldsymbol{0} \quad \text{on } \partial R, \tag{3.10}$$

where $\boldsymbol{f} = \boldsymbol{F} - M\boldsymbol{v}$.

Let $L_2(R)$ be the Hilbert space of all vector fields $\boldsymbol{u} = (\varphi, \psi)$ whose components are square-integrable on R. The norm of this space is generated by the scalar product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_R (\varphi\varphi' + \psi\psi')\, dv,$$

where $\boldsymbol{u} = (\varphi, \psi)$, $\boldsymbol{v} = (\varphi', \psi')$.

Let $W_0^2(R)$ be the set of elements of $L_2(R)$ belonging to $C^2(R)$ and satisfying the condition (3.10).

We now consider the operator A: $W_0^2(R) \to L_2(R)$. In what follows we assume that $\boldsymbol{f} \in L_2(R)$.

THEOREM 3.4. If the function $\hat{\zeta}$ has continuous derivatives of second order with respect to $\boldsymbol{E}$ and $\boldsymbol{H}$, and satisfies the inequality

$$\int_R \left(\frac{\partial^2 \hat{\zeta}}{\partial E_i \partial E_j} G_i G_j + 2 \frac{\partial^2 \hat{\zeta}}{\partial E_i \partial H_j} G_i K_j + \frac{\partial^2 \hat{\zeta}}{\partial H_i \partial H_j} K_i K_j \right) dv > 0, \tag{3.11}$$

for all $\boldsymbol{w} = (\varphi, \psi)$, $\boldsymbol{g} = (\alpha, \beta) \in W_0^2(R)$, $g \neq 0$, $E_i = \varphi_{,i}$, $H_i = \psi_{,i}$, $G_i = \alpha_{,i}$, $K_j = \beta_{,j}$, then

α) if a solution $\boldsymbol{w}_0 \in W_0^2(R)$ of the equation (3.9) exists, it is unique and attains on $W_0^2(R)$ the minimum of the functional

$$\Phi(w) = \int_0^1 \langle A(tw), w\rangle dt - \langle f, w\rangle ; \tag{3.12}$$

β) conversely, if an element $w_0 \in W_0^2(R)$ attains on $W_0^2(R)$ the minimum of the functional (3.12), then it is a solution of (3.9).

PROOF. Let us show that the hypotheses of Theorem 3.1 are satisfied

i) $W_0^2(R)$ is a linear set, dense in $L_2(R)$ (see *e.g.* [9]).

ii) For all $w, g \in W_0^2(R)$, A has the linear Gâteaux differential

$$(DA_1)(w)g = \lim_{t\to 0} \frac{1}{t}[A_1(w+tg) - A_1(w)] =$$

$$= \lim_{t\to 0} \frac{1}{t}[M_1(w+tg+v) - M_1(w+v)] = -\left(\frac{\partial^2 \hat{\zeta}}{\partial E_i \partial E_j} G_j + \frac{\partial^2 \hat{\zeta}}{\partial E_i \partial H_j} K_j\right)_{,i},$$

$$(DA_2)(w)g = -\left(\frac{\partial^2 \hat{\zeta}}{\partial H_j \partial E_i} G_j + \frac{\partial^2 \hat{\zeta}}{\partial H_i \partial H_j} K_j\right)_{,i}.$$

It is easy to see that for a given g, $(DA)(w)g$ is a continuous mapping of w in every hyperplane which contains the point w.

iii) It follows from (3.8) that $A(0) = 0$.

iv) For all $w, g, h \in W_0^2(R)$, $h = (\gamma, \eta)$, $Q_i = \gamma_{,i}$, $S_i = \eta_{,i}$ we get

$$\langle (DA)(w)g, h\rangle = \tag{3.13}$$

$$= -\int_R \left[\left(\frac{\partial^2 \hat{\zeta}}{\partial E_i \partial E_j} G_j + \frac{\partial^2 \hat{\zeta}}{\partial E_i \partial H_j} K_j\right)_{,i}\gamma + \left(\frac{\partial^2 \hat{\zeta}}{\partial E_j \partial H_i} G_j + \frac{\partial^2 \hat{\zeta}}{\partial H_i \partial H_j} K_j\right)_{,i}\eta\right] dv =$$

$$= \int_R \left(\frac{\partial^2 \hat{\zeta}}{\partial E_i \partial E_j} G_j Q_i + \frac{\partial^2 \hat{\zeta}}{\partial E_i \partial H_j} K_j Q_i + \frac{\partial^2 \hat{\zeta}}{\partial E_j \partial H_i} G_j S_i + \frac{\partial^2 \hat{\zeta}}{\partial H_i \partial H_j} K_j S_i\right) dv =$$

$$= \langle (DA)(w)h, g\rangle .$$

v) It follows from (3.11) and (3.13), that $\langle (DA)(w)h, h\rangle > 0$, for all $w, h \in W_0^2(R)$, $h \neq 0$. This completes the proof.

THEOREM 3.5. Assume that (3.11) holds. If there exists a solution $u \in C^2(R)$ for the boundary-value problem (2.2), (2.5) and (2.6) then this solution is unique.

PROOF. Let Z be the set of all vector fields $u = (\varphi, \psi)$ of class $C^2(R)$ that satisfy the boundary conditions (2.6). We begin by establishing that the operator M defined by (3.5) and (3.6) is strictly monotone on Z. To prove this assertion we use the following result [9]: «if $D(P)$ is convex, then a sufficient condition for P to be strictly monotone on $D(P)$ is that the derivative

$$\frac{d}{dt}[\langle P(\boldsymbol{u}+t\boldsymbol{g}),\boldsymbol{g}\rangle]_{t=0}$$

exists and is strictly positive for all $\boldsymbol{u},\boldsymbol{v}\in D(P)$, $\boldsymbol{g}=\boldsymbol{v}-\boldsymbol{u}$, $\boldsymbol{g}\neq 0$».

Let $\boldsymbol{u},\boldsymbol{v}\in Z$, $0\leqslant t\leqslant 1$. It is easy to see that $t\boldsymbol{u}+(1-t)\,\boldsymbol{v}\in Z$. Next, with the aid of (3.11) and (3.13) we have

$$\frac{d}{dt}[\langle M(\boldsymbol{u}+t\boldsymbol{g}),\boldsymbol{g}\rangle]_{t=0}=\langle (DA)(\boldsymbol{u})\,\boldsymbol{g},\boldsymbol{g}\rangle>0,$$

for all $\boldsymbol{u},\boldsymbol{v}\in Z$, $\boldsymbol{g}=\boldsymbol{v}-\boldsymbol{u}$, $\boldsymbol{g}=\boldsymbol{0}$ on ∂R. Thus, we conclude that M is strictly monotone on Z. Then, for two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we have $\langle M\boldsymbol{u}_1-M\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2\rangle=\langle 0,\boldsymbol{u}_1-\boldsymbol{u}_2\rangle=0$, so that $\boldsymbol{u}_1=\boldsymbol{u}_2$.

The following proposition is a direct consequence of Theorem 3.2.

Theorem 3.6. Assume that the hypotheses of Theorem 3.4 hold, and

$$W(\boldsymbol{w})=\int_R\left(\frac{\partial^2\hat{\zeta}}{\partial E_i\partial E_j}G_iG_j+2\frac{\partial^2\hat{\zeta}}{\partial E_i\partial H_j}G_iK_j+\frac{\partial^2\hat{\zeta}}{\partial H_i\partial H_j}K_iK_j\right)dv\geqslant \geqslant c\int_R(\eta^2+\gamma^2)\,dv, \tag{3.14}$$

for all $\boldsymbol{w},\boldsymbol{g}\in W_0^2(R)$ with $\boldsymbol{w}=(\varphi,\psi)$, $\boldsymbol{g}=(\eta,\gamma)$, $E_i=\varphi_{,i}$, $H_i=\psi_{,i}$, $G_i=\eta_{,i}$, $K_i=\gamma_{,i}$, and $c=$ const., $c>0$. Then

i) the functional (3.12) is bounded below on $W_0^2(R)$;

ii) the functional (3.12) is strictly convex on $W_0^2(R)$;

iii) any minimizing sequence of the functional (3.12) is convergent in $L_2(R)$; and the limit is generalized solution of (3.9), (3.10);

iv) the generalized solution is unique.

Remark. If there exists a positive constant c' such that for all $\boldsymbol{w}=(\varphi,\psi)$, $\boldsymbol{g}=(\eta,\gamma)\in W_0^2(R)$,

$$\frac{\partial^2\hat{\zeta}}{\partial E_i\partial E_j}G_iG_j+2\frac{\partial^2\hat{\zeta}}{\partial E_i\partial H_j}G_iK_j+\frac{\partial^2\hat{\zeta}}{\partial H_i\partial H_j}K_iK_j>c'\,(\boldsymbol{G}^2+\boldsymbol{K}^2), \tag{3.15}$$

where $E_i=\varphi_{,i}$, $H_i=\psi_{,i}$, $G_i=\eta_{,i}$, $K_i=\gamma_{,i}$, then the condition (3.14) is satisfied. Indeed, by Friedrichs' inequality, there exists a real constant d^2 such that

$$d^{-2}\int_R(\boldsymbol{G}^2+\boldsymbol{K}^2)\,dv\geqslant\int_R(\eta^2+\gamma^2)\,dv. \tag{3.16}$$

Clearly, (3.15) and (3.16) imply (3.14).

The convexity of thermodynamical potentials for electromagnetic materials has been studied by Fabrizio (see [10, 11]).

The next result is an immediate consequence of Theorem 3.3.

Theorem 3.7. If there exists $\boldsymbol{w}_0 \in W_0^2(R)$ and the positive constants c_1, c_2 such that

$$W(\boldsymbol{w}) \geqslant c_1 W(\boldsymbol{w}_0) \geqslant c_2 \int_R (\eta^2 + \gamma^2)\, dv,$$

for all $\boldsymbol{w}, \boldsymbol{g} \in W_0^2(R)$, $\boldsymbol{g} = (\eta, \gamma)$, then the generalized solution of the boundary-value problem (3.9), (3.10) belongs to the energetic space of linear operator $(DA)(\boldsymbol{w}_0)$.

We note that a variational formulation for nonlinear dielectrics has been established by Morro [12].

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