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Adrian Constantin

Some observations on a Conti's result

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Equazioni differenziali ordinarie. — *Some observations on a Conti's result.* Nota (*) di Adrian Constantin, presentata dal Corrisp. R. Conti.

Abstract. — An extension of a result of R. Conti is given from which some integro-differential inequalities of the Gronwall-Bellman-Bihari type and a criterion for the continuation of solutions of a system of ordinary differential equations are deduced.

Key words: Ordinary differential equations; Comparison method; Inequalities.

Riassunto. — *Alcune osservazioni su un risultato di Conti.* Si dà una estensione di un risultato di R. Conti dal quale risultano alcune disuguaglianze integro-differenziali di tipo Gronwall-Bellman-Bihari e un criterio di prolungabilità delle soluzioni di un sistema di equazioni differenziali ordinarie.

1. R. Conti [3] has proved that bounds of the norms of the solutions of a system of ordinary differential equations can be obtained by comparison with a related first order differential equation. Most of the known explicit bounds, as well as criteria for global existence and boundedness or stability, can be obtained from such comparison theorems, together with a detailed analysis of the resulting first order differential equation.

In the following we want to show that the same arguments used by R. Conti [3, 4], provide, with minor modifications, a more general result which can be used to obtain a criterion for the continuation of solutions of a system of ordinary differential equations as well as to obtain some well known Gronwall-Bellman-Bihari like inequalities. This enables us also to give a more direct proof of the stability result for hidden variables obtained by A. Morro [8].

2. Consider the ordinary differential equation

$$x' = f(t, x) \tag{E}$$

where $f: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous. We suppose that for each $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ there is a unique solution $x(t)$ of this equation defined in a neighborhood of t_0 such that $x(t_0) = x_0$.

Lemma. *Let $g(t, u)$ be a continuous real valued function defined in $\mathbf{R} \times \mathbf{R}^n$ and $V(t, x)$ be a real valued locally Lipschitz function defined on $\mathbf{R} \times \mathbf{R}^n$ such that for all $t \in \mathbf{R}$, except perhaps a denumerable set, and any $x \in \mathbf{R}^n$ we have*

$$\min\Big\{\limsup_{h\to 0^+}\,[V(t+h, x+hf(t,x)) - V(t,x)]/h,\; \liminf_{h\to 0^+}\,[V(t,x) - V(t-h, x-hf(t,x))]/h\Big\} \le g(t, V(t,x)). \tag{1}$$

(*) Pervenuta all'Accademia il 4 settembre 1990.

If $u_0 \geqslant V(t_0, x_0)$ and if T_0^+ is the supremum of the values of t for which both the solution $x(t)$ of (E) with $x(t_0) = x_0$ and the maximum solution $u_0(t)$ of the comparison equation

$$u' = g(t, u) \tag{2}$$

with initial condition $u_0(t_0) = u_0$ are defined, we have

$$V(t, x(t)) \leqslant u_0(t), \qquad t \in [t_0, T_0^+). \tag{3}$$

PROOF. We closely follow Conti's arguments up to a certain point. We take t_1 arbitrary so that $t_0 < t_1 < T_0^+$; there is an $\varepsilon_1 > 0$, depending on t_1, so that the maximum solution $u_0(t, \varepsilon)$ of the equation $u' = g(t, u) + \varepsilon$ with initial condition $u_0(t_0, \varepsilon) = u_0$ is defined for $t_0 \leqslant t \leqslant t_1$ for every $0 < \varepsilon < \varepsilon_1$ and

$$\lim_{\varepsilon \to 0^+} u_0(t, \varepsilon) = u_0(t)$$

uniformly with respect to $t \in [t_0, t_1]$.

To show (3) it is sufficient to prove that for every $0 < \varepsilon_2 < \varepsilon_1$ there is an $0 < \varepsilon < \varepsilon_2$ such that

$$V(t, x(t)) \leqslant u_0(t, \varepsilon), \qquad t_0 \leqslant t \leqslant t_1. \tag{4}$$

From this, letting $\varepsilon \to 0^+$, it follows that $V(t, x(t)) \leqslant u_0(t)$ for $t_0 \leqslant t \leqslant t_1$. Since t_1 was taken arbitrary, we have that (3) holds for $t_0 \leqslant t < T_0^+$.

Let us suppose now that for a given $t_1 \in (t_0, T_0^+)$ the relation (4) does not hold. There is an ε_2, $0 < \varepsilon_2 < \varepsilon_1$ such that for every $0 < \varepsilon < \varepsilon_2$ there are values of $t \in (t_0, t_1)$ such that $V(t, x(t)) > u_0(t, \varepsilon)$. For each such ε let t_ε be the infimum of the values $t \in (t_0, t_1)$ with this property. From the continuity of $V(t, x(t))$ we deduce that $V(t_\varepsilon, x(t_\varepsilon)) = u_0(t_\varepsilon, \varepsilon)$ and there are numbers $h > 0$, as small as we wish, so that $V(t_\varepsilon + h, x(t_\varepsilon + h)) > u_0(t_\varepsilon + h, \varepsilon)$ from which we deduce that

$$\limsup_{h \to 0^+} [V(t_\varepsilon + h, x(t_\varepsilon + h)) - V(t_\varepsilon, x(t_\varepsilon))]/h \geqslant \left.\frac{du(t, \varepsilon)}{dt}\right|_{t = t_\varepsilon}.$$

Since for every $h > 0$ sufficiently close to 0, we have $V(t_\varepsilon - h, x(t_\varepsilon - h)) < u_0(t_\varepsilon - h, \varepsilon)$ we deduce that

$$\liminf_{h \to 0^+} [V(t_\varepsilon, x(t_\varepsilon)) - V(t_\varepsilon - h, x(t_\varepsilon - h))]/h \geqslant \left.\frac{du(t, \varepsilon)}{dt}\right|_{t = t_\varepsilon}.$$

Since V is locally Lipschitz on $\mathbf{R} \times \mathbf{R}^n$ we have that

$$\limsup_{h \to 0^+} [V(t_\varepsilon + h, x(t_\varepsilon + h)) - V(t_\varepsilon, x(t_\varepsilon))]/h =$$
$$= \limsup_{h \to 0^+} [V(t_\varepsilon + h, x(t_\varepsilon) + hf(t_\varepsilon, x(t_\varepsilon))) - V(t_\varepsilon, x(t_\varepsilon))]/h,$$

$$\liminf_{h \to 0^+} [V(t_\varepsilon, x(t_\varepsilon)) - V(t_\varepsilon - h, x(t_\varepsilon - h))]/h =$$
$$= \liminf_{h \to 0^+} [V(t_\varepsilon, x(t_\varepsilon)) - V(t_\varepsilon - h, x(t_\varepsilon) - hf(t_\varepsilon, x(t_\varepsilon)))]/h,$$

as T. Yoshizawa proved in [14] and so we deduce that

(5) $$\min\{\limsup_{h\to 0^+}[V(t_\varepsilon+h,x(t_\varepsilon)+hf(t_\varepsilon,x(t_\varepsilon)))-V(t_\varepsilon,x(t_\varepsilon))]/h,$$
$$\liminf_{h\to 0^+}[V(t_\varepsilon,x(t_\varepsilon))-V(t_\varepsilon-h,x(t_\varepsilon)-hf(t_\varepsilon,x(t_\varepsilon)))]/h\}\geqslant \left.\frac{du(t,\varepsilon)}{dt}\right|_{t=t_\varepsilon}.$$

We will show that if $\varepsilon'\neq\varepsilon''$ then $t_{\varepsilon'}\neq t_{\varepsilon''}$.

Let us suppose that there are $0<\varepsilon'<\varepsilon''$ such that $t_{\varepsilon'}=t_{\varepsilon''}=t'$. We then have that $u_0(t',\varepsilon')=u_0(t',\varepsilon'')=V(t',x(t'))$. Since
$$\left.\frac{d[u_0(t,\varepsilon'')-u_0(t,\varepsilon')]}{dt}\right|_{t=t_0}=\varepsilon''-\varepsilon'>0,$$
we deduce that for every $t>t_0$ near t_0 we have $u_0(t,\varepsilon'')-u_0(t,\varepsilon')>0$.

If the preceding relation does not hold for all values of $t\in(t_0,t_1)$, there is a point $t_2\in(t_0,t_1)$ such that

(6) $$u_0(t_2,\varepsilon'')-u_0(t_2,\varepsilon')=0$$

and $u_0(t,\varepsilon'')-u_0(t,\varepsilon')>0$ for $t\in(t_0,t_2)$. We deduce that there are numbers $h>0$, as small as we wish, such that
$$\left.\frac{d(u_0(t,\varepsilon'')-u_0(\varepsilon'))}{dt}\right|_{t=t_2-h}\leqslant 0.$$

On the other hand we have from (6) that
$$\left.\frac{d(u_0(t,\varepsilon'')-u_0(t,\varepsilon'))}{dt}\right|_{t=t_2}=\varepsilon''-\varepsilon'>0$$
and so the preceding two relations are contradictory since the function $d(u_0(t,\varepsilon'')+-u_0(t,\varepsilon'))/dt$ is continuous in $t=t_2$.

We deduce that if $\varepsilon'\neq\varepsilon''$, then $t_{\varepsilon'}\neq t_{\varepsilon''}$

Since (1) holds for all $t\in(t_0,t_1)$ except perhaps a denumerable set, we have that there exists at least a t_{ε_0} such that (1) and (5) hold simultaneously. Since
$$\left.\frac{du(t,\varepsilon_0)}{dt}\right|_{t=t_{\varepsilon_0}}=g(t_{\varepsilon_0},u_0(t_{\varepsilon_0},\varepsilon_0))+\varepsilon_0=g(t_{\varepsilon_0},V(t_{\varepsilon_0},x(t_{\varepsilon_0})))+\varepsilon_0>$$
$$>g(t_{\varepsilon_0},V(t_{\varepsilon_0},V(t_{\varepsilon_0},x(t_{\varepsilon_0}))))$$
we deduce that (1) and (5) cannot hold simultaneously.

THEOREM 1. *If the hypotheses of the Lemma are satisfied for a function* $V:\mathbf{R}\times\times\mathbf{R}^n\to\mathbf{R}$ *that satisfies also the condition*

(7) $$V(t,x)\to\infty \quad \textit{as} \quad \|x\|\to\infty \quad \textit{for each fixed } t,$$

and if the maximum solution of the comparison equation (2) exists in the future (i.e., *it exists for all* $t\geqslant t_0$), *then every solution of* (E) *exists in the future.*

PROOF. If the solution $x(t)$ of (E) with $x(t_0)=x_0$ fails to exist in the future, we have that [5] $|x(t)|\to\infty$ as $t\to T_0^+$, for some finite T_0^+.

From the Lemma we deduce, in a way similar to that of A. Strauss [11], that $V(t,x(t))\to\infty$ as $t\to T_0^+$.

Since $V(t,x(t)) \leqslant u_0(t)$, $t_0 \leqslant t < T_0^+$, we have a contradiction to the existence assumption on the maximum solution $u_0(t)$ of (2).

REMARK. If condition (7) is replaced by

$$(7^*) \qquad V(t,x) \to \infty \quad \text{as} \quad \|x\| \to \infty$$

uniformly in t for t in any compact interval, and if condition (1) is replaced by

$$(1^*) \qquad \limsup_{h \to 0^+} [V(t+h, x+hf(t,x)) - V(t,x)]/h \leqslant g(t, V(t,x))$$

for every $(t,x) \in \mathbf{R} \times \mathbf{R}^n$ we obtain a result of J. P. La Salle and S. Lefschetz [9]. Furthermore, J. Kato and A. Strauss [6] have shown that if all solutions of (E) exist in the future, then there exists a locally Lipschitz $V: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ satisfying (1*) and (7*). For functions that satisfy only (7) and (1*), R. Conti [4] proved that all solutions of (E) exist in the future. In [11] A. Strauss shows that Conti's conditions are less restrictive than (1*) and (7*) and establish conditions under which (7) and (7*) are equivalent. In our Lemma we replaced condition (7*) by (7) and condition (1*) by the less restrictive condition (1).

3. In a way which is similar to the one followed for proving the Lemma, we can prove that:

THEOREM 2. *Let $g(t,u)$ be a continuous real valued function defined in $\mathbf{R} \times \mathbf{R}^n$ and let $u_0(t)$ be the maximum solution of the equation*

$$(2) \qquad u' = g(t,u)$$

with initial condition $u_0(t_0) = u_0$ defined on $[t_0, T_0^+)$ where T_0^+ is the supremum of the values t for which $u_0(t)$ is defined. Let $x = (x_1, x_2, \ldots, x_n): [t_0, T_0^+) \to \mathbf{R}^n$ be a differentiable function and $V(t,x)$ be a real valued continuous function defined on $\mathbf{R} \times \mathbf{R}^n$ such that $V(t_0, x(t_0)) \leqslant u_0$ and for all $t \in (t_0, T_0^+)$ except perhaps a denumerable set

$$(8) \qquad \min\Big\{\limsup_{h \to 0^+} [V(t+h, x(t+h)) - V(t,x(t))]/h,$$
$$\liminf_{h \to 0^+} [V(t,x(t)) - V(t-h, x(t-h))]/h\Big\} \leqslant g(t, V(t,x(t))).$$

We have then that $V(t,x(t)) \leqslant u_0(t)$, $t \in [t_0 T_0^+)$.

Theorem 2 can be used in proving stability properties of solutions of a system of ordinary differential equations as well as to obtain some well known Gronwall-Bellman-Bihari like inequalities which can be used in proving the unicity of solutions of ordinary differential equations.

Let us suppose that $n = 1$, $V(t,x) = x$ and that (8) holds for all $t \in [t_0, T_0^+)$. We obtain

COROLLARY 1 (J. S. W. Wong [13]). *We consider the first order equation*

$$(9) \qquad u' = f(t,u), \qquad u(0) = c,$$

where $f(t,u)$ is continuous in the region $S_1 = [0, \infty) \times (-\infty, \infty)$ and its corresponding *differential inequality*

$$v' \leqslant f(t, v), \qquad v(0) = c, \tag{10}$$

also defined in the region S_1. *If* $u(t)$ *is the maximum solution of* (9), *then it follows that* $v(t) \leqslant u(t)$ *for each* $t \geqslant 0$.

COROLLARY 2 (B. Viswanathan [12]). *If* $w(t) \leqslant c(t) + \int_0^t f(s, w(s))\, ds$ *where* $f(t, w)$ *is continuous and monotonic increasing in* w *in the region* S_1 *and* $w(t)$, $c(t)$ *are continuous for* $t \geqslant 0$, *then* $w(t) \leqslant c(t) + u(t)$ *where* $u(t)$ *is the maximum solution of the equation* $u' = f(t, u + c(t))$ *with initial condition* $u(0) = w(0)$.

PROOF. Let $y(t) = \int_0^t f(s, w(s))\, ds$. We have then that $y'(t) = f(t, w(t)) \leqslant f(t, y(t) + c(t))$ by the monotonicity of f. The conclusion follows from Theorem 2.

As particular cases of Corollary 2 we have Corollaries 3 and 4:

COROLLARY 3. (T. H. Gronwall-R. Bellman). *Let* $v(t)$, $c(t)$ *and* $g(t)$ *be real valued continuous functions defined on* $[0, \infty)$ *and let* g *be nonnegative on this interval. If*

$$v(t) \leqslant c(t) + \int_0^t g(s)\, v(s)\, ds, \qquad 0 \leqslant t,$$

then

$$v(t) \leqslant c(t) + \int_0^t g(s)\, c(s) \exp\left(\int_s^t g(u)\, du \right) ds, \qquad 0 \leqslant t.$$

PROOF. We take $f(t, w) = g(t)\, w$ in Corollary 2. The solution of the equation $u'(t) = g(t)\, (c(t) + u(t))$, $u(0) = 0$, $t \geqslant 0$, is

$$u(t) = \int_0^t g(s)\, c(s) \exp\left(\int_s^t g(u)\, du \right) ds, \qquad t \geqslant 0,$$

and so we deduce Corollary 3.

COROLLARY 4 (I. Bihari [2]). *Let* v, h *be positive, continuous functions in* $[t_0, t_1]$, *and let* a, b *be nonnegative constants; further, let* g *be a positive nondecreasing function on* $[0, \infty)$. *Then the inequality*

$$v(t) \leqslant a + b \int_{t_0}^t h(s)\, g(v(s))\, ds, \qquad t_0 \leqslant t \leqslant d,$$

implies the inequality

$$v(t) \leqslant G^{-1}\left(G(a) + b \int_{t_0}^t h(s)\, ds \right), \qquad t_0 \leqslant t \leqslant d',$$

where

$$G(x)=\int_{x_0}^{x}\frac{ds}{g(s)},\qquad x_0>0,\quad x>0,$$

and d' is defined so that $G(a)+b\int_{t_0}^{t}h(s)\,ds$ lies within the domain of definition of G^{-1}, for $t_0\le t\le d'$.

PROOF. Let us take $f(s,w)=bh(s)\,g(w)$. The equation $u'=bh(t)\,g(u+a)$, $u(t_0)=0$, has the solution $u=G^{-1}\left(G(a)+b\int_{t_0}^{t}h(s)\,ds\right)-a$ and so the conclusion follows.

COROLLARY 5 (B. G. Pachpatte [10]). *Let $u,\ v,\ w\in\mathcal{C}(I,\mathbf{R}_+)$, $f,\ h\in\mathcal{C}(\mathbf{R}_+,\mathbf{R}_+)$, f strictly increasing and h be nondecreasing, $k\in\mathcal{C}(I\times I,\mathbf{R}_+)$, and suppose further that the inequality*

$$f(u(t))\le v(t)+w(t)\,h\left[c(t)+\int_{t_0}^{t}k(\tau,s)\,g(s,u(s)),Tu(s)\,ds\right]\tag{11}$$

is satisfied for all $t\in I$, $0\le t_0\le\tau\le t$; where $g(t,u,v)\in\mathcal{C}(I\times\mathbf{R}_+\times\mathcal{C}(I,\mathbf{R}_+),\mathbf{R}_+)$ and monotone increasing in u and v for each fixed $t\in I$, and T is a continuous operator on $\mathcal{C}(I,\mathbf{R}_+)$ such that $u(t)\le v(t)$, $0\le t\le t_1$, $t_1\in I$, implies $Tu\le Tv$ for $t=t_1$. Then for $t\in I_0$

$$u(t)\le f^{-1}(v(t)+w(t)\,h(c(t)+r(t))),\tag{12}$$

where $I_0=\{t\in I;\ v(t)+w(t)\,h(c(t)+r(t))\in\operatorname{dom}(f^{-1})\}$ and $r(t)$ is the maximum solution of

$$r'(t)=k(\tau,t)\,g\big(t,f^{-1}(v(t)+w(t)\,h(c(t)+r(t))),\ T\big(f^{-1}(v(t)+w(t)\,h(c(t)+r(t)))\big)\big)\tag{12'}$$

with $r(t_0)=0$, existing on $I_0\subset I$.

PROOF. Denoting

$$z(t)=c(t)+\int_{t_0}^{t}k(\tau,s)\,g(s,u(s),\ Tu(s))\,ds,\qquad z(t_0)=c(t_0),\tag{13}$$

we have that

$$f(u(t))\le v(t)+w(t)\,h(z(t)).\tag{14}$$

From the monotone property of g we deduce, denoting $m(t)=z(t)-c(t)$ in (13), that

$$m'(t)\le k(\tau,t)\,g\big(t,f^{-1}(v(t)+w(t)\,h(c(t)+m(t))),\ T\big(f^{-1}(v(t)+w(t)\,h(c(t)+m(t)))\big)\big).$$

Applying Corollary 1 we deduce that $m(t)\le r(t)$, $t\in I_0$, where $r(t)$ is the maximum solution of (12) such that $r(t_0)=m(t_0)=0$. From this the desired result follows.

REMARK. If the operator T is defined by $Tu=u$, then (12) reduces to the inequality studied by V. Lakshmikantham [7].

As a particular case of Theorem 2 we deduce also the mean value theorem for real valued fuctions [15, 16].

MEAN VALUE THEOREM. *Let $I=[a,b]$ be a compact interval in* $\mathbf{R}$, φ *a continuous mapping of I into* $\mathbf{R}$, *such that there is a denumerable subset D of I with the property that at every point $t\in I-D$, φ has a right derivative with respect to I and $m\leqslant\varphi'_d(t)\leqslant M$. Then $m(b-a)\leqslant\varphi(b)-\varphi(a)\leqslant M(b-a)$.*

PROOF. We take in Theorem 2 $V(t,x)=\varphi(t)$, $g(t,x)=M$. The maximum solution $u_0(t)$ of the equation $u'=M$ with initial condition $u_0(a)=\varphi(a)$ defined on $[a,b]$ is $u_0(t)=M(t-t_0)+\varphi(a)$ and so we deduce that $\varphi(b)-\varphi(a)\leqslant M(b-a)$.

For the second part we take $V(t,x)=-\varphi(t)$, $g(t,x)=-m$ in Theorem 2 and in a similar way we obtain that $m(b-a)\leqslant\varphi(b)-\varphi(a)$.

As pointed out by N. Bourbaki [16] the result of the mean value theorem is no longer valid if D is a nondenumerable subset of I.

4. A. Morro [8] has introduced a precise mathematical structure of the hidden variable model. Also a new Gronwall-like inequality and its application to the asymptotic stability of the solution of the evolution equation is given.

In the proof of the asymtotic stability, A. Morro [8] used an estimate of the norm h of the solution for the difference of two evolution equations. This estimate is of the form

$$(15)\qquad h(t)\leqslant \exp(-m(t-t_0))\,h(t_0)+\varepsilon\int_{t_0}^{t}\exp(-m(t-s))\,\omega(s)\,ds+ \\ +\delta\int_{t_0}^{t}\exp(-m(t-s))\,h(s)\,ds,\qquad t\geqslant t_0,$$

where m, δ, $\varepsilon\in\mathbf{R}_+-\{0\}$ and $\omega:\mathbf{R}\to\mathbf{R}_+$ is a piecewise continuous function.

We will show that an application of Theorem 2 yields, in a more direct way, a bound that can be used as in [8] to give new insights into the asymptotic stability of the evolution equation for hidden variables and to lend precision to the assertion that the hidden variables are independent of the present value of the physical variables.

Let us denote $y(t)=h(t)\exp(mt)$, $t\geqslant t_0$.

We have then that

$$y(t)\leqslant\exp(mt_0)\,h(t_0)+\varepsilon\int_{t_0}^{t}\exp(ms)\,\omega(s)\,ds+\delta\int_{t_0}^{t}y(s)\,ds.$$

We consider the equation

$$u'=g(t,u),\qquad t\geqslant t_0,$$

where $g(t,u)=\varepsilon\exp(mt)\,\omega(t)+\delta u$.

The maximum solution of this equation with initial value $u_0(t_0)=h(t_0)\exp(mt_0)$ is given by

$$u_0(t) = \exp(\delta t)\, b(t_0) \exp((m-\delta)\, t_0) + \int_{t_0}^{t} \exp((m-\delta)\, s)\, \omega(s)\, ds.$$

Applying the preceding results we deduce that $y(t) \leq u_0(t)$, $t \geq t_0$, and since $y(t) = b(t) \exp(mt)$, $t \geq t_0$, we have

$$b(t) \leq b(t_0) \exp(-(m-\delta)(t-t_0)) + \int_{t_0}^{t} \exp(-(m-\delta)(t-s))\, \omega(s)\, ds. \tag{16}$$

Letting $\omega(t) = 0$ we obtain that

$$b(t) \leq \exp(-(m-\delta)(t-t_0))\, b(t_0). \tag{17}$$

If $m - \delta > 0$ we have that (17) ensures the asymptotic stability of the solution to the evolution equation for the hidden variables. The bound obtained in (16) can be used in studying the independence property of the hidden variables of the present value of the physical variables (see [8]).

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Universitatea din Timişoara
Bulev. V. Pârvan, 4
Timişoara 1900 (Romania)