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Nonvariational basic parabolic systems of second order


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Analisi matematica. — Non variational basic parabolic systems of second order. Nota(*) del Corrisp. SERGIO CAMPAANATO.

ABSTRACT. — \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), of class \( C^2 \) and \( T>0 \). In the cylinder \( \Omega = \Omega \times (0,T) \) we consider non variational basic operator \( a(H(u)) - \partial u / \partial t \) where \( a(\xi) \) is a vector in \( \mathbb{R}^N \), \( N \geq 1 \), which is continuous in \( \xi \) and satisfies the condition (A). It is shown that \( \forall f \in L^2(\Omega) \) the Cauchy-Dirichlet problem \( u \in W^{2,1}_0(\Omega), a(H(u)) - \partial u / \partial t = f \) in \( \Omega \), has a unique solution. It is further shown that if \( u \in W^{2,1}(\Omega) \) is a solution of the basic system \( a(H(u)) - \partial u / \partial t = 0 \) in \( \Omega \), then \( H(u) \) and \( \partial u / \partial t \) belong to \( H^1_{\text{loc}}(\Omega) \). From this the Hölder continuity in \( \Omega \) of the vectors \( u \) and \( Du \) are deduced respectively when \( n \leq 4 \) and \( n = 2 \).

KEY WORDS: Nonlinear non variational systems; (A) condition; Existence theorem.

RIASSUNTO. — Sistemi parabolici base non variazionali del 2° ordine. \( \Omega \) è un aperto limitato di \( \mathbb{R}^n \) di classe \( C^2 \) e \( T>0 \). Nel cilindro \( \Omega = \Omega \times (0,T) \) si considera l'operatore non variazionale base \( a(H(u)) - \partial u / \partial t \) dove \( a(\xi) \) è un vettore di \( \mathbb{R}^N \), \( N \geq 1 \), continuo in \( \xi \) il quale verifica la condizione (A). Si dimostra che \( \forall f \in L^2(\Omega) \) il problema di Cauchy-Dirichlet \( u \in W^{2,1}_0(\Omega), a(H(u)) - \partial u / \partial t = f \) in \( \Omega \), ha una e una sola soluzione. Si dimostra inoltre che se \( u \in W^{2,1}(\Omega) \) è una soluzione del sistema base \( a(H(u)) - \partial u / \partial t = 0 \) in \( \Omega \), allora \( H(u) \) e \( \partial u / \partial t \) appartengono ad \( H^1_{\text{loc}}(\Omega) \). Se ne deduce l'holderianità in \( \Omega \) dei vettori \( u \) e \( Du \) rispettivamente quando \( n \leq 4 \) e \( n = 2 \).

1. — INTRODUCTION

Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), \( n \geq 1 \), of class \( C^2 \) and let \( x \) be a generic point in it. \( N \) is an integer \( \geq 1 \) and \( Q \) is the cylinder \( \Omega \times (0,T) \) with \( T>0 \), \( X = (x, t) \) a point of \( \mathbb{R}^n \times \mathbb{R} \), and \( u(X) \) is a vector \( \mathbb{Q} \rightarrow \mathbb{R}^N \). We shall set \( Du = (D_1 u, \ldots, D_n u) \), \( H(u) = \{D_{ij} u\} \), \( i, j = 1, \ldots, n \). \( Du \) is a vector in \( \mathbb{R}^{nN} \) and \( H(u) \) is an element of \( \mathbb{R}^{nN} \), that is, it is an \( n \times n \) matrix of vectors in \( \mathbb{R}^N \). If \( \tau \in \mathbb{R}^{nN} \) we set as usual

\[
\text{Tr. } \tau = \sum_{i=1}^{n} \tau_{ii}.
\]

It is well known that \( H^2 \cap H^1_0(\Omega) \) is a Hilbert space with the norm \( \|H(u)\|_{L^2(\Omega)} \).

We shall denote by \( W^{2,1}(\Omega) \) and \( W^{2,1}_0(\Omega) \), respectively, the Hilbert spaces of vectors \( u: \Omega \rightarrow \mathbb{R}^N \) such that

(1.1) \( u \in L^2(0,T,H^2(\Omega)) \), \( \partial u / \partial t \in L^2(Q) \)

and

(1.2) \( u \in L^2(0,T,H^2 \cap H^1_0(\Omega)) \), \( \partial u / \partial t \in L^2(Q), \quad u(x,0) = 0 \) in \( \Omega \).

We shall provide \( W^{2,1}_0(\Omega) \) with the norm

(1.3) \( \|u\|_{(2)}^2 = \int_{\Omega} \left[ \|H(u)\|^2 + \alpha^2 \left\| \frac{\partial u}{\partial t} \right\|^2 \right] \, dx \, dt \)

where \( \alpha \) is a positive constant.

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Let $a(\xi)$ be a vector in $\mathbb{R}^N$, continuous onto $\mathbb{R}^{N^2}$ such that $a(0) = 0$. Suppose that the vector $a(\xi)$ satisfies the following condition

(A) There exist three positive constants $\alpha, \gamma$ and $\delta$ with $(\gamma + \delta) < 1$ such that, $\forall \xi, \tau, \in \mathbb{R}^{N^2}$ we have

\begin{equation}
\|\text{Tr. } \tau - \alpha [a(\tau + \xi) - a(\xi)]\|_N \leq \gamma \|\tau\| + \delta \|\text{Tr. } \tau\|_N.
\end{equation}

One shows that if the vector $a(\xi)$ is of class $C^1$ with its derivatives

$$
\partial a(\xi)/\partial \xi_{ij} = \{\partial^2 a(\xi)/\partial \xi_{ij}^2\} \quad b, k = 1, \ldots, N,
$$

bounded, then the fact that $a(\xi)$ satisfies the condition (A) implies that $a(\xi)$ is elliptic (see [5]).

It follows, in particular, from the condition (1.4) that $\forall \tau \in \mathbb{R}^{N^2}$ we have

\begin{equation}
\|a(\tau)\| \leq c(n) \|\tau\|/\alpha.
\end{equation}

We shall consider the basic operator

\begin{equation}
a(H(u)) - \partial u/\partial t
\end{equation}

and consider the Cauchy-Dirichlet problem:

Given $f \in L^2(Q)$ to find $u \in W^{2,1}_0(Q)$ such that

\begin{equation}
a(H(u)) - \partial u/\partial t = f \quad \text{in } Q.
\end{equation}

We shall prove the following

**Theorem 1.1.** If $Q$ is of class $C^2$ and is convex and the vector $a(\xi)$ satisfies the condition (A), $\forall f \in L^2(Q)$ the Cauchy-Dirichlet problem (1.7) has a unique solution.

If $X_0 = (x^0, t_0)$ and $\sigma > 0$ we set $B(x^0, \sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\}$, $Q(X_0, \sigma) = B(x^0, \sigma) \times (t_0 - \sigma^2, t_0)$.

We say that $Q(X_0, \sigma) \subseteq Q$ if $B(x^0, \sigma) \subseteq Q$ and $\sigma^2 < t_0 \leq T$.

Let $u \in W^{2,1}(Q)$ be a solution of the basic system

\begin{equation}
a(H(u)) - \partial u/\partial t = 0 \quad \text{in } Q.
\end{equation}

We shall prove the following

**Theorem 1.2.** If the vector $a(\xi)$ satisfies the condition (A) then $H(u) \in H^{1}_{\text{loc}}(Q)$, $\partial u/\partial t \in H^{1}_{\text{loc}}(Q)$ and $\forall Q(2\sigma) \subseteq Q$ we have the following estimates

\begin{equation}
\int_{Q(\sigma)} \left[ \|DH(u)\|^2 + \left\|D \frac{\partial u}{\partial t}\right\|^2 \right] dx \, dt \leq c(\sigma) \int_{Q(2\sigma)} \left[ \|Du\|^2 + \|H(u)\|^2 \right] dx \, dt;
\end{equation}

\begin{equation}
\int_{Q(\sigma)} \left[ \left\|\frac{\partial}{\partial t} H(u)\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2 \right] dx \, dt \leq c(\sigma) \int_{Q(2\sigma)} \left[ \left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial Du}{\partial t}\right\|^2 \right] dx \, dt.
\end{equation}

In view of the Sobolev imbedding theorem it follows from the implication of the Theorem 1.2 that, if $u$ is the solution of the system (1.8), then the vector $Du$ is Hölder continuous in $Q$ if $n = 2$, the vector $u$ is Hölder continuous in $Q$ if $n \leq 4$. 

2. - Preliminaries

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two real Banach spaces, eventually two finite dimensional Hilbert spaces. Let $A$ and $B$ be two mappings $\mathcal{B}_1 \rightarrow \mathcal{B}_2$.

**Definition 2.1.** We shall say that $A$ is near $B$ if there exist two positive constants $\alpha$ and $K$, with $0 < K < 1$, such that $\forall u, v \in \mathcal{B}_1$ we have

\begin{equation}
\| B(u) - B(v) - \alpha [A(u) - A(v)] \|_{\mathcal{B}_2} \leq K \| B(u) - B(v) \|_{\mathcal{B}_2}.
\end{equation}

We have the following

**Theorem 2.1.** If $B: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a bijection and $A$ is near $B$ with constants $\alpha$ and $K$ then $A$ is also a bijection and $\forall u \in \mathcal{B}_1$ we have the estimate

\begin{equation}
\| B(u) - B(0) \|_{\mathcal{B}_2} \leq \alpha \| A(u) - A(0) \|_{\mathcal{B}_2} / (1 - K)
\end{equation}

([3] Theorem 2.1).

Since $\Omega$ is of class $C^2$ and is convex we have the following estimate due to C. Miranda and G. Talenti: $\forall u \in H^2 \cap H_0^1(\Omega)$

\[ \int_\Omega \| H(u) \|^2 \, dx \leq \int_\Omega \| \Delta u \|^2 \, dx. \]

As a consequence, we have, if $\Omega$ is of class $C^2$ and is convex and if $\Omega = \Omega \times (0, T)$, the following

**Lemma 2.1.** For each $\alpha > 0$ and $\forall u \in W_0^{2,1}(\Omega)$

\begin{equation}
\| u \|_{(a)}^2 \leq \int_\Omega \left[ \| \Delta u \|^2 + \alpha^2 \left\| \frac{\partial u}{\partial t} \right\|^2 \right] \, dx \, dt.
\end{equation}

We have the following Lemma ([6] Lemma 1.1)

**Lemma 2.2.** For each $u \in W_0^{2,1}(\Omega)$ the following estimate holds

\begin{equation}
\int_\Omega \left( \Delta u \left| \frac{\partial u}{\partial t} \right| \right)_N \, dx \, dt \leq 0.
\end{equation}

As a consequence we obtain

**Lemma 2.3.** If $\Omega$ is of class $C^2$ and is convex then $\forall u \in W_0^{2,1}(\Omega)$ and $\forall \alpha > 0$ we have

\begin{equation}
\| u \|_{(a)}^2 \leq \int_\Omega \left( \Delta u - \alpha \frac{\partial u}{\partial t} \right) \, dx \, dt.
\end{equation}

This is a trivial consequence of (2.3) and (2.4).
**Lemma 2.4.** If $\Omega$ is of class $C^2$ and is convex, $\forall u \in W^{2,1}_0(\Omega)$ and $\forall \alpha > 0$ we have

$$\int_Q \|\Delta u\|^2_{\text{ess}} \, dx \, dt \leq \int_Q \|\Delta u - \alpha \frac{\partial u}{\partial t}\|^2_N \, dx \, dt. \tag{2.6}$$

This is a trivial consequence of the estimate (2.4).

### 3. - Proof of the Theorem 1.1

In view of the estimate (1.5) the operator

$$a(H(u)) - \frac{\partial u}{\partial t} \tag{3.1}$$

maps $W^{2,1}_0(\Omega)$ into $L^2(\Omega)$. On the other hand it is well known that the linear operator

$$\Delta u - \alpha \frac{\partial u}{\partial t} \tag{3.2}$$

is an isomorphism of $W^{2,1}_0(\Omega)$ into $L^2(\Omega)$. We choose as $\alpha$ in (3.2) to be exactly the positive constant that appears in the condition (A) for the vector $a(\xi)$. In virtue of Theorem 2.1, in order to show that, $\forall f \in L^2(\Omega)$, the Cauchy-Dirichlet problem (1.7) has a unique solution one is reduced to show that the operator (3.1) is near the operator (3.2), which means that we should show that there exists a constant $K \in (0, 1)$ such that, $\forall u, v \in W^{2,1}_0(\Omega)$, we have

$$\int_Q \|\Delta (u - v) - \alpha [a(H(u)) - a(H(v))]\|^2_{\text{ess}} \, dx \, dt \leq K^2 \int_Q \|\Delta (u - v) - \alpha \frac{\partial (u - v)}{\partial t}\|^2_N \, dx \, dt. \tag{3.3}$$

On the other hand, it follows from the condition (A) that, $\forall \varepsilon > 0$ and $\forall \xi, \tau \in \mathbb{R}^{2N}$, we have $\|\text{Tr. } \tau - \alpha [a(\tau + \xi) - a(\xi)]\|^2 \leq (1 + \varepsilon) \gamma^2 \|\tau\|^2 + (1 + 1/\varepsilon) \delta^2 \|	ext{Tr. } \tau\|^2$. Assuming $\varepsilon = \delta / \gamma$ we find that

$$\|\text{Tr. } \tau - \alpha [a(\tau + \xi) - a(\xi)]\|^2 \leq \gamma (\gamma + \delta) \|\tau\|^2 + \delta (\gamma + \delta) \|	ext{Tr. } \tau\|^2. \tag{3.4}$$

We observe that

$$\gamma (\gamma + \delta) + \delta (\gamma + \delta) = (\gamma + \delta)^2 < 1. \tag{3.5}$$

It now follows, from the estimate (3.4) and the Lemmas 2.3 and 2.4, that,
∀u, v ∈ W^{2,1}_0(Q), we have
\[
\int_Ω \|Δ(u - v) - α[a(H(u)) - a(H(v))]\|^2 dx dt \leq \\
≤ γ(γ + δ) \int_Ω \|H(u - v)\|^2 dx dt + δ(γ + δ) \int_Ω \|Δ(u - v)\|^2 dx dt \leq \\
≤ γ(γ + δ) \|u - v\|^2_{a(Ω)} + δ(γ + δ) \int_Ω \|Δ(u - v)\|^2 dx dt \leq \\
≤ (γ + δ)^2 \int_Ω \left\| Δ(u - v) - \frac{∂(u - v)}{∂t} \right\|^2_N dx dt.
\]
And we thus have (3.3) with K = (γ + δ) < 1. Hence the Theorem (1.1) is proved.

We observe that, in view of the estimate (2.2) and the Lemma 2.3, the solution u of the Cauchy-Dirichlet problem (1.7) satisfies the following estimate
\[
(3.6) \quad \|u\|_{a(Ω)} \leq \|Δu - α \frac{∂u}{∂t}\|_{L^2(Q)} \leq α \|f\|_{L^2(Q)} / [1 - (γ + δ)].
\]

4. - Proof of Theorem 1.2

Let u ∈ W^{2,1}(Q) be a solution of the basic system
\[
(4.1) \quad a(H(u)) - \frac{∂u}{∂t} = 0 \quad \text{in} \ Ω
\]
where a(ξ) is a vector of R^N, continuous into R^n, a(0) = 0 and a(ξ) satisfies the condition (A).

Let us fix a cylinder Q(2°) = Q(x_0, 2°) cc Q.

Let θ(x) and g(t) be two C^∞ functions, respectively in R^n and R, with the following properties: 0 ≤ θ ≤ 1, θ = 1 on B(x_0, σ), θ = 0 in R^n \ B(3σ/2), |D^i θ| ≤ cσ^{-|i|} ∀ multi-indices α, and 0 ≤ g ≤ 1, g = 1 for t ≥ t_0, g = 0 for t ≤ t_0 - 3σ^2, |g'(t)| ≤ cσ^{-2}.

We set ρ_{s,b} u(X) = u(x + bε^i, t) - u(X), s = 1, ..., n and |b| < σ/4; ρ_{s,b} u(X) = u(x, t + + b) - u(X), |b| < σ/4 and let us first consider the increments with respect to the variables x_s, s = 1, ..., n.

Let
\[
(4.2) \quad \mathcal{U}(X) = θ(x) g(t) ρ_{s,b} u.
\]

Obviously u ∈ W^{2,1}_0(Q(3σ/2)). From the system (4.1) we have \(ρ_{s,b} a(H(u)) - \rho_{s,b} \frac{∂u}{∂t} = 0\) in Q(3σ/2).

On the other hand \(ρ_{s,b} a(H(u)) = a(H(ρ_{s,b} u) + H(u)) - a(H(u))\) and hence also \(Δ(ρ_{s,b} u) - αρ_{s,b} \frac{∂u}{∂t} = Δ(ρ_{s,b} u) - α[a(H(ρ_{s,b} u) + H(u)) - a(H(u))]\), where α is the positive constant which appears in the condition (A).
In view of the condition (A), we get from this that

\[(4.3) \quad \|\theta g[\Delta(\varphi_{ib} u) - \alpha\varphi_{ib} \partial u / \partial t]\|_{N} \leq \theta g[\gamma H(\varphi_{ib} u)] + \delta \|\Delta(\varphi_{ib} u)\|_{N} . \]

On the other hand

\[(4.4) \quad \Delta U = \theta g \Delta(\varphi_{ib} u) + A(u), \]
\[H(U) = \theta g H(\varphi_{ib} u) + B(u), \]
\[\partial U / \partial t = \theta g \varphi_{ib} \partial u / \partial t + \theta g' \varphi_{ib} u, \]

where

\[(4.5) \quad A(u) = g \Delta \varphi_{ib} u + 2g \sum_{i} D_{i} \theta \varphi_{ib} D_{i} u, \quad B(u) = g \sum_{i} D_{i} \theta \varphi_{ib} D_{i} (\varphi_{ib} u). \]

Hence, \(\forall \varepsilon > 0\), we have

\[(4.6) \quad \|\Delta U - \alpha \partial U / \partial t\|_{N} \leq (1 + \varepsilon)^{2} \{\gamma(\gamma + \delta) \|H(U)\|^{2} + \delta(\gamma + \delta) \|\Delta U\|^{2}\} +
+c(\varepsilon) \{\|A(u)\|^{2} + \|B(u)\|^{2} + \theta^{2} g^{2} \|\varphi_{ib} u\|^{2}\} . \]

On integrating (4.6) on \(Q(3\sigma/2)\), taking into account that \(U \in W_{0}^{2,1}(Q(3\sigma/2))\) and taking into consideration the Lemmas 2.3 and 2.4, we obtain, for \(\varepsilon\) sufficiently small, that

\[\int_{Q(3\sigma/2)} \left[\|H(U)\|^{2} + \alpha^{2} \left\|\frac{\partial U}{\partial t}\right\|^{2}\right] dx dt \leq \int_{Q(3\sigma/2)} \left\|\Delta U - \alpha \frac{\partial U}{\partial t}\right\|^{2} dx dt \leq c \int_{Q(3\sigma/2)} \left[\|A(u)\|^{2} + \|B(u)\|^{2} + g^{2} \|\varphi_{ib} u\|^{2}\right] dx dt . \]

and also

\[(4.7) \quad \int_{Q(\sigma)} \left\|\varphi_{ib} \left[H(u) + \alpha \frac{\partial u}{\partial t}\right]\right\|^{2} dx dt \leq c \int_{Q(3\sigma/2)} \left[\|A(u)\|^{2} + \|B(u)\|^{2} + g^{2} \|\varphi_{ib} u\|^{2}\right] dx dt . \]

We evaluate the right hand side of (4.7) using the Lemmas of Nirenberg (see for example [4]).

\[(4.8) \quad \int_{Q(3\sigma/2)} \|A(u)\|^{2} dx dt \leq c \sigma^{-4} \int_{Q(3\sigma/2)} \|\varphi_{ib} u\|^{2} dx dt + c \sigma^{-2} \int_{Q(3\sigma/2)} \|\varphi_{ib} Du\|^{2} dx dt \leq
\leq c |b|^{2} \left\{\sigma^{-4} \int_{Q(2\sigma)} \|Du\|^{2} dx dt + \sigma^{-2} \int_{Q(2\sigma)} \|H(u)\|^{2} dx dt\right\} . \]

One estimates

\[\int_{Q(3\sigma/2)} \|B(u)\|^{2} dx dt \]

in a similar way. Finally

\[(4.9) \quad \int_{Q(3\sigma/2)} \theta^{2} g^{2} \|\varphi_{ib} u\|^{2} dx dt \leq c \sigma^{-4} |b|^{2} \int_{Q(2\sigma)} \|Du\|^{2} dx dt .\]
From the estimates (4.7), (4.8) and (4.9) we conclude that \( DH(u) \) and \( D\frac{du}{dt} \) exist in \( L^2(Q(\sigma)) \) and we have the following estimate

\[
(4.10) \quad \int_{Q(\sigma)} \left[ \|H(Du)\|^2 + \alpha^2 \left\| \frac{\partial Du}{\partial t} \right\|^2 \right] \, dx \, dt \leq c(\sigma) \int_{Q(3\sigma)} \left[ \|Du\|^2 + \|H(u)\|^2 \right] \, dx \, dt.
\]

The same procedure can be repeated to estimate the increment with respect to the variable \( t \) and one obtains

\[
(4.11) \quad \int_{Q(\sigma)} \left[ \|\partial_{\tau} H(u)\|^2 + \alpha^2 \left\| \partial_{\tau}^2 u \right\|^2 \right] \, dx \, dt \leq c(\sigma) \int_{Q(3\sigma/2)} \left[ \left\| \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial Du}{\partial t} \right\|^2 \right] \, dx \, dt.
\]

From this, in view of the Lemmas of Nirenberg, we conclude that \( \partial H(u)/\partial t \) and \( \partial^2 u/\partial t^2 \) also exist and belong to \( L^2(Q(\sigma)) \) and we have the following estimate

\[
(4.12) \quad \int_{Q(\sigma)} \left[ \left\| \frac{\partial}{\partial t} H(u) \right\|^2 + \alpha^2 \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2 \right] \, dx \, dt \leq c(\sigma) \int_{Q(3\sigma/2)} \left[ \left\| \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial Du}{\partial t} \right\|^2 \right] \, dx \, dt.
\]

5. - A RESULT ON HÖLDER CONTINUITY

A result on Hölder continuity of the vector \( Du \) follows from the Theorem 1.2.

**Theorem 5.1.** If \( u \in W^{2,1}(Q) \) is a solution of the basic system

\[
(5.1) \quad a(H(u)) - \frac{du}{dt} = 0 \quad \text{in } Q
\]

and if \( n = 2 \) then the vector \( Du \) is Hölder continuous in \( Q \).

In fact, \( \forall Q(\sigma) \subset Q \) we have the Poincaré inequality

\[
(5.2) \quad \int_{Q(\sigma)} \|Du - (Du)_{Q(\sigma)}\|^2 \, dx \, dt \leq C(\sigma) \sigma^2 \int_{Q(\sigma)} \left[ \|H(u)\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 \right] \, dx \, dt
\]

(see for example, Lemma 2.11 in [7]).

On the other hand \( H(u) \in H^1_{\text{loc}}(Q) \) and \( \frac{du}{dt} \in H^1_{\text{loc}}(Q) \) and hence, by the Sobolev imbedding Theorem, we have \( H(u) \in L^p_{\text{loc}}(Q) \) and \( \frac{du}{dt} \in L^p_{\text{loc}}(Q) \) where \( 1/p = (n-1)/(2(n+1)) \). We also have the estimate

\[
(5.3) \quad \int_{Q(\sigma)} \|H(u)\|^2 \, dX \leq c \left( \int_{Q(\sigma)} \|H(u)\|^p \, dX \right)^{2/p} \sigma^{(n+2)(1-2/p)},
\]

\[
(5.4) \quad \int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 \, dX \leq c \left( \int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^p \, dX \right)^{2/p} \sigma^{(n+2)(1-2/p)}.
\]

It follows from (5.2), (5.3) and (5.4) that

\[
(5.5) \quad Du \in L^{2,2}_{\text{loc}} + (n+2)(1-2/p) (Q)
\]

and hence \( Du \) is Hölder continuous in \( Q \) if \( n = 2 \).
For each $Q(\sigma) \subset Q$ we also have the following Poincaré inequality

$$
\int_{Q(\sigma)} \| u - (u)_{Q(\sigma)} \|^2 \, dX \leq c \sigma^2 \int_{Q(\sigma)} \| D(u) \|^2 \, dX + c \sigma^4 \int_{Q(\sigma)} \left\| \frac{\partial u}{\partial t} \right\|^2 \, dX
$$

(5.6)

(see for instance Lemma 2.1 in [7]).

Using (5.4) and (5.5), it follows from this that

$$
u \in L^{2,4 + (n + 2)(1 - 2/p)}_{\text{loc}}(Q)
$$

(5.7)

and hence $u$ is Hölder continuous in $Q$ if $n \leq 4$.

Remembering what happens for solutions of a basic non variational elliptic system [5], we think that the Hölder continuity result of this last section is not optimal.

REFERENCES


